Functionaal Analyse

Homework Assignment 2

23 Feb 2017

Problem 2.1

The 2-norm on $C_{\mathbb{F}}[0,1]$ is defined by

$$\|f\|_2 = \left(\int_0^1 |f(t)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}, \quad f \in C_{\mathbb{F}}[0,1].$$

a. Show that $(C_{\mathbb{F}}[0,1], \|\cdot\|_2)$ is not a Banach space.
(Hint: consider piecewise linear functions which are equal to zero from 0 to slightly below $\frac{1}{2}$ and equal to one from slightly above $\frac{1}{2}$ to 1.)

b. Conclude that the 2-norm is not equivalent to the standard norm on $C_{\mathbb{F}}[0,1]$

Problem 2.2

Let $(X, \|\cdot\|)$ be a normed space. Suppose that X has the property that a series $\sum_{n=1}^{\infty} x_n$ converges in X whenever $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

a. Show that X is a Banach space. (This is a converse to R&Y 2.30).

Problem 2.3

Let X be a vector space over $\mathbb{R}$ and let $\|\cdot\|$ be a norm on X satisfying the parallelogram rule,

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2) \text{ for all } x, y \in X.$$

Show that there exists an inner product $(\cdot,\cdot)$ on X such that $\|x\|^2 = (x,x)$ for all $x \in X$.
(Hint: first find a candidate for $(\cdot,\cdot)$, then show that $(x_1+x_2, y) = (x_1,y) + (x_2,y)$ and then show that $(\alpha x, y) = \alpha(x,y)$ consecutively for $\alpha \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, $\alpha \in \mathbb{Q}$, and $\alpha \in \mathbb{R}$.)

PROBLEM 2.4

Let $\mathscr{H}$ be a Hilbert space over $\mathbb{R}$ and let $a, b \in \mathscr{H}$ be such that $(a, b) > 0$. Prove that there exists a unique element $x \in \mathcal{H}$ of minimal norm for which both conditions

$$(x, a) \geq 1 \quad \text{and} \quad (x, b) \geq 1,$$

are satisfied.