Euclidean ϕ_4^4-theory with random inverse limit measures: existence, quantization and renormalization

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Quantum field theory

Quantum field theory (QFT) is based on the *Wightman axioms*, properties for n-point functions,

$$\big\langle \phi(x_1)\phi(x_2)\ldots\phi(x_n)\big\rangle \in S'(\mathbb{R}^{4n})$$

like **temperedness**, **Poincaré invariance**, **symmetry** and **positivity** [1, 2]. Thus far there is no mathematical theory satisfying the Wightman axioms although the question has been studied extensively, especially in a simple $d=4$ interacting QFT known as **Euclidean ϕ_4^4-theory** [3, 4].

Random inverse limit measures

For a directed set $\mathscr{A}$ of Borel measurable partitions $\alpha = (A_1,\ldots,A_{|\alpha|})$ of $\mathbb{R}^4$, define **random histograms** Φ_α with probability distributions Π_α,

$$\Phi_\alpha = \big(\Phi_\alpha(A_1),\ldots,\Phi_\alpha(A_{|\alpha|})\big) \sim \Pi_\alpha.$$

If, for all α and β that refine α,

$$\Big(\sum_{B\subset A_1}\Phi_\beta(B)\,,\ \ldots\ ,\ \sum_{B\subset A_{|\alpha|}}\Phi_\beta(B)\Big) \sim \Pi_\alpha$$

then Π_α, $(\alpha\in\mathscr{A})$ forms a **coherent inverse system of measures**. Let $\varphi_\alpha : M(\mathbb{R}^4)\to\mathbb{R}^{|\alpha|} : \mu\mapsto\mu_\alpha$ denote **histogram projections**.

> **Thm** (Bourbaki, 1967) *Let $\mathscr{A}$ be rich enough and let the Π_α form a coherent inverse system. Then there exists a Radon probability measure Π on $M(\mathbb{R}^4)$ such that $\Pi_\alpha = \Pi\circ\varphi_\alpha^{-1}$ for all α, if and only if, for all $\epsilon>0$, there exist compact $H\subset M(\mathbb{R}^4)$ such that,*
>
> $$\Pi_\alpha(\varphi_\alpha(H)) \geq 1-\epsilon,$$
>
> *for all $\alpha\in\mathscr{A}$.*

$\Pi\in M^1(M(\mathbb{R}^4))$ is the **histogram limit** of the inverse system. [5] uses the above for the *Dirichlet* and *Pólya tree* random probability measures, Kingman's *completely random measures* [6] and a new stochastic process of *Gaussian inverse limit measures* useful for QFT.

Gaussian inverse limit measures

Let λ be a signed measure on $\mathbb{R}^4$. Let Σ be a signed symmetric measure on $\mathbb{R}^4\times\mathbb{R}^4$, such that for every $\alpha\in\mathscr{A}$, the $|\alpha|\times|\alpha|$-matrix,

$$\Sigma_{\alpha,ij} = \Sigma(A_i\times A_j),$$

is **semi-positive definite**. The

$$\Phi_\alpha = \big(\Phi_\alpha(A_1),\ldots,\Phi_\alpha(A_{|\alpha|})\big) \sim N_{|\alpha|}(\lambda_\alpha,\Sigma_\alpha)$$

form a (coherent) **Gaussian inverse limit system**. Gaussian histogram limits exist under conditions on λ and Σ [5].

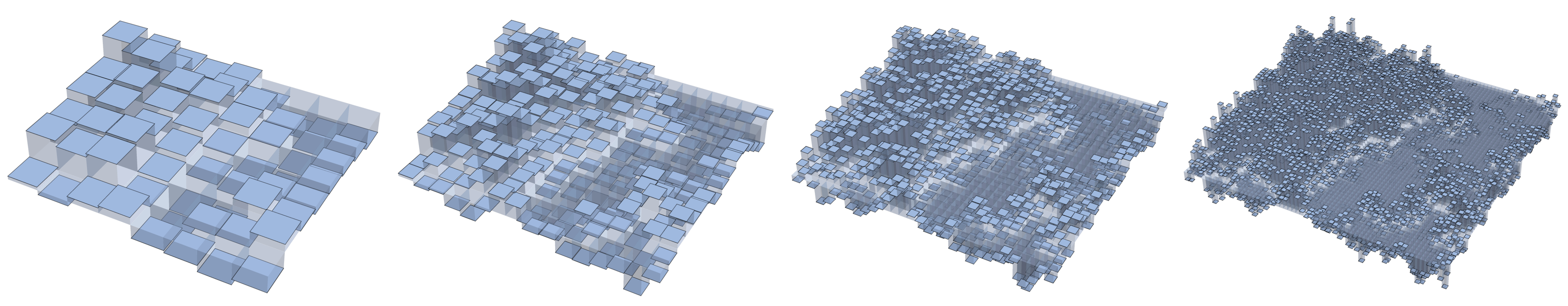

Figure 1. A sampled Gaussian histogram on a 64x64 partition (far right), and its 32x32, 16x16 and 8x8 coarsened histograms. Coherence says that the distributions of the random 8x8, 16x16 and 32x32 histograms must equal the distributions implied by coarsenings of the 64x64 histogram distribution. The histogram limit is the random object obtained by infinite refinement. (From [5].)

> **Thm** *For any **bounded Moore–Aronszajn kernel** k, there exists a centred Gaussian histogram limit Π_k, with $\lambda=0$ and,*
>
> $$\Sigma_k(A\times B) = \int_{A\times B} k(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

For example, take $k(x,y) = \triangle_\epsilon^{-1}(x,y)$, (an ϵ-regularized version of) the **Green's function for the Laplacian in $d=4$**, leading to a random measure $\Phi\sim\Pi$ that represents the **free massless scalar boson**.

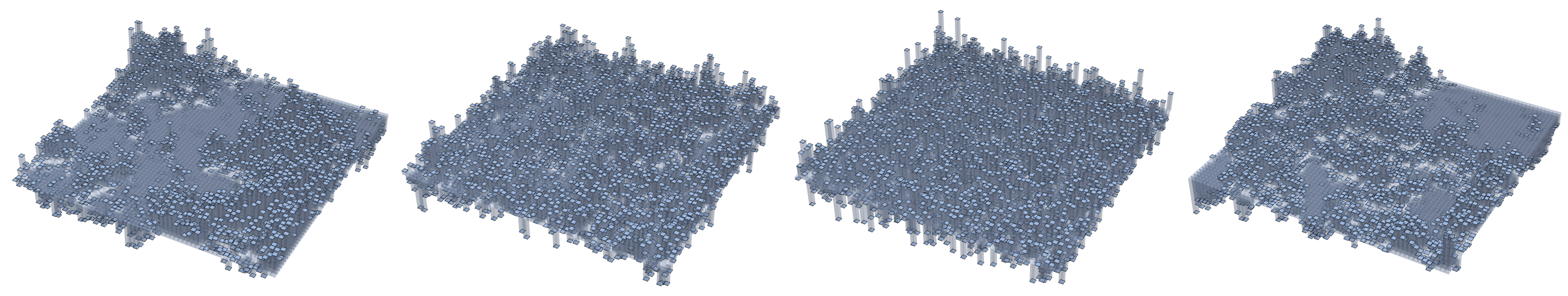

Figure 2. Sampled random histograms on a 64x64 partition, with a (regularized) Green's function $\triangle_\epsilon^{-1}(x,y)$ for the Laplacian to define Σ, in $d=2$; in $d=3$; and in $d=4$; and a sample with the Yukawa potential in $d=4$. (From [5].)

Interaction Lagrangians

For α's in a subset of $\mathscr{A}$, and with $\phi_\alpha(A) = \Phi_\alpha(A)/\mu(A)$, the quantities,

$$L_{4,\alpha}(\Phi_\alpha) = \sum_{A\in\alpha}\phi_\alpha(A)^4\,\mu(A),\quad L_{2,\alpha}(\Phi_\alpha) = \sum_{A\in\alpha}\phi_\alpha(A)^2\,\mu(A)$$

are **sub-martingales** with almost-sure limits L_4 and L_2; linear combinations serve as **interaction Lagrangians**, e.g. for ϕ_4^4-theory:

$$\Pi_I(B) = \frac{1}{Z}\int_B e^{-gL_4(\Phi)+m^2L_2(\Phi)}\,\mathrm{d}\Pi(\Phi)$$

Π_I induces a probability measure on $S'(\mathbb{R}^4)$, and,

$$G_{I,n}(f_1,\ldots,f_n) = \mathrm{E}_I(\Phi_{f_1}\ldots\Phi_{f_n}),$$

($\Phi_f = \int f\,\mathrm{d}\Phi$, for $f\in S(\mathbb{R}^4)$) defines the **n-point functions**.

Fourier domain and quantization

In the Fourier domain $\triangle^{-1}(p,q)\ \propto\ |p|^{-2}\delta(p-q)$, and the Gaussian histogram limit $\tilde{\Pi}$ describes a **completely random measure**,

$$\tilde{\Phi} = \sum_i w_i\,\delta_{p_i},$$

a **marked Poisson process** with $\tilde{\Pi}$-probability one.

> **Thm** *In the Fourier domain **quantization** in terms of particles emerges; **interacting theories** $\tilde{\Pi}_I$ are dominated by $\tilde{\Pi}$ and manifest in the same quantized form.*

This resolves the paradox posed by **Haag's theorem** [7]. Expansion of n-point functions and **Feynman calculus** match.

Renormalization and Kadanoff effective action

Refining partitions imply scaling and induce effective actions [8]:

$$\mathrm{Res}_{\sigma(\Phi_\alpha)}(\Pi_I)(B) = \frac{1}{Z}\int_B e^{-L_\alpha(\Phi_\alpha)}\,\mathrm{d}\Pi_\alpha(\Phi_\alpha).$$

with,

$$L_\alpha(\Phi_\alpha) = -\log\mathrm{E}\big[\,e^{-gL_4(\Phi)+m^2L_2(\Phi)}\,\big|\,\Phi_\alpha\,\big],$$

the **Kadanoff effective interaction Lagrangian** for partition α. Coarsening induces **renormalization group** transformations.

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