KdV institute	Nonparametric Bayesian Statistics, assignment set 2
UvA	October 30, 2014

- **Deadline: November 13, 2014.**
- **Send a pdf file with your answers to B.Kleijn@uva.nl.**
- **Your name and student number should be on your submission.**

1. Let $\mathscr{P} = \{P_\theta : \theta \in \Theta\}$ be a dominated model of distributions on the interval $[0,1]$. Assume that the parameter space Θ consists of continuous functions $\theta : [0,1] \to \mathbb{R}$ and that there exists a constant $C > 0$ such that Hellinger metric distances (see definition (8.2) of the lecture notes) satisfy,
$$h(P_{\theta_1}, P_{\theta_2}) \leq C\,\|\theta_1 - \theta_2\|_\infty,$$
for all $\theta_1, \theta_2 \in \Theta$. In addition, assume that Θ is uniformly bounded (*i.e.*, there exists a constant $M > 0$ such that $\|\theta\|_\infty < M$ for all $\theta \in \Theta$) and that the family Θ is equicontinuous.

    a. Show that $N(\delta, \mathscr{P}, h) < \infty$ for all $\delta > 0$.

    Let $P_0 \in \mathscr{P}$ be given. As a consequence of the Minimax theorem (see theorem 8.3), there exists a constant $L > 0$ such that, for every Hellinger ball W there is a test sequence (ϕ_n) such that,
$$P_0^n \phi_n + \sup_{P \in W} P^n(1 - \phi_n) \leq e^{-nLh(P_0,W)^2},$$
for all $n \geq 1$. (Here, $h(P_0, W) = \inf_{P \in W} d(P_0, P)$.) Fix some $\epsilon > 0$ and consider the complement $V = \{P \in \mathscr{P} : h(P, P_0) \geq \epsilon\}$ of the Hellinger ball of radius ϵ around P_0. For parts *b.* and *c.* below, assume that Θ is uniformly bounded and equicontinuous.

    b. Show that there exists a test sequence (ψ_n) and a constant $L' > 0$ such that,
$$P_0^n \psi_n + \sup_{P \in V} P^n(1 - \psi_n) \leq e^{-nL'\epsilon^2}.$$

    c. Vary on the proof of theorem 7.8 to show that the posterior associated with a KL-prior on Θ is consistent.

2. Approximation in measure from within by compact subsets has a deep background in analysis. Central is the following notion: for a given Hausdorff topological space Θ, a *Radon measure* Π is a Borel measure that is *locally finite* (meaning that any $\theta \in \Theta$ has a neighbourhood U such that $\Pi(U) < \infty$) and *inner regular* (meaning that for any measurable subset $S \subset \Theta$ and any $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu(S \setminus K) < \epsilon$).

a. Let Θ be a Hausdorff topological space with a finite Borel measure Π. Denote by $\mathcal{R}$ the collection of all Borel sets S for which,

$$\Pi(S) = \sup\{\Pi(K) \,:\, K \subset S,\ K \text{ compact}\},$$
$$\Pi(\Theta \setminus S) = \sup\{\Pi(K) \,:\, K \subset \Theta \setminus S,\ K \text{ compact}\}.$$

Show that $\mathcal{R}$ is a σ-algebra iff $\Theta \in \mathcal{R}$.

b. Show that any probability measure on a Polish space is Radon.

c. Keeping in mind that most (almost all, in fact) models in non-parametric statistics are parametrized by Polish spaces, comment on the nature of conditions like the first in the displayed inequalities of theorem 7.8, *(i)* of Theorem 7.9 and inequality (9.2).

3. Prove metric entropy bound (8.3) of lemma 8.6.