Functionaal Analyse

Exercises 4

Thursday 2 Mar 2017

Exercise 4.1

Let X be the space of all sequences $x = \{\xi_j\}$ of components $\xi_j \in \mathbb{C}$ such that only a finite number of components ξ_j differ from zero.

a. Show that $\|x\| = \sup\{|\xi_j| : j \geq 1\}$ is a norm on X. Is X a Banach space?

For given $\alpha \geq 0$, define $T_\alpha : X \to X$ by $(T_\alpha \xi)_j = j^{-\alpha}\xi_j$ for all $j \geq 1$.

b. Show that T_α is a bounded linear operator. Also show that T_α has an inverse $T_\alpha^{-1} : X \to X$ which is linear but not bounded.

Let c_0 denote the space of all sequences $x = \{\xi_j\}$ that converge to zero. Let $(\ell^p, \|\cdot\|_p)$ $(1 \leq p < \infty)$ denote the normed space of all sequences $x = \{\xi_j\}$ such that $\|x\|_p := (\sum_j |\xi_j|^p)^{1/p} < \infty$.

c. Show that $X \subset \ell^p \subset c_0 \subset \ell^\infty$, for any $1 \leq p < \infty$. Also show that c_0 is a closed subspace of ℓ^∞.

d. Show that X is a dense subspace in c_0 but not in ℓ^∞. Conclude that T_α has a continuous extension $S_\alpha : c_0 \to c_0$.

e. Show that X is a dense subspace of $(\ell^p, \|\cdot\|_p)$, for any $1 \leq p < \infty$. Conclude that T_α has a continuous extension $R_\alpha : \ell^p \to \ell^p$.

f. Show that if $q < p$ and α is such that $\alpha > q^{-1} - p^{-1}$, then $R_\alpha : \ell^p \to \ell^p$ has range $R_\alpha(\ell^p) = \ell^q$.

Exercise 4.2

Let c be the space of all sequences $x = \{\xi_j\}$ of components $\xi_j \in \mathbb{C}$ that converge.

a. Show that c is isomorphic to c_0. in other words, that there exists a one-to-one, continuous, linear $T : c \to c_0$ with continuous inverse T^{-1} defined on the range of T.

b. Prove that for every $x \in B_{c_0} := \{x \in c_0 : \|x\|_\infty = 1\}$, there exist $x_1, x_2 \in B_{c_0}$, $x = \frac{1}{2}(x_1 + x_2)$, while $x_1 \neq x_2$.

c. Show that there exists an $x \in B_c := \{x \in c : \|x\|_\infty = 1\}$ such that $x_1, x_2 \in B_{c_0}$, $x = \frac{1}{2}(x_1 + x_2)$ implies $x_1 = x_2$.

d. Prove from b. and c. above that there does not exist an isometric isomorphism $T : c \to c_0$.

EXERCISE 4.3

Let X be a Banach space, let $A : X \to X$ be an element of $B(X, X)$ and let $t \in \mathbb{R}$. Consider the sequence of operators $p_n : X \to X$ defined by,

$$p_n(t, A) = \sum_{k=0}^{n} \frac{t^k A^k}{k!}.$$

a. Show that $p_n \in B(X, X)$. Show that the limit $e^{tA} : X \to X$, $(t, A) \mapsto \lim_{n\to\infty} p_n(t, A)$ lies in $B(X, X)$.

b. Make sense of the identity,

$$\frac{\partial}{\partial t} e^{tA} \Big|_{t=0} = A,$$

in $B(X, X)$.
