Functionaal Analyse

Exercises 2

Thursday 16 Feb 2017

Exercise 2.1

a. Let H be a Hilbert space and suppose that $f, g \in H$ are linearly independent with norms $\|f\| = \|g\| = 1$. Show that for all $0 < t < 1$, the norms of the convex combinations $\|tf + (1-t)g\| < 1$.

Exercise 2.2

a. Let L be a non-empty linear subspace of a Hilbert space H. Show that L is dense in H, if and only if, $L^{\perp} = \{0\}$. *(In other words, you are required to give a direct proof of the equivalence of (two things very much akin of) parts* (a) *and* (b) *of theorem 3.47 in Rynne and Youngson.)*

Exercise 2.3

a. Consider the elements $f_0, \ldots, f_4 \in L^2_{\mathbb{R}}[-1,1]$ corresponding to the functions $x \mapsto 1$, $x \mapsto x, \ldots, x \mapsto x^4$. Orthogonalize by Gram-Schmidt.

b. Show that the resulting elements $p_0, \ldots, p_4$ correspond with the first four functions in the sequence,

$$p_n(x) = \frac{1}{2^n n!}\Big(\frac{d}{dx}\Big)^n (x^2-1)^n,$$

(Rodrigues' formula for the Legendre polynomials.)

Exercise 2.4

a. Study the proof that the Fourier functions form an orthonormal basis for $L^2_{\mathbb{R}}[0,\pi]$ (the proof of theorem 3.54 in Rynne and Youngson). Discuss it with your fellow students, until you understand all aspects in full detail.
