Functionaal Analyse

Exercises 1

Thursday 9 Feb 2017

Exercise 1.1

Let S be a nonempty set, let

$$B(S) = \{f\colon S \to \mathbb{R}\colon\ f \text{ is bounded}\},$$

and endow $B(S)$ with the uniform metric, that is,

$$d(f,g) = \sup_{x\in S} |f(x) - g(x)|, \quad f,g \in B(S).$$

a. Prove that if $f,g \in B(S)$ and $\lambda \in \mathbb{R}$ then $f+g \in B(S)$, $fg \in B(S)$, and $\lambda f \in B(S)$.

b. Prove that $(B(S), d)$ is complete.

c. Suppose that S is a metric space and consider the subset $C_b(S)$ of $B(S)$ consisting of continuous bounded functions. Prove that $C_b(S)$ is a closed subspace of $B(S)$.

d. Does it follow from *b.* and *c.* that $C_b(S)$ endowed with the uniform metric is complete?

Exercise 1.2

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $C[a,b]$ which is pointwise bounded, that is, for every $x \in [a,b]$ there exists M_x such that $|f_n(x)| \le M_x$ for all $n \in \{1,2,3,\ldots\}$.

a. Prove that there exists a subinterval of $[a,b]$ on which the f_n are uniformly bounded, that is, there exist $c,d \in [a,b]$ with $c < d$ and $M \ge 0$ such that $|f_n(x)| \le M$ for all $x \in [c,d]$ and all $n = 1,2,3,\ldots$.

EXERCISE 1.3

Which of the following formulas defines a metric? Give for each case a proof or a counterexample.

a. $d(f,g) = \sup_{x\in[0,1]} x|f(x)-g(x)|$ (for $f,g \in B(S)$) on $B[0,1]$.
(Recall: $B[0,1]$ is the set of all bounded functions from $[0,1]$ to $\mathbb{R}$.)

b. $d(f,g) = \int_0^1 x|f(x)-g(x)|\,\mathrm{d}x$ (for $f,g \in C[0,1]$) on $C[0,1]$.

c. $d(f,g) = \int_{-1}^1 x|f(x)-g(x)|\,\mathrm{d}x$ (for $f,g \in C[-1,1]$) on $C[-1,1]$.

d. $d(f,g) = \int_a^b |f'(x)-g'(x)|\,\mathrm{d}x$ (for $f,g \in C^1[a,b]$) on $C^1[a,b]$.