Functionaal Analyse

Exercises 3

Thursday 23 Feb 2017

Exercise 3.1

Let X be a normed space and let $\{a_n\}$ be a sequence in X. The vectors $\{a_n\}$ are said to be *linearly independent* if, for any $n \geq 1$, a_n cannot be written as a finite linear combination from $\{a_m : m \neq n\}$. By constrast, we say that $\{a_n\}$ is *topologically independent* if, for any $n \geq 1$, a_n does not lie in the *closed* linear span $\overline{\mathrm{Sp}}\{a_m : m \neq n\}$.

a. Prove that topological independence implies linear independence.

In the rest of this problem, let X be the Banach space $(C_{\mathbb{R}}[0,1], \|\cdot\|_\infty)$. First we consider the sequence $\{f_n\} \subset C_{\mathbb{R}}[0,1]$, $f_n(x) = (2/\pi)^{1/2}\sin(nx)$.

b. For any $n \geq 1$ and all $f \in \mathrm{Sp}\{f_m : m \neq n\}$,

$$\int_0^1 \big(f_n(x) - f(x)\big)^2 dx \geq 1. \tag{1}$$

Prove this by switching to $L^2_{\mathbb{R}}[0,1]$ and using Bessel's inequality.

c. Show that it follows from (1) that for any $n \geq 1$,

$$\inf\{\, \|f_n - f\|_\infty \;:\; f \in \mathrm{Sp}\{f_m : m \neq n\}\,\} > 0, \tag{2}$$

Explain that (2) implies that $\{f_n\}$ is topologically independent.

Next we consider the sequence $\{g_n\} \subset C_{\mathbb{R}}[0,1]$, $g_n(x) = x^n$. Although clearly linearly independent, we shall see that $\{g_n\}$ is *not* topologically independent.

d. Consider sequences of real-valued polynomials $\{p_n\} \in C_{\mathbb{R}}[0,1]$ with only even powers of x, that is, for every $n \geq 1$, p_n is of the form,

$$p_n(x) = \sum_{k \in S_n} a_{n,k}\, x^{2k}$$

(for some finite subset S_n of $\{0, 1, \ldots\}$ and some finite collection of coefficients $\{a_{n,k} : k \in S_n\} \subset \mathbb{R}$). Prove that there exists a sequence $\{p_n\}$ of such polynomials that converges to the function $[0,1] \to \mathbb{R} : x \mapsto x$ in $\|\cdot\|_\infty$-norm. *(Hint: note that the function $s(y) = \sqrt{y}$ is continuous on $[0,1]$, so the Stone-Weierstrass theorem applies.)*

e. Assume that there exists a sequence $\{p_n\}$ that converges to the identity function $x \mapsto x$, as proved in part *d.*. Show that $\{g_n\}$ is linearly independent but *not* topologically independent.

EXERCISE 3.2

a. Let α, β be such that $-1 < \alpha, \beta < 1$ and define $L : \ell^2_{\mathbb{R}} \to \ell^2_{\mathbb{R}}$ component-wise for every $k \geq 1$, as follows,

$$L\big(\{\xi_n\}\big)_k = \sum_{j \geq 1} \alpha^k \, \beta^j \, \xi_j,$$

for all $\{\xi_n\} \in \ell^2_{\mathbb{R}}$. First show that for every $\xi \in \ell^2_{\mathbb{R}}$, the image $L\xi$ lies in $\ell^2_{\mathbb{R}}$. Next, show that L is bounded, with norm:

$$\|L\| = \frac{|\alpha\beta|}{\sqrt{1-\alpha^2}\sqrt{1-\beta^2}}.$$

EXERCISE 3.3

b. Let $(H_1, \langle \cdot, \cdot \rangle_1)$ and $(H_2, \langle \cdot, \cdot \rangle_2)$ be real Hilbert spaces and let $T : H_1 \to H_2$ be a bounded linear operator. Suppose that $\{e_j\}$ is an orthonormal basis for H_2. Show that there exists a sequence $\{f_j\}$ in H_1 such that, for all $x \in H_1$,

$$T\,x = \sum_{j \geq 1} \langle x, f_j \rangle_1 \, e_j.$$
