Exercises 7

Thursday 23 Mar 2017

Exercise 7.1

Let c_0 denote the linear space of all sequences $x = \{x_i\}$ in $\mathbb{C}$ such that $x_i \neq 0$ only for a finite number of values of i, equipped with the supremum norm $\|x\| = \sup\{|x_i| : i \geq 1\}$. Let ℓ_1 denote the space of all absolutely summable sequences $y = \{y_i\}$ in $\mathbb{C}$ with norm $\|y\|_1 = \sum_{i\geq 1} |y_i|$.

a. Show that for every $y \in \ell_1$, the map $x' : c_0 \to \mathbb{C}$,

$$x'_y(x) = \sum_{i\geq 1} y_i\, x_i,$$

defines an element of c'_0. Also show that the operator T that maps y to x'_y is an element of $B(\ell_1, c'_0)$.

b. Conversely, show that any $x' \in c'$ is mapped to an element $y_{x'}$ of ℓ^1 by the inverse of T.

c. Combining *a.* and *b.*, show that the dual c'_0 is isometrically isomorphic to ℓ_1.

d. Show that if $\lim_{n\to\infty} \sum_{i\geq 1} y_{n,i}\, x_i = 0$ for all $x \in c_0$, then $\sup\{\|y_n\|_1 : n \geq 1\} < \infty$.

Exercise 7.2

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces. Suppose that $\{T_n\}$ is a sequence in $B(X, Y)$ such that $\sup\{\|T_n\| : n \geq 1\} = \infty$.

a. Show that there exists a point $x \in X$ such that $\sup\{\|T_n x\|_Y : n \geq 1\} = \infty$.
