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Evaluation Trees for Guzmán-Squier Proposition Algebra

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Abstract

The combination of Hoare's conditional connective and McCarthy's three-valued propositional logic led to three-valued proposition algebra, which was introduced in 2021 in order to define three-valued 'short-circuit logics', including definitions of U-evaluation trees to give simple semantics. In this paper, we focus on a particular three-valued proposition algebra which we call 'Guzmán-Squier proposition algebra'. The strongest congruence on U-evaluation trees is 'Guzmán-Squier se^{U}-congruence', GSse^{U}-congruence for short, which is axiomatised by seven simple, equational axioms and is based on Conditional Logic introduced by Guzmán and Squier in 1990. The weaker 'memorisation se^{U}-congruence' is axiomatised by omitting one axiom of these seven, and serves as a stepping stone to the definition of GSse^{U}-congruence. The weakest congruence we discuss is 'se^{U}-congruence', which is axiomatised by omitting one more axiom. GSse^{U}-congruence is based on an ordering of propositional variables and we provide normal forms and normalisation functions for GSse^{U}-congruence. For the sake of completeness, we also define evaluation trees for two-valued Guzmán-Squier proposition algebra and the corresponding evaluation trees and se-congruence. Finally, we prove that all the axiomatisations considered are independent.

Keywords: short-circuit evaluation, evaluation tree, conditional logic.

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1 Introduction

Two-valued 'proposition algebras' were introduced in 2011 in [4] by Bergstra and Ponse, and are based on the constants T and F (for truth and falsehood), and the ternary connective $P \lhd Q \rhd R$, which expresses that 'short-circuit evaluation' is prescribed: first Q is evaluated; if Q evaluates to *true*, P is evaluated and determines the evaluation result of $P \lhd Q \rhd R$, and if Q evaluates to *false*, R is evaluated and this determines the evaluation result. A typical axiom is $x \lhd \mathsf{T} \rhd y = x$, which represents **if true then** x **else** $y = x$. Proposition algebra's are the mathematical structures we have used to define a family 'short-circuit logics'; we discuss this family briefly in Section 6.

In 1990, Guzmán and Squier introduced in [14] three-valued 'conditional logic' and provided a complete, equational axiomatisation of this non-commutative logic. A typical example is the asymmetry of the conditional expression $x \neq 0$ **and** $y/x > 17$. The message we wish to convey in this paper is twofold: a) to provide a systematic framework of proposition algebras for the definition of sequential connectives, in particular for three-valued short-circuit logics, and b) to introduce the specific three-valued proposition algebra that underlies conditional logic.

In 2021, we introduced 'three-valued proposition algebras' in [8], using a constant U for the truth value *undefined* and adopting the equational axiom $x \lhd \mathsf{U} \rhd y = \mathsf{U}$. In 2025, Bergstra and Ponse published the paper *Conditional logic as a short-circuit logic* [5], in which 'conditional short-circuit logic' is defined. This paper is a follow-up to [5] and we focus on a specific, three-valued proposition algebra which we now call *Guzmán-Squier proposition algebra*, 'GS proposition algebra' for short. We define *U-evaluation trees* and the congruence induced by these trees as a relatively simple semantics for GS proposition algebra. In [5], we provided axioms for GS proposition algebra, but we did not define U-evaluation trees and instead used the completeness result proved by Guzmán and Squier in [14]. Below, we first provide some context for GS proposition algebra, also for its two-valued version, and then describe the further contents of the paper.

The two-valued setting. In 1985, Hoare introduced in [15] the *conditional* connective with notation $x \lhd y \rhd z$, which, together with the constants T and F, models **if** y **then** x **else** z. In that paper, Hoare provided eleven equational axioms to show that the conditional characterises the propositional calculus.[3]

[3] In 1948, a similar result was proved by Church in [10], where he introduced *conditioned disjunction*, notation $[x, y, z]$, which also models **if** y **then** x **else** z. Moreover, in [10], the duality property of $[x, y, z]$ is highlighted and even older related work is mentioned. For

All truth functions can be defined using the conditional, in particular

$$\neg P = \mathsf{F} \lhd P \rhd \mathsf{T} \quad \text{and} \quad P \wedge Q = Q \lhd P \rhd \mathsf{F}.$$

With the conditional, algebraic properties can be elegantly expressed, e.g.,

$$\begin{aligned} &x \lhd \mathsf{T} \rhd y = x, \quad x \lhd \mathsf{F} \rhd y = y, \quad \mathsf{T} \lhd x \rhd \mathsf{F} = x \ , \quad \text{and} \\ &x \lhd (y \lhd z \rhd u) \rhd v = (x \lhd y \rhd v) \lhd z \rhd (x \lhd u \rhd v). \end{aligned} \tag{1}$$

In 2011, Bergstra and Ponse introduced *Proposition algebra* [4], motivated by the observation that Hoare's conditional connective $P \lhd Q \rhd R$ both naturally prescribes short-circuit evaluation and allows for different semantics or so-called *valuation congruences*, the strongest of which is 'static valuation congruence' that is axiomatised by Hoare's axioms, and the weakest of which is called 'free valuation congruence', which is axiomatised by the four axioms in (1) above. A 'proposition algebra' is an algebra that satisfies these four axioms. The most basic proposition algebra, axiomatised by (1), was used to define 'Free Short-Circuit Logic', the weakest short-circuit logic we defined and that is immune to side effects. In [4], several more valuation congruences in between free and static valuation congruence are defined, and it is shown that for most of these, the conditional primitive cannot be expressed by the sequential versions of binary and unary primitives.

In 2012, Staudt introduced in [23] *evaluation trees* as a simple semantics for both the axioms of proposition algebra that define free valuation congruence, and for two sequential logics; for an overview of this work see [20]. Evaluation trees are reminiscent of truth tables for propositional logic, but take sequentiality into account. At the same time, they are mathematical objects that admit transformations and compositions. In [9], we defined various *se-congruences* based on the equality of two-valued evaluation trees, where 'se' is the abbreviation for short-circuit evaluation, and we proved their equivalence with the corresponding valuation congruences defined in [4].

The three-valued setting. In 1963, McCarthy extended in [17, 18] the connectives $\wedge$ and $\vee$ to a three-valued setting by prescribing a sequential, short-circuited reading:

$$x \wedge y = y \lhd x \rhd \mathsf{F} \quad \text{and} \quad x \vee y = \mathsf{T} \lhd x \rhd y, \tag{2}$$

and provided truth tables for this reading. Using U for the truth value *undefined*, McCarthy's extension can be represented by the equations in (2) and

the conditioned disjunction, reference [11] (1956) is often used.

the equational axiom $x \trianglelefteq \mathsf{U} \trianglerighteq y = \mathsf{U}$. In [8], where we introduced three-valued proposition algebra and U-evaluation trees, we axiomatised two different se-congruences with U.

In his 1976-book *A Discipline of Programming* [12], Dijkstra introduced the connectives 'conditional and', notation **cand**, and 'conditional or', notation **cor**, that coincide with McCarthy's reading of the classical connectives and have the great advantage of having their own notation and name.

Gries discussed in his 1981-book [13] some laws for **cand** and **cor** (and some exercises on these connectives), using McCarthy's truth tables (for his reading of $\wedge$ and $\vee$, respectively). Guzmán and Squier write in [14]: *Following [13], we call this algebra "conditional logic".*

Contents of the paper. We discuss the three three-valued proposition algebras that we distinguish and define their corresponding U-evaluation trees and congruences. The weakest congruence on U-evaluation trees is 'se^{U}-congruence', which is axiomatised by the axioms in (1) and $x \trianglelefteq \mathsf{U} \trianglerighteq y = \mathsf{U}$. 'Memorising proposition algebra' is axiomatised by adding one axiom that captures the memorisation of evaluations of propositional variables and excludes side effects. For this setting, definitions of'memorising U-evaluation trees' and 'memorising se^{U}-congruence' are provided. Adding one more axiom defines 'GS proposition algebra'. Adopting an ordering of propositional variables, we define 'Guzmán-Squier U-evaluation trees' and 'Guzmán-Squier se^{U}-congruence'. Furthermore, we provide normal forms and normalisation functions for closed terms for all proposition algebras considered.

Novel aspects of the paper. In Sections 2 and 3, the tree composition function $_ \mathbin{\widetilde{\trianglelefteq}} _ \mathbin{\widetilde{\trianglerighteq}} _$ is new and used to prove completeness results (Theorems 2.12 and 3.16).[4] In Sections 4 and 5, all definitions of 'Guzmán-Squier notions' are new. Finally, the independence result in Appendix A is new.

Structure of the paper. In Section 2, we recall U-evaluation trees, the axiom system CP^{U}, and se^{U}-congruence. In Section 3, we extend $\mathrm{CP}^{\mathsf{U}}$to $\mathrm{CP}^{\mathsf{U}}_{mem}$, define 'memorising U-evaluation trees', and prove that $\mathrm{CP}^{\mathsf{U}}_{mem}$ axiomatises 'memorising se^{U}-congruence'. In Section 4, we extend $\mathrm{CP}^{\mathsf{U}}_{mem}$ to $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$, define GS U-evaluation trees, and show that the corresponding $\mathrm{GS}se^{\mathsf{U}}$-congruence is axiomatised by $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$. Then, in Section 5, we restrict the contents of Section 4 to the two-valued setting. In Section 6, we give a brief overview

[4] This function is related to the functions $\widetilde{\neg}$, $\widetilde{\bigwedge}$ and $\widetilde{\bigvee}$ from [8], which were used to prove a completeness result for a different set of primitives.

to our work on short-circuit logics and conclude that Conditional Logic and GS proposition algebra model an important congruence in the field of three-valued logic. Finally, we establish that two-valued GS proposition algebra also models an important congruence. In Appendix A, we give a detailed proof of the independence of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$, and hence of all axiomisations considered in this paper. In Appendix B we repeat a non-trivial proof from [5].

2 Evaluation Trees for se^{U}-congruence

In this section, we recall the axiomatisation CP^{U} of se^{U}-congruence for three-valued conditional statements, which is needed to define the subsequent more complex se^{U}-congruences in the next two sections. We note that this congruence and also Theorem 2.12 already appear in [8, Def.7.1 & Prn.7.13], but here we use a new approach that is much simpler. First, we introduce the tree composition function (Definition 2.4) and secondly, we provide a normalisation function for three-valued conditional statements, which is also used in the following sections.

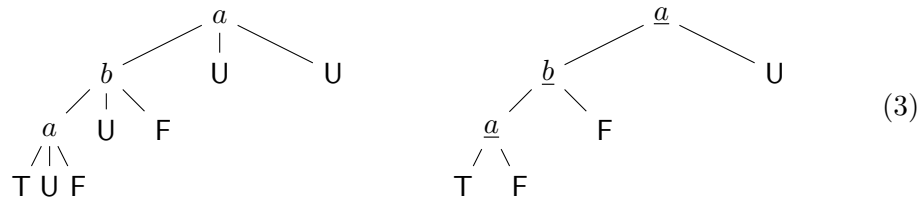
Let A be a set of 'atoms' (propositional variables).

Definition 2.1. *The set $\mathcal{T}^{\mathsf{U}}_A$ of U-evaluation trees over A with leaves in $\{\mathsf{T},\mathsf{F},\mathsf{U}\}$ is defined inductively by*

$$B \in \mathcal{T}^{\mathsf{U}}_A \text{ for } B \in \{\mathsf{T},\mathsf{F},\mathsf{U}\}, \text{ and } X \trianglelefteq \underline{a} \trianglerighteq Y \in \mathcal{T}^{\mathsf{U}}_A \text{ if } X,Y \in \mathcal{T}^{\mathsf{U}}_A \text{ and } a \in A.$$

The operator $_ \trianglelefteq \underline{a} \trianglerighteq _$ is called U-tree composition over a. In the evaluation tree $X \trianglelefteq \underline{a} \trianglerighteq Y$, the root is represented by a, the left branch by X, the right branch by Y, and the underlining of the root represents a middle branch to the leaf U.

U-evaluation trees were first defined in [8], and in that paper we provided two types of graphical representation of these trees, which we introduce here using the example $((\mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F}) \trianglelefteq \underline{b} \trianglerighteq \mathsf{F}) \trianglelefteq \underline{a} \trianglerighteq \mathsf{U}$:



(3)

We prefer the image on the right and will henceforth illustrate other U-evaluation trees with underlined atoms. The following auxiliary function on U-evaluation trees plays a major role.

Definition 2.2 (Leaf replacement)**.** *Given U-evaluation trees $Y, Z \in \mathcal{T}_A^{\mathsf{U}}$, the* ***leaf replacement function*** $[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ *for which postfix notation*

$$X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]$$

is adopted, is defined as follows, where $a \in A$:

$$\begin{aligned}
\mathsf{T}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] &= Y,\\
\mathsf{F}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] &= Z,\\
\mathsf{U}[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] &= \mathsf{U},\\
(X \trianglelefteq \underline{a} \trianglerighteq X')[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] &= X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z] \trianglelefteq \underline{a} \trianglerighteq X'[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z].
\end{aligned}$$

We note that the order in which the replacements of leaves of X is listed is irrelevant and adopt the convention of not listing identities inside the brackets, e.g., $X[\mathsf{F} \mapsto Z] = X[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto Z]$. Furthermore, repeated replacements satisfy the following equation, which follows easily by structural induction on X:

$$\begin{aligned}
&X[\mathsf{T} \mapsto Y_1, \mathsf{F} \mapsto Z_1]\ [\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2]\\
&= X[\mathsf{T} \mapsto Y_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2],\ \mathsf{F} \mapsto Z_1[\mathsf{T} \mapsto Y_2, \mathsf{F} \mapsto Z_2]].
\end{aligned}$$

Given a set A of atoms, we define the signature

$$\Sigma^{\mathsf{U}}_{\mathrm{CP}(A)} = \{_ \trianglelefteq _ \trianglerighteq _, \mathsf{T}, \mathsf{F}, \mathsf{U}, a \mid a \in A\},$$

where the constants T, F and U represent the truth values *true*, *false* and *undefined*. Let $\mathcal{C}_A^{\mathsf{U}}$ be the set of closed terms over $\Sigma^{\mathsf{U}}_{\mathrm{CP}(A)}$. We now have the terminology and notation to define the interpretation of conditional statements in $\mathcal{C}_A^{\mathsf{U}}$ as U-evaluation trees by a function se^{U} (abbreviating short-circuit evaluation with undefinedness).

Definition 2.3. *The unary* ***U-evaluation function***

$$se^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$$

yields ***U-evaluation trees*** *and is defined as follows ($a \in A$):*

$$\begin{aligned}
se^{\mathsf{U}}(B) &= B \quad \textit{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\},\\
se^{\mathsf{U}}(a) &= \mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F},\\
se^{\mathsf{U}}(P \trianglelefteq Q \trianglerighteq R) &= se^{\mathsf{U}}(Q)[\mathsf{T} \mapsto se^{\mathsf{U}}(P), \mathsf{F} \mapsto se^{\mathsf{U}}(R)].
\end{aligned}$$

A simple example is $se^{\mathsf{U}}((a \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq \mathsf{U}) = ((\mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F}) \trianglelefteq \underline{b} \trianglerighteq \mathsf{F}) \trianglelefteq \underline{a} \trianglerighteq \mathsf{U}$ (see also picture (3)).

Table 1: The set CP^{U} of axioms for proposition algebra with undefinedness

$$x \lhd \mathsf{T} \rhd y = x \tag{CP1}$$
$$x \lhd \mathsf{F} \rhd y = y \tag{CP2}$$
$$\mathsf{T} \lhd x \rhd \mathsf{F} = x \tag{CP3}$$
$$x \lhd (y \lhd z \rhd u) \rhd v = (x \lhd y \rhd v) \lhd z \rhd (x \lhd u \rhd v) \tag{CP4}$$
$$x \lhd \mathsf{U} \rhd y = \mathsf{U} \tag{CP-U}$$

Definition 2.4. *The* ***tree composition function***

$$_ \mathbin{\tilde{\lhd}} _ \mathbin{\tilde{\rhd}} _ : \mathcal{T}_A^{\mathsf{U}} \times \mathcal{T}_A^{\mathsf{U}} \times \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$$

is defined by $X \mathbin{\tilde{\lhd}} Y \mathbin{\tilde{\rhd}} Z = Y[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z]$.

It immediately follows that $se^{\mathsf{U}}(P \lhd Q \rhd R) = se^{\mathsf{U}}(P) \mathbin{\tilde{\lhd}} se^{\mathsf{U}}(Q) \mathbin{\tilde{\rhd}} se^{\mathsf{U}}(R)$. In the following we introduce the set CP^{U} of axioms, and we will use the tree composition function to prove the validity of these axioms.

Definition 2.5. CP^{U} *is the set of axioms in Table 1, and* CP *is the set* $\mathrm{CP}^{\mathsf{U}} \setminus \{\text{(CP-U)}\}$.

The axiom system CP (the name abbreviates Conditional Propositions) was first defined in [4], and CP^{U} was first defined in [8].

The *dual* P^d of $P \in \mathcal{C}_A^{\mathsf{U}}$ is defined as follows, where $a \in A$:

$$\mathsf{T}^d = \mathsf{F}, \quad \mathsf{F}^d = \mathsf{T}, \quad \mathsf{U}^d = \mathsf{U}, \quad a^d = a, \quad (P \lhd Q \rhd R)^d = R^d \lhd Q^d \rhd P^d.$$

Observe that CP^{U} is a self-dual axiomatisation: when defining $x^d = x$ for each variable x, the dual of each axiom is also in CP^{U}, and hence

$$\mathrm{CP}^{\mathsf{U}} \vdash P = Q \iff \mathrm{CP}^{\mathsf{U}} \vdash P^d = Q^d.$$

With CP^{U} at our disposal, we can provide the following formal definition.

Definition 2.6. *A* ***three-valued proposition algebra*** *is an algebra with signature* $\Sigma^{\mathsf{U}}_{\mathrm{CP}(A)}$ *that satisfies the axioms of* CP^{U}.

It will prove useful to define the following set of closed terms in $\mathcal{C}_A^{\mathsf{U}}$.

Definition 2.7. ***Basic forms over*** *A are defined by the following grammar:*

$$t ::= \mathsf{T} \mid \mathsf{F} \mid \mathsf{U} \mid t \lhd a \rhd t \quad \textit{for } a \in A.$$

We write BF_A^{U} for the set of basic forms over A.

The following lemmas exploit the structure of basic forms.[5]

Lemma 2.8. *For each $P \in \mathcal{C}_A^{\mathsf{U}}$ there exists $Q \in BF_A^{\mathsf{U}}$ such that*

$$\mathrm{CP}^{\mathsf{U}} \vdash P = Q.$$

Proof. First, we establish an auxiliary result: if P, Q, R are basic forms, then there is a basic form S such that $\mathrm{CP}^{\mathsf{U}} \vdash P \lhd Q \rhd R = S$. This follows by structural induction on Q.

The lemma's statement follows by structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}, a \mid a \in A\}$ are trivial, and if $P = P_1 \lhd P_2 \rhd P_3$ there exist by induction basic forms Q_i such that $\mathrm{CP}^{\mathsf{U}} \vdash P_i = Q_i$, hence $\mathrm{CP}^{\mathsf{U}} \vdash P_1 \lhd P_2 \rhd P_3 = Q_1 \lhd Q_2 \rhd Q_3$. Now apply the auxiliary result. □

Lemma 2.9. *For all basic forms P and Q, $se^{\mathsf{U}}(P) = se^{\mathsf{U}}(Q)$ implies $P = Q$.*

Proof. By structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial. If $P = P_1 \lhd a \rhd P_2$, then $Q \notin \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ and $Q \neq Q_1 \lhd b \rhd Q_2$ if $b \neq a$, so $Q = Q_1 \lhd a \rhd Q_2$ and $se^{\mathsf{U}}(P_i) = se^{\mathsf{U}}(Q_i)$. By induction we find $P_i = Q_i$, and hence $P = Q$. □

The relation on $\mathcal{C}_A^{\mathsf{U}}$ that identifies propositional statements that have identical U-evaluation trees is a congruence. First, we give a formal definition of this relation and then prove this fact.

Definition 2.10. *The binary relation* ***se^{U}-congruence****, notation $=_{se^{\mathsf{U}}}$, is defined on $\mathcal{C}_A^{\mathsf{U}}$ as follows:*

$$P =_{se^{\mathsf{U}}} Q \iff se^{\mathsf{U}}(P) = se^{\mathsf{U}}(Q).$$

Lemma 2.11. *The relation $=_{se^{\mathsf{U}}}$ is a congruence.*

[5] We use the term "basic form" instead of "normal form" in order to avoid intuitions from term rewriting: for example, the basic form of atom a is $\mathsf{T} \lhd a \rhd \mathsf{F}$, whereas one would expect that the normal form of the latter is a.

Proof. Let $P, P', Q, Q', R, R' \in \mathcal{C}_A^{\mathsf{U}}$ and $P =_{se^{\mathsf{U}}} P'$, $Q =_{se^{\mathsf{U}}} Q'$ and $R =_{se^{\mathsf{U}}} R'$. Then

$$\begin{aligned} se^{\mathsf{U}}(P \lhd Q \rhd R) &= se^{\mathsf{U}}(P) \mathbin{\tilde{\lhd}} se^{\mathsf{U}}(Q) \mathbin{\tilde{\rhd}} se^{\mathsf{U}}(R) \\ &= se^{\mathsf{U}}(P') \mathbin{\tilde{\lhd}} se^{\mathsf{U}}(Q') \mathbin{\tilde{\rhd}} se^{\mathsf{U}}(R') \\ &= se^{\mathsf{U}}(P' \lhd Q' \rhd R'). \end{aligned}$$

□

Theorem 2.12 (Completeness of CP^{U})**.** *For all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$,*

$$\mathrm{CP}^{\mathsf{U}} \vdash P = Q \iff P =_{se^{\mathsf{U}}} Q.$$

Moreover, CP^{U} *is an independent axiomatisation.*

Proof. ($\Rightarrow$)[6] By Lemma 2.11, $=_{se^{\mathsf{U}}}$ is a congruence relation. Furthermore, it easily follows that closed instances of CP^{U}-axioms are valid and we only show this for axiom (CP4). For $i = 1, 2$, let $X_i, Y_i, Z \in \mathcal{T}_A^{\mathsf{U}}$. Then

$$\begin{aligned} &X_1 \mathbin{\tilde{\lhd}} (Y_1 \mathbin{\tilde{\lhd}} Z \mathbin{\tilde{\rhd}} Y_2) \rhd X_2 \\ &\quad = (Y_1 \mathbin{\tilde{\lhd}} Z \mathbin{\tilde{\rhd}} Y_2)[\mathsf{T} \mapsto X_1, \mathsf{F} \mapsto X_2] \\ &\quad = \big(Z[\mathsf{T} \mapsto Y_1, \mathsf{F} \mapsto Y_2]\big)\,[\mathsf{T} \mapsto X_1, \mathsf{F} \mapsto X_2] \\ &\quad = Z[\mathsf{T} \mapsto Y_1[\mathsf{T} \mapsto X_1, \mathsf{F} \mapsto X_2], \mathsf{F} \mapsto Y_2[\mathsf{T} \mapsto X_1, \mathsf{F} \mapsto X_2]] \\ &\quad = Z[\mathsf{T} \mapsto (X_1 \mathbin{\tilde{\lhd}} Y_1 \mathbin{\tilde{\rhd}} X_2), \mathsf{F} \mapsto (X_1 \mathbin{\tilde{\lhd}} Y_2 \mathbin{\tilde{\rhd}} X_2)] \\ &\quad = (X_1 \mathbin{\tilde{\lhd}} Y_1 \mathbin{\tilde{\rhd}} X_2) \mathbin{\tilde{\lhd}} Z \mathbin{\tilde{\rhd}} (X_1 \mathbin{\tilde{\lhd}} Y_2 \mathbin{\tilde{\rhd}} X_2). \end{aligned}$$

Hence, for arbitrary $P_i, Q_i, R \in \mathcal{C}_A^{\mathsf{U}}$,

$$se^{\mathsf{U}}(P_1 \lhd (Q_1 \lhd R \rhd Q_2) \rhd P_2) = se^{\mathsf{U}}((P_1 \lhd Q_1 \rhd P_2) \lhd R \rhd (P_1 \lhd Q_2 \rhd P_2)).$$

($\Leftarrow$) Let $P =_{se^{\mathsf{U}}} Q$. According to Lemma 2.8 there exist basic forms P' and Q' such that $\mathrm{CP}^{\mathsf{U}} \vdash P = P'$ and $\mathrm{CP}^{\mathsf{U}} \vdash Q = Q'$. By ($\Rightarrow$) we find $P =_{se^{\mathsf{U}}} P'$ and $Q =_{se^{\mathsf{U}}} Q'$, and because $=_{se^{\mathsf{U}}}$ is a congruence, $P' =_{se^{\mathsf{U}}} Q'$. By Lemma 2.9, $P' = Q'$. Hence, $\mathrm{CP}^{\mathsf{U}} \vdash P = P' = Q' = Q$.

The independence of CP^{U} follows from Theorem 4.16. □

In [9], we defined the so-called basic form function $bf(P)$, which we extend here to the three-valued setting by means of the following two definitions. This function is a normalisation function and will also prove useful in the following sections.

[6] Without loss of generality it can be assumed that substitutions happen first in equational proofs (see, e.g., [1]).

Definition 2.13. *Given $Q, R \in BF_A^{\mathsf{U}}$, the auxiliary function*

$$[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] : BF_A^{\mathsf{U}} \to BF_A^{\mathsf{U}}$$

for which postfix notation $P[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]$ is used, is defined as follows:

$$\begin{aligned}
\mathsf{T}[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= Q,\\
\mathsf{F}[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= R,\\
\mathsf{U}[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= \mathsf{U},\\
(P_1 \lhd a \rhd P_2)[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] &= P_1[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] \lhd a \rhd P_2[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R].
\end{aligned}$$

(The notational overloading with the leaf replacement function on U-evaluation trees is harmless.)

Definition 2.14. *The **basic form function** $bf^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to BF_A^{\mathsf{U}}$ is defined as follows, where $a \in A$:*

$$\begin{aligned}
bf^{\mathsf{U}}(B) &= B \quad \textit{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\},\\
bf^{\mathsf{U}}(a) &= \mathsf{T} \lhd a \rhd \mathsf{F} \quad \textit{for } a \in A,\\
bf^{\mathsf{U}}(P \lhd Q \rhd R) &= bf^{\mathsf{U}}(Q)[\mathsf{T} \mapsto bf^{\mathsf{U}}(P), \mathsf{F} \mapsto bf^{\mathsf{U}}(R)].
\end{aligned}$$

These two definitions immediately imply the following characterisation of the basic form function bf^{U}.

Proposition 2.15 (bf^{U} is a normalisation function)**.** *For all $P \in \mathcal{C}_A^{\mathsf{U}}$, $bf^{\mathsf{U}}(P)$ is a basic form, and for each basic form $P \in BF_A^{\mathsf{U}}$, $bf^{\mathsf{U}}(P) = P$.*

The following result links the function bf^{U} to the U-evaluation function se^{U} and is used to define one more characterisation of se^{U}-congruence.

Lemma 2.16. *For all $P \in \mathcal{C}_A^{\mathsf{U}}$, $se^{\mathsf{U}}(bf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(P)$.*

Proof. First, we prove an auxiliary result:

$$\begin{aligned}
&\text{For all } P, Q, R \in BF_A^{\mathsf{U}},\\
&se^{\mathsf{U}}(P[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]) = se^{\mathsf{U}}(P)[\mathsf{T} \mapsto se^{\mathsf{U}}(Q), \mathsf{F} \mapsto se^{\mathsf{U}}(R)].
\end{aligned} \tag{4}$$

Result (4) follows by structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$

are trivial. If $P = P_1 \lhd a \rhd P_2$, then derive

$$\begin{aligned}
&se^{\mathsf{U}}((P_1 \lhd a \rhd P_2)[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]) \\
&\quad = se^{\mathsf{U}}(P_1[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R] \lhd a \rhd P_2[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]) \\
&\quad = se^{\mathsf{U}}(a)[\mathsf{T} \mapsto se^{\mathsf{U}}(P_1[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R]), \mathsf{F} \mapsto se^{\mathsf{U}}(P_2[\mathsf{T} \mapsto Q, \mathsf{F} \mapsto R])] \\
&\quad = se^{\mathsf{U}}(a)[\mathsf{T} \mapsto se^{\mathsf{U}}(P_1)[\mathsf{T} \mapsto se^{\mathsf{U}}(Q), \mathsf{F} \mapsto se^{\mathsf{U}}(R)], \\
&\qquad\qquad \mathsf{F} \mapsto se^{\mathsf{U}}(P_2)[\mathsf{T} \mapsto se^{\mathsf{U}}(Q), \mathsf{F} \mapsto se^{\mathsf{U}}(R)]] \qquad \text{by IH (twice)} \\
&\quad = (se^{\mathsf{U}}(a)[\mathsf{T} \mapsto se^{\mathsf{U}}(P_1), \mathsf{F} \mapsto se^{\mathsf{U}}(P_2)])[\mathsf{T} \mapsto se^{\mathsf{U}}(Q), \mathsf{F} \mapsto se^{\mathsf{U}}(R)] \\
&\quad = se^{\mathsf{U}}(P_1 \lhd a \rhd P_2)[\mathsf{T} \mapsto se^{\mathsf{U}}(Q), \mathsf{F} \mapsto se^{\mathsf{U}}(R)].
\end{aligned}$$

The lemma's statement follows by structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial, and if $P = a \in A$, then $se^{\mathsf{U}}(bf^{\mathsf{U}}(a)) = se^{\mathsf{U}}(\mathsf{T} \lhd a \rhd \mathsf{F}) = se^{\mathsf{U}}(a)[\mathsf{T} \mapsto \mathsf{T}, \mathsf{F} \mapsto \mathsf{F}] = se^{\mathsf{U}}(a)$. If $P = P_1 \lhd P_2 \rhd P_3$, then

$$\begin{aligned}
&se^{\mathsf{U}}(bf^{\mathsf{U}}(P_1 \lhd P_2 \rhd P_3)) \\
&\quad = se^{\mathsf{U}}(bf^{\mathsf{U}}(P_2)[\mathsf{T} \mapsto bf^{\mathsf{U}}(P_1), \mathsf{F} \mapsto bf^{\mathsf{U}}(P_3)]) \\
&\quad = se^{\mathsf{U}}(bf^{\mathsf{U}}(P_2))[\mathsf{T} \mapsto se^{\mathsf{U}}(bf^{\mathsf{U}}(P_1)), \mathsf{F} \mapsto se^{\mathsf{U}}(bf^{\mathsf{U}}(P_3))] \qquad \text{by (4)} \\
&\quad = se^{\mathsf{U}}(P_2)[\mathsf{T} \mapsto se^{\mathsf{U}}(P_1), \mathsf{F} \mapsto se^{\mathsf{U}}(P_3)] \qquad \text{by IH} \\
&\quad = se^{\mathsf{U}}(P_1 \lhd P_2 \rhd P_3).
\end{aligned}$$

□

Definition 2.17. *The binary relation $=_{bf^{\mathsf{U}}}$ on $\mathcal{C}_A^{\mathsf{U}}$ is defined by $P =_{bf^{\mathsf{U}}} Q$ if $bf^{\mathsf{U}}(P) = bf^{\mathsf{U}}(Q)$.*

Theorem 2.12 and Lemma 2.16 imply the following consequences.

Corollary 2.18. *1. For all $P \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}} \vdash bf^{\mathsf{U}}(P) = P$, and hence for all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}} \vdash P = Q \iff P =_{bf^{\mathsf{U}}} Q$.*

2. For all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$, $P =_{bf^{\mathsf{U}}} Q \iff P =_{se^{\mathsf{U}}} Q$.

Proof. 1. By Lemma 2.16, $se^{\mathsf{U}}(bf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(P)$, so by Theorem 2.12, $\mathrm{CP}^{\mathsf{U}} \vdash bf^{\mathsf{U}}(P) = P$. Consequently, if $\mathrm{CP}^{\mathsf{U}} \vdash P = Q$, then $\mathrm{CP}^{\mathsf{U}} \vdash bf^{\mathsf{U}}(P) = bf^{\mathsf{U}}(Q)$, hence $se^{\mathsf{U}}(bf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(bf^{\mathsf{U}}(Q))$, so according to Lemma 2.16, $bf^{\mathsf{U}}(P) = bf^{\mathsf{U}}(Q)$, so $P =_{bf^{\mathsf{U}}} Q$. Conversely, if $P =_{bf^{\mathsf{U}}} Q$, then $se^{\mathsf{U}}(bf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(bf^{\mathsf{U}}(Q))$, so according to Theorem 2.12, $\mathrm{CP}^{\mathsf{U}} \vdash bf^{\mathsf{U}}(P) = bf^{\mathsf{U}}(Q)$, so by $\mathrm{CP}^{\mathsf{U}} \vdash bf^{\mathsf{U}}(R) = R$ for all $R \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}} \vdash P = Q$.

2. This consequence follows immediately from 1 and Theorem 2.12. □

3 Evaluation Trees for Memorising se^{U}-congruence

In this section, we define memorising se^{U}-congruence for three-valued conditional statements, which is a stepping stone to the more complex se^{U}-congruence that we define in Section 4. In [8], we introduced $\mathrm{CP}^{\mathsf{U}}_{mem}$, "memorising CP with undefinedness", as a means to define *memorising short-circuit logic with undefinedness*, but we refrained from defining the corresponding se^{U}-congruence.[7] In this section, we study memorising se^{U}-congruence without using previous results on short-circuit logic.

In [4], Bergstra and Ponse defined CP_{mem}, so-called *memorising* CP, as the extension of CP with the following axiom:

$$x \lhd y \rhd (z \lhd v \rhd (z' \lhd y \rhd w)) = x \lhd y \rhd (z \lhd v \rhd w). \tag{CPmem}$$

Axiom (CPmem) expresses that the first evaluation value of y is memorised. More precisely, a "memorising evaluation" is one with the property that upon the evaluation of a compound propositional statement, the first evaluation value of each atom is memorised throughout the evaluation.

Definition 3.1. $\mathrm{CP}^{\mathsf{U}}_{mem}$ *is the set of axioms* $\mathrm{CP}^{\mathsf{U}} \cup \{(\mathrm{CPmem})\}$*, and* $\mathrm{CP}_{mem} = \mathrm{CP}^{\mathsf{U}}_{mem} \setminus \{(\mathrm{CP\text{-}U})\}$.

Replacing the variable y in axiom (CPmem) by $\mathsf{F} \lhd y \rhd \mathsf{T}$ and/or the variable v by $\mathsf{F} \lhd v \rhd \mathsf{T}$ yields all other memorising patterns:

$$(z \lhd v \rhd (w \lhd y \rhd z')) \lhd y \rhd x = (z \lhd v \rhd w) \lhd y \rhd x, \tag{CPm1}$$
$$x \lhd y \rhd ((z' \lhd y \rhd w) \lhd v \rhd z) = x \lhd y \rhd (w \lhd v \rhd z), \tag{CPm2}$$
$$((w \lhd y \rhd z') \lhd v \rhd z) \lhd y \rhd x = (w \lhd v \rhd z) \lhd y \rhd x. \tag{CPm3}$$

Observe that (CPmem) and (CPm3) are each other's duals (just as (CPm1) and (CPm2) are), so the duality principle also holds in $\mathrm{CP}^{\mathsf{U}}_{mem}$. Furthermore, if we replace in axiom (CPmem) v by F and change some variable names, we find the *contraction law*

$$x \lhd y \rhd (z \lhd y \rhd v) = x \lhd y \rhd v, \tag{5}$$

and replacing y by $\mathsf{F} \lhd y \rhd \mathsf{T}$ yields the dual contraction law

$$(v \lhd y \rhd z) \lhd y \rhd x = v \lhd y \rhd x. \tag{6}$$

We define a restricted subset of basic forms with the property that under memorising evaluations, every propositional statement can be proved equal to such a basic form.

[7] We could do without this congruence for conditional statements because the conditional connective can be expressed in (two-valued) memorising short-circuit logic.

Definition 3.2. *Let A' be a subset of A.* ***Mem-basic forms over*** *A' are inductively defined:*

- T, F *and* U *are mem-basic forms over A', and*
- $P \lhd a \rhd Q$ *is a mem-basic form over A' if $a \in A'$ and P and Q are mem-basic forms over $A' \setminus \{a\}$.*

P is a ***mem-basic form*** *if for some $A' \subset A$, P is a mem-basic form over A'. We write MBF_A^{U} for the set of mem-basic forms over A.*

Note that if A is finite, the number of mem-basic forms is also finite.

We define a transformation on U-evaluation trees and we will prove that equality of two such transformed trees characterises the congruence that is axiomatized by CP_{mem}.

Definition 3.3. *The unary* ***memorising U-evaluation function***

$$mse^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$$

yields ***memorising U-evaluation trees*** *and is defined by*

$$mse^{\mathsf{U}}(P) = m^{\mathsf{U}}(se^{\mathsf{U}}(P)).$$

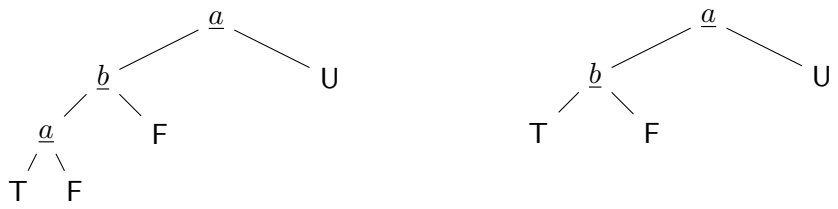
The auxiliary function $m^{\mathsf{U}} : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ is defined as follows ($a \in A$):

$$m^{\mathsf{U}}(B) = B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, \text{ and}$$
$$m^{\mathsf{U}}(X \trianglelefteq \underline{a} \trianglerighteq Y) = m^{\mathsf{U}}(L_a^{\mathsf{U}}(X)) \trianglelefteq \underline{a} \trianglerighteq m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y)).$$

For $a \in A$, the auxiliary functions $L_a^{\mathsf{U}} : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ ("Left a-reduction") and $R_a^{\mathsf{U}} : \mathcal{T}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$ ("Right a-reduction") are defined by

$$L_a^{\mathsf{U}}(B) = R_a^{\mathsf{U}}(B) = B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\},$$
$$L_a^{\mathsf{U}}(X \trianglelefteq \underline{b} \trianglerighteq Y) = \begin{cases} L_a^{\mathsf{U}}(X) & \text{if } b = a, \\ L_a^{\mathsf{U}}(X) \trianglelefteq \underline{b} \trianglerighteq L_a^{\mathsf{U}}(Y) & \text{otherwise,} \end{cases}$$
$$R_a^{\mathsf{U}}(X \trianglelefteq \underline{b} \trianglerighteq Y) = \begin{cases} R_a^{\mathsf{U}}(Y) & \text{if } b = a, \\ R_a^{\mathsf{U}}(X) \trianglelefteq \underline{b} \trianglerighteq R_a^{\mathsf{U}}(Y) & \text{otherwise.} \end{cases}$$

As a simple example we depict $se^{\mathsf{U}}((a \lhd b \rhd \mathsf{F}) \lhd a \rhd \mathsf{U})$ (see also picture (3)) and the memorising U-evaluation tree $mse^{\mathsf{U}}((a \lhd b \rhd \mathsf{F}) \lhd a \rhd \mathsf{U})$:



The relation on $\mathcal{C}_A^{\mathsf{U}}$ that identifies propositional statements that have the same memorising U-evaluation trees is a congruence. We first give a formal definition of this relation and then prove this fact in Lemma 3.6.

Definition 3.4. ***Memorising se$^{\mathsf{U}}$-congruence****, notation $=_{mse^{\mathsf{U}}}$, is defined on $\mathcal{C}_A^{\mathsf{U}}$ as follows:*

$$P =_{mse^{\mathsf{U}}} Q \iff mse^{\mathsf{U}}(P) = mse^{\mathsf{U}}(Q).$$

We start with an auxiliary result (Lemma 3.5) that is identical to Lemma A.9.(2) in [21], except that we now include the case $\mathsf{U} \in \mathcal{T}_A^{\mathsf{U}}$. However, this is resolved by adding a trivial base case to the inductive proof of the cited lemma (namely, the base case $X = \mathsf{U}$).

Lemma 3.5. *For all $a \in A$, $f \in \{L^{\mathsf{U}}, R^{\mathsf{U}}\}$ and $X, Y, Z \in \mathcal{T}_A^{\mathsf{U}}$,*

$$f_a(X[\mathsf{T} \mapsto Y, \mathsf{F} \mapsto Z]) = f_a(X)[\mathsf{T} \mapsto f_a(Y), \mathsf{F} \mapsto f_a(Z)].$$

Lemma 3.6. *Memorising se$^{\mathsf{U}}$-congruence is a congruence on $\mathcal{C}_A^{\mathsf{U}}$.*

Proof. It suffices to prove that for all $X, X', Y, Y', Z, Z' \in \mathcal{T}_A^{\mathsf{U}}$,

$$\left.\begin{array}{l} m^{\mathsf{U}}(X) = m^{\mathsf{U}}(X') \\ m^{\mathsf{U}}(Y) = m^{\mathsf{U}}(Y') \\ m^{\mathsf{U}}(Z) = m^{\mathsf{U}}(Z') \end{array}\right\} \Rightarrow m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} Z) = m^{\mathsf{U}}(X' \mathbin{\tilde{\triangleleft}} Y' \mathbin{\tilde{\triangleright}} Z'). \tag{7}$$

First, observe that for all $a \in A$ and $V \in \mathcal{T}_A^{\mathsf{U}}$,

$$m^{\mathsf{U}}(V) = m^{\mathsf{U}}(V') \Rightarrow \begin{cases} m^{\mathsf{U}}(L_a^{\mathsf{U}}(V)) = m^{\mathsf{U}}(L_a^{\mathsf{U}}(V')), \\ m^{\mathsf{U}}(R_a^{\mathsf{U}}(V)) = m^{\mathsf{U}}(R_a^{\mathsf{U}}(V')). \end{cases} \tag{8}$$

We prove (7) by induction on the depth $d(Y)$ of Y. If $d(Y) = 0$, then $d(Y') = 0$ (necessarily), so both are in $\{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ and are therefore equal.

If $d(Y) = n+1$, then $Y = Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2$ and by $m^{\mathsf{U}}(Y) = m^{\mathsf{U}}(Y')$ it must be that $Y' = Y_1' \trianglelefteq \underline{a} \trianglerighteq Y_2'$, hence $m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1)) = m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1'))$ and $m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y_2)) =$

$m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y_2'))$. Note that $d(L_a^{\mathsf{U}}(Y_1)) < d(Y)$ and $d(R_a^{\mathsf{U}}(Y_2)) < d(Y)$, and derive

$$
\begin{aligned}
& m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} Z) \\
&\quad = m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} (Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2) \mathbin{\tilde{\triangleright}} Z) \\
&\quad = m^{\mathsf{U}}((Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2)[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z]) \\
&\quad = m^{\mathsf{U}}(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z] \trianglelefteq \underline{a} \trianglerighteq Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z]) \\
&\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z])) \trianglelefteq \underline{a} \trianglerighteq \\
&\qquad m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto Z])) \\
&\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1)[\mathsf{T} \mapsto L_a^{\mathsf{U}}(X), \mathsf{F} \mapsto L_a^{\mathsf{U}}(Z)]) \trianglelefteq \underline{a} \trianglerighteq \\
&\qquad m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y_2)[\mathsf{T} \mapsto R_a^{\mathsf{U}}(X), \mathsf{F} \mapsto R_a^{\mathsf{U}}(Z)]) && \text{by Lemma 3.5} \\
&\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(X) \mathbin{\tilde{\triangleleft}} L_a^{\mathsf{U}}(Y_1) \mathbin{\tilde{\triangleright}} L_a^{\mathsf{U}}(Z)) \trianglelefteq \underline{a} \trianglerighteq \\
&\qquad m^{\mathsf{U}}(R_a^{\mathsf{U}}(X) \mathbin{\tilde{\triangleleft}} R_a^{\mathsf{U}}(Y_2) \mathbin{\tilde{\triangleright}} R_a^{\mathsf{U}}(Z)) \\
&\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(X') \mathbin{\tilde{\triangleleft}} L_a^{\mathsf{U}}(Y_1') \mathbin{\tilde{\triangleright}} L_a^{\mathsf{U}}(Z')) \trianglelefteq \underline{a} \trianglerighteq \\
&\qquad m^{\mathsf{U}}(R_a^{\mathsf{U}}(X') \mathbin{\tilde{\triangleleft}} R_a^{\mathsf{U}}(Y_2') \mathbin{\tilde{\triangleright}} R_a^{\mathsf{U}}(Z')) && \text{by IH and (8)} \\
&\quad \vdots \\
&\quad = m^{\mathsf{U}}(X' \mathbin{\tilde{\triangleleft}} (Y_1' \trianglelefteq \underline{a} \trianglerighteq Y_2') \mathbin{\tilde{\triangleright}} Z') && \text{as above} \\
&\quad = m^{\mathsf{U}}(X' \mathbin{\tilde{\triangleleft}} Y' \mathbin{\tilde{\triangleright}} Z').
\end{aligned}
$$

□

Later on, in Theorem 3.16, we prove that memorising se^{U}-congruence is axiomatised by $\mathrm{CP}^{\mathsf{U}}_{mem}$. We start with a soundness result.

Theorem 3.7 (Soundness of $\mathrm{CP}^{\mathsf{U}}_{mem}$)**.** *For all $P, Q \in \mathcal{C}_A$,*

$$\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P = Q \;\Rightarrow\; P =_{mse^{\mathsf{U}}} Q.$$

Proof. With Lemma 3.6 and the soundness of CP^{U} (Theorem 3.7), it suffices to show that the axiom (CPmem) is valid, which is a consequence of the following property:

$$
\begin{aligned}
&\text{For all } X, Y, Z, Z', V, W \in \mathcal{T}_A^{\mathsf{U}}, \\
&m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))) = m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)).
\end{aligned} \tag{9}
$$

We prove (9) by induction on $d(Y)$. The base cases $d(Y) = 0$, thus $Y \in \{\mathsf{T}, \mathsf{F}.\mathsf{U}\}$, are trivial. For the case $Y = Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2$, first note that Lemma 3.5 implies that for $f \in \{L^{\mathsf{U}}, R^{\mathsf{U}}\}$,

$$f_a(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} Z) = f_a(X) \mathbin{\tilde{\triangleleft}} f_a(Y) \mathbin{\tilde{\triangleright}} f_a(Z). \tag{10}$$

Derive

$$\begin{aligned}
&m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W)))\\
&\quad = m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} (Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2) \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W)))\\
&\quad = m^{\mathsf{U}}((Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2)[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))])\\
&\quad = m^{\mathsf{U}}(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))] \trianglelefteq \underline{a} \trianglerighteq\\
&\qquad\quad Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))])\\
&\quad = m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))])) \trianglelefteq \underline{a} \trianglerighteq\\
&\qquad m^{\mathsf{U}}(R^{\mathsf{U}}_a(Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))]))\\
&\quad = m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1)[\mathsf{T} \mapsto L^{\mathsf{U}}_a(X), \mathsf{F} \mapsto L^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))]) \trianglelefteq \underline{a} \trianglerighteq\\
&\qquad m^{\mathsf{U}}(R^{\mathsf{U}}_a(Y_2)[\mathsf{T} \mapsto R^{\mathsf{U}}_a(X), \mathsf{F} \mapsto R^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))]).
\end{aligned}$$

In the last two leaf replacements, replace Y by $Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2$ and apply (10):

$$\begin{aligned}
&m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1)[\mathsf{T} \mapsto L^{\mathsf{U}}_a(X), \mathsf{F} \mapsto L^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))])\\
&= m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1)[\mathsf{T} \mapsto L^{\mathsf{U}}_a(X),\\
&\qquad\qquad \mathsf{F} \mapsto L^{\mathsf{U}}_a(Z) \mathbin{\tilde{\triangleleft}} L^{\mathsf{U}}_a(V) \mathbin{\tilde{\triangleright}} (L^{\mathsf{U}}_a(Z') \mathbin{\tilde{\triangleleft}} L^{\mathsf{U}}_a(Y_1) \mathbin{\tilde{\triangleright}} L^{\mathsf{U}}_a(W))])\\
&= m^{\mathsf{U}}(L^{\mathsf{U}}_a(X) \mathbin{\tilde{\triangleleft}} L^{\mathsf{U}}_a(Y_1) \mathbin{\tilde{\triangleright}} (L^{\mathsf{U}}_a(Z) \mathbin{\tilde{\triangleleft}} L^{\mathsf{U}}_a(V) \mathbin{\tilde{\triangleright}} (L^{\mathsf{U}}_a(Z') \mathbin{\tilde{\triangleleft}} L^{\mathsf{U}}_a(Y_1) \mathbin{\tilde{\triangleright}} L^{\mathsf{U}}_a(W)))),
\end{aligned} \tag{11}$$

and likewise,

$$\begin{aligned}
&m^{\mathsf{U}}(R^{\mathsf{U}}_a(Y_2)[\mathsf{T} \mapsto R^{\mathsf{U}}_a(X), \mathsf{F} \mapsto R^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} (Z' \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} W))])\\
&= m^{\mathsf{U}}(R^{\mathsf{U}}_a(X) \mathbin{\tilde{\triangleleft}} R^{\mathsf{U}}_a(Y_2) \mathbin{\tilde{\triangleright}} (R^{\mathsf{U}}_a(Z) \mathbin{\tilde{\triangleleft}} R^{\mathsf{U}}_a(V) \mathbin{\tilde{\triangleright}} (R^{\mathsf{U}}_a(Z') \mathbin{\tilde{\triangleleft}} R^{\mathsf{U}}_a(Y_2) \mathbin{\tilde{\triangleright}} R^{\mathsf{U}}_a(W)))).
\end{aligned} \tag{12}$$

In order to apply the induction hypothesis, expand the right-hand side of (9):

$$\begin{aligned}
&m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} Y \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W))\\
&= m^{\mathsf{U}}(X \mathbin{\tilde{\triangleleft}} (Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2) \mathbin{\tilde{\triangleright}} (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W))\\
&= m^{\mathsf{U}}((Y_1 \trianglelefteq \underline{a} \trianglerighteq Y_2)[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)])\\
&= m^{\mathsf{U}}(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)] \trianglelefteq \underline{a} \trianglerighteq Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)])\\
&= m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)])) \trianglelefteq \underline{a} \trianglerighteq\\
&\quad m^{\mathsf{U}}(R^{\mathsf{U}}_a(Y_2[\mathsf{T} \mapsto X, \mathsf{F} \mapsto (Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)]))\\
&= m^{\mathsf{U}}(L^{\mathsf{U}}_a(Y_1)[\mathsf{T} \mapsto L^{\mathsf{U}}_a(X), \mathsf{F} \mapsto L^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W))]) \trianglelefteq \underline{a} \trianglerighteq\\
&\quad m^{\mathsf{U}}(R^{\mathsf{U}}_a(Y_2)[\mathsf{T} \mapsto R^{\mathsf{U}}_a(X), \mathsf{F} \mapsto R^{\mathsf{U}}_a(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)]).
\end{aligned}$$

Expand the last two leaf replacements using (10),

$$\begin{aligned} &m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1)[\mathsf{T} \mapsto L_a^{\mathsf{U}}(X), \mathsf{F} \mapsto L_a^{\mathsf{U}}(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)]) \\ &\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(Y_1)[\mathsf{T} \mapsto L_a^{\mathsf{U}}(X), \mathsf{F} \mapsto L_a^{\mathsf{U}}(Z) \mathbin{\tilde{\triangleleft}} L_a^{\mathsf{U}}(V) \mathbin{\tilde{\triangleright}} L_a^{\mathsf{U}}(W)]) \\ &\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(X) \mathbin{\tilde{\triangleleft}} L_a^{\mathsf{U}}(Y_1) \mathbin{\tilde{\triangleright}} (L_a^{\mathsf{U}}(Z) \mathbin{\tilde{\triangleleft}} L_a^{\mathsf{U}}(V) \mathbin{\tilde{\triangleright}} L_a^{\mathsf{U}}(W))), \end{aligned} \tag{13}$$

and secondly,

$$\begin{aligned} &m^{\mathsf{U}}(R_a^{\mathsf{U}}(Y_2)[\mathsf{T} \mapsto R_a^{\mathsf{U}}(X), \mathsf{F} \mapsto R_a^{\mathsf{U}}(Z \mathbin{\tilde{\triangleleft}} V \mathbin{\tilde{\triangleright}} W)]) \\ &\quad = m^{\mathsf{U}}(R_a^{\mathsf{U}}(X) \mathbin{\tilde{\triangleleft}} R_a^{\mathsf{U}}(Y_2) \mathbin{\tilde{\triangleright}} (R_a^{\mathsf{U}}(Z) \mathbin{\tilde{\triangleleft}} R_a^{\mathsf{U}}(V) \mathbin{\tilde{\triangleright}} R_a^{\mathsf{U}}(W))). \end{aligned} \tag{14}$$

By $d(L_a^{\mathsf{U}}(Y_1)) < d(Y)$ and $d(R_a^{\mathsf{U}}(Y_2)) < d(Y)$ and induction (twice), it follows that (11) = (13) and (12) = (14), which together prove property (9). □

The completeness of $\mathrm{CP}^{\mathsf{U}}_{mem}$ is less straightforward to prove than that of CP^{U} (Theorem 2.12). First, we introduce a normalisation function that transforms conditional statements into mem-basic forms.

Definition 3.8. *The* ***mem-basic form function*** $mbf^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to BF_A^{\mathsf{U}}$ *is defined by*

$$mbf^{\mathsf{U}}(P) = mf^{\mathsf{U}}(bf^{\mathsf{U}}(P)).$$

The auxiliary function $mf^{\mathsf{U}} : BF_A^{\mathsf{U}} \to BF_A^{\mathsf{U}}$ *is defined as follows:*

$$\begin{aligned} mf^{\mathsf{U}}(B) &= B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, \\ mf^{\mathsf{U}}(P \mathbin{\triangleleft} a \mathbin{\triangleright} Q) &= mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(P)) \mathbin{\triangleleft} a \mathbin{\triangleright} mf^{\mathsf{U}}(r_a^{\mathsf{U}}(Q)). \end{aligned}$$

For $a \in A$*, the auxiliary functions* $\ell_a^{\mathsf{U}} : BF_A \to BF_A$ *and* $r_a^{\mathsf{U}} : BF_A^{\mathsf{U}} \to BF_A^{\mathsf{U}}$ *are defined by*

$$\begin{aligned} \ell_a^{\mathsf{U}}(B) = r_a^{\mathsf{U}}(B) &= B \quad \text{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, \\ \ell_a^{\mathsf{U}}(P \mathbin{\triangleleft} b \mathbin{\triangleright} Q) &= \begin{cases} \ell_a^{\mathsf{U}}(P) & \text{if } b = a, \\ \ell_a^{\mathsf{U}}(P) \mathbin{\triangleleft} b \mathbin{\triangleright} \ell_a^{\mathsf{U}}(Q) & \text{otherwise}, \end{cases} \\ r_a^{\mathsf{U}}(P \mathbin{\triangleleft} b \mathbin{\triangleright} Q) &= \begin{cases} r_a^{\mathsf{U}}(Q) & \text{if } b = a, \\ r_a^{\mathsf{U}}(P) \mathbin{\triangleleft} b \mathbin{\triangleright} r_a^{\mathsf{U}}(Q) & \text{otherwise}. \end{cases} \end{aligned}$$

Thus, mbf^{U} maps a conditional statement P to $bf^{\mathsf{U}}(P)$ and then transforms $bf^{\mathsf{U}}(P)$ according to the auxiliary functions mf^{U}, ℓ_a^{U}, and r_a^{U}. We will use the following inequalities.

Lemma 3.9. *For all $a \in A$ and $P \in BF_A^{\mathsf{U}}$, $d(P) \geq d(\ell_a^{\mathsf{U}}(P))$ and $d(P) \geq d(r_a^{\mathsf{U}}(P))$.*

Proof. Fix some $a \in A$. We prove these inequalities by structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial.

For the inductive case $P = Q \lhd b \rhd R$ we have to distinguish the cases $b = a$ and $b \neq a$. If $b = a$, then

$$\begin{aligned} d(Q \lhd a \rhd R) &= 1 + \max\{d(Q), d(R)\} \\ &\geq 1 + d(Q) \\ &\geq 1 + d(\ell_a^{\mathsf{U}}(Q)) && \text{by IH} \\ &= 1 + d(\ell_a^{\mathsf{U}}(Q \lhd a \rhd R)), \end{aligned}$$

and $d(Q \lhd a \rhd R) \geq d(r_a^{\mathsf{U}}(Q \lhd a \rhd R))$ follows in a similar way. If $b \neq a$, then

$$\begin{aligned} d(Q \lhd b \rhd R) &= 1 + \max\{d(Q), d(R)\} \\ &\geq 1 + \max\{d(\ell_a^{\mathsf{U}}(Q)), d(\ell_a^{\mathsf{U}}(R))\} && \text{by IH} \\ &= d(\ell_a^{\mathsf{U}}(Q) \lhd b \rhd \ell_a^{\mathsf{U}}(R)) \\ &= d(\ell_a^{\mathsf{U}}(Q \lhd b \rhd R)), \end{aligned}$$

and $d(Q \lhd b \rhd R) \geq d(r_a^{\mathsf{U}}(Q \lhd b \rhd R))$ follows in a similar way. □

The following lemnma is analogous to Lemma 2.8, except for the use of the mem-basic form function, which is crucial for its proof.

Lemma 3.10. *For all $P \in \mathcal{C}_A^{\mathsf{U}}$, $mbf^{\mathsf{U}}(P)$ is a mem-basic form.*

Proof. First, we prove an auxiliary result:

$$\text{For all } P \in BF_A^{\mathsf{U}},\ mf^{\mathsf{U}}(P) \text{ is a mem-basic form.} \tag{15}$$

This follows by induction on the depth $d(P)$ of P. If $d(P) = 0$, then $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$, and hence $mf^{\mathsf{U}}(P) = P$ is a mem-basic form. For the inductive case $d(P) = n + 1$ it must be the case that $P = Q \lhd a \rhd R$. We find

$$mf^{\mathsf{U}}(Q \lhd a \rhd R) = mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd a \rhd mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R)),$$

which is a mem-basic form because by Lemma 3.9, $\ell_a^{\mathsf{U}}(Q)$ and $r_a^{\mathsf{U}}(R)$ are basic forms with depth smaller than or equal to n, so by the induction hypothesis, $mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q))$ is a mem-basic form over A_Q and $mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R))$ is a mem-basic form over A_R for suitable subsets A_Q and A_R of A. Note that by definition of ℓ_a^{U} and r_a^{U}, the atom a does not occur in $A_Q \cup A_R$. Hence,

$mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd a \rhd mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R))$ is a mem-basic form over $A_Q \cup A_R \cup \{a\}$, which completes the proof of (15).

The lemma's statement follows by structural induction: the base cases (comprising a single atom a) are again trivial, and for the inductive case, we find

$$mbf^{\mathsf{U}}(P \lhd Q \rhd R) = mf^{\mathsf{U}}(bf^{\mathsf{U}}(P \lhd Q \rhd R)) \overset{\text{La.2.9}}{=} mf^{\mathsf{U}}(S)$$

for some basic form S. By (15), $mf^{\mathsf{U}}(S)$ is a mem-basic form. □

With Lemma 3.10 we can easily prove the following result, which will be used to prove that for all $P \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}_{mem}^{\mathsf{U}} \vdash P = mbf^{\mathsf{U}}(P)$ (Theorem 3.13).

Proposition 3.11 (mbf^{U} is a normalisation function)**.** *For all $P \in \mathcal{C}_A$, $mbf^{\mathsf{U}}(P)$ is a mem-basic form, and for each mem-basic form P, $mbf^{\mathsf{U}}(P) = P$.*

Proof. The first statement is Lemma 3.10. For the second statement, it suffices by Proposition 2.15 to prove that for each mem-basic form P, $mf^{\mathsf{U}}(P) = P$. We prove this by induction on $d(P)$.

The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial. For the inductive case, assume $P = Q \lhd a \rhd R$, thus

$$mf^{\mathsf{U}}(P) = mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd a \rhd mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R)).$$

Because P is a mem-basic form, Q and R are mem-basic forms in which a does not occur, and thus $\ell_a^{\mathsf{U}}(Q) = Q$ and $r_a^{\mathsf{U}}(R) = R$. By induction, $mf^{\mathsf{U}}(Q) = Q$ and $mf^{\mathsf{U}}(R) = R$, and thus $mf^{\mathsf{U}}(P) = P$. □

Lemma 3.12. *For all $P \in BF_A^{\mathsf{U}}$, $\mathrm{CP}_{mem}^{\mathsf{U}} \vdash P = mf^{\mathsf{U}}(P)$.*

Proof. First, we prove an auxiliary result:

$$\text{For all } a \in A \text{ and } P, Q \in BF_A^{\mathsf{U}},\ \mathrm{CP}_{mem}^{\mathsf{U}} \vdash P \lhd a \rhd Q = P \lhd a \rhd r_a^{\mathsf{U}}(Q), \quad (16)$$
$$P \lhd a \rhd Q = \ell_a^{\mathsf{U}}(P) \lhd a \rhd Q. \quad (17)$$

Fix some $a \in A$. We prove (16) by structural induction on Q. The base cases $Q \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial. For the inductive case $Q = Q_1 \lhd b \rhd Q_2$ we have to distinguish the cases $b = a$ and $b \neq a$. If $b = a$, then $r_a^{\mathsf{U}}(Q) = r_a^{\mathsf{U}}(Q_2)$ and

$$\begin{aligned} \mathrm{CP}_{mem}^{\mathsf{U}} \vdash P \lhd a \rhd (Q_1 \lhd a \rhd Q_2) &= P \lhd a \rhd Q_2 && \text{by (5)} \\ &= P \lhd a \rhd r_a^{\mathsf{U}}(Q_2) && \text{by IH} \\ &= P \lhd a \rhd r_a^{\mathsf{U}}(Q). \end{aligned}$$

If $b \neq a$, then $r_a^{\mathsf{U}}(Q) = r_a^{\mathsf{U}}(Q_1) \lhd b \rhd r_a^{\mathsf{U}}(Q_2)$ and

$$\begin{aligned}
&\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P \lhd a \rhd (Q_1 \lhd b \rhd Q_2)\\
&= P \lhd a \rhd ((\mathsf{T} \lhd a \rhd Q_1) \lhd b \rhd (\mathsf{T} \lhd a \rhd Q_2)) && \text{by (CPmem), (CPm2)}\\
&= P \lhd a \rhd ((\mathsf{T} \lhd a \rhd r_a^{\mathsf{U}}(Q_1)) \lhd b \rhd (\mathsf{T} \lhd a \rhd r_a^{\mathsf{U}}(Q_2))) && \text{by IH (twice)}\\
&= P \lhd a \rhd (r_a^{\mathsf{U}}(Q_1) \lhd b \rhd r_a^{\mathsf{U}}(Q_2)) && \text{by (CPmem), (CPm2)}\\
&= P \lhd a \rhd r_a^{\mathsf{U}}(Q).
\end{aligned}$$

Auxiliary result (17) follows in a similar way with help of axioms (CPm1) and (CPm3).

The lemma's statement follows by induction on $d(P)$. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial. For the inductive case, assume $P = Q \lhd a \rhd R$. Derive

$$\begin{aligned}
\mathrm{CP}_{mem} \vdash Q \lhd a \rhd R&\\
&= \ell_a^{\mathsf{U}}(Q) \lhd a \rhd r_a^{\mathsf{U}}(R) && \text{by (16), (17)}\\
&= mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd a \rhd mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R)) && \text{by IH (and Lemma 3.9)}\\
&= mf^{\mathsf{U}}(Q \lhd a \rhd R).
\end{aligned}$$

□

Theorem 3.13. *For all* $P \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P = mbf^{\mathsf{U}}(P)$.

Proof. By Corollary 2.18.1, $\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P = bf^{\mathsf{U}}(P)$, and by Lemma 3.12, $\mathrm{CP}^{\mathsf{U}}_{mem} \vdash bf^{\mathsf{U}}(P) = mf^{\mathsf{U}}(bf^{\mathsf{U}}(P))$, and $mf^{\mathsf{U}}(bf^{\mathsf{U}}(P)) = mbf^{\mathsf{U}}(P)$. □

The similarities between mse^{U} and the function mbf^{U} will be exploited.

Lemma 3.14. *For all* $P \in BF_A^{\mathsf{U}}$, $m^{\mathsf{U}}(se^{\mathsf{U}}(P)) = se^{\mathsf{U}}(mf^{\mathsf{U}}(P))$.

Proof. First, we prove an auxiliary result:

$$\begin{aligned}
&\text{For all } a \in A \text{ and } P \in BF_A^{\mathsf{U}},\\
&L_a^{\mathsf{U}}(se^{\mathsf{U}}(P)) = se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(P)) \text{ and } R_a^{\mathsf{U}}(se^{\mathsf{U}}(P)) = se^{\mathsf{U}}(r_a^{\mathsf{U}}(P)).
\end{aligned} \tag{18}$$

Fix some $a \in A$. We prove (18) by structural induction on P. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ are trivial. For the inductive case $P = Q \lhd b \rhd R$ we have to distinguish the cases $b = a$ and $b \neq a$. If $b = a$, then

$$\begin{aligned}
L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q \lhd a \rhd R)) &= L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q) \trianglelefteq \underline{a} \trianglerighteq se^{\mathsf{U}}(R))\\
&= L_a^{\mathsf{U}}(se(Q))\\
&= se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) && \text{by IH}\\
&= se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q \lhd a \rhd R)),
\end{aligned}$$

and if $b \neq a$, then

$$\begin{aligned} L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q \lhd b \rhd R)) &= L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q) \trianglelefteq \underline{b} \trianglerighteq se^{\mathsf{U}}(R)) \\ &= L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q)) \trianglelefteq \underline{b} \trianglerighteq L_a^{\mathsf{U}}(se^{\mathsf{U}}(R)) \\ &= se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd b \rhd se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(R)) \qquad \text{by IH (twice)} \\ &= se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q \lhd b \rhd R)). \end{aligned}$$

The second equality can be derived in a similar way, which finishes the proof of (18).

We prove the lemma's statement by induction on $d(P)$. The base cases $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ follow immediately. Assume $P = Q \lhd a \rhd R$, then

$$\begin{aligned} &m^{\mathsf{U}}(se^{\mathsf{U}}(Q \lhd a \rhd R)) \\ &\quad = m^{\mathsf{U}}(se^{\mathsf{U}}(Q) \trianglelefteq \underline{a} \trianglerighteq se^{\mathsf{U}}(R)) \\ &\quad = m^{\mathsf{U}}(L_a^{\mathsf{U}}(se^{\mathsf{U}}(Q))) \trianglelefteq \underline{a} \trianglerighteq m^{\mathsf{U}}(R_a^{\mathsf{U}}(se^{\mathsf{U}}(R))) \\ &\quad = m^{\mathsf{U}}(se^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q))) \trianglelefteq \underline{a} \trianglerighteq m^{\mathsf{U}}(se^{\mathsf{U}}(r_a^{\mathsf{U}}(R))) \qquad \text{by (18)} \\ &\quad = se^{\mathsf{U}}(mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q))) \trianglelefteq \underline{a} \trianglerighteq se^{\mathsf{U}}(mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R))) \quad \text{by IH (and Lemma 3.9)} \\ &\quad = se^{\mathsf{U}}(mf^{\mathsf{U}}(\ell_a^{\mathsf{U}}(Q)) \lhd a \rhd mf^{\mathsf{U}}(r_a^{\mathsf{U}}(R))) \\ &\quad = se^{\mathsf{U}}(mf^{\mathsf{U}}(Q \lhd a \rhd R)). \end{aligned}$$

□

Corollary 3.15. *For all $P \in \mathcal{C}_A^{\mathsf{U}}$, $mse^{\mathsf{U}}(P) = se^{\mathsf{U}}(mbf^{\mathsf{U}}(P))$.*

Theorem 3.16 (Completeness of $\mathrm{CP}_{mem}^{\mathsf{U}}$)**.** *For all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$,*

$$\mathrm{CP}_{mem}^{\mathsf{U}} \vdash P = Q \iff P =_{mse^{\mathsf{U}}} Q.$$

Moreover, $\mathrm{CP}_{mem}^{\mathsf{U}}$ is an independent axiomatisation.

Proof. According to Theorem 3.7 (Soundness of $\mathrm{CP}_{mem}^{\mathsf{U}}$), only $\Leftarrow$ needs to be proved. Assume $P =_{mse^{\mathsf{U}}} Q$. By Corollary 3.15, $se^{\mathsf{U}}(mbf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(mbf^{\mathsf{U}}(Q))$, so by Lemma 2.9. $P =_{mbf^{\mathsf{U}}} Q$. According to Theorem 3.13, $\mathrm{CP}_{mem}^{\mathsf{U}} \vdash P = Q$.

The independence of $\mathrm{CP}_{mem}^{\mathsf{U}}$ follows from Theorem 4.16. □

Theorems 3.13, 3.16 and Corollary 3.15 can be used to estabish a completeness result for $\mathrm{CP}_{mem}^{\mathsf{U}}$ in terms of the normalisation function mbf^{U}.

Corollary 3.17.

1. *For all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}} \vdash P = Q \iff P =_{mbf^{\mathsf{U}}} Q$.*

2. For all $P, Q \in \mathcal{C}_A^{\mathsf{U}}$, $P =_{mse^{\mathsf{U}}} Q \iff P =_{mbf^{\mathsf{U}}} Q$.

Proof. 1. ($\Rightarrow$) If $\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P = Q$, then according to Theorem 3.16, $P =_{mse^{\mathsf{U}}} Q$, so by Corollary 3.15, $se^{\mathsf{U}}(mbf^{\mathsf{U}}(P)) = se^{\mathsf{U}}(mbf^{\mathsf{U}}(Q))$, so by Lemma 2.9, $mbf^{\mathsf{U}}(P) = mbf^{\mathsf{U}}(Q)$.

($\Leftarrow$) If $mbf^{\mathsf{U}}(P) = mbf^{\mathsf{U}}(Q)$, then the se^{U}-images of both are also equal, hence by Corollary 3.15, $mse^{\mathsf{U}}(P) = mse^{\mathsf{U}}(Q)$. According to Theorem 3.16, $\mathrm{CP}^{\mathsf{U}}_{mem} \vdash P = Q$.

Consequence 2 follows immediately from 1 and Theorem 3.16. □

4 Evaluation Trees for Guzmán-Squier se^{U}-congruence

In 1990, *Conditional Logic* was defined by Guzmán and Squier in [14]. A typical axiom for this logic is

$$(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y),$$

where, following Gries [13] and McCarthy [17, 18], the connectives $\wedge$ and $\vee$ prescribe the sequential, short-circuited reading mentioned in (2) in Section 1, i.e.,

$$x \wedge y = y \lhd x \rhd \mathsf{F} \quad \text{and} \quad x \vee y = \mathsf{T} \lhd x \rhd y.$$

In [5], the following axioms are defined:

$$\mathsf{T} \lhd (y \lhd x \rhd \mathsf{F}) \rhd (x \lhd y \rhd \mathsf{F}) = \mathsf{T} \lhd (x \lhd y \rhd \mathsf{F}) \rhd (y \lhd x \rhd \mathsf{F}), \tag{CL1}$$

$$(x \lhd y \rhd z) \lhd u \rhd (v \lhd y \rhd w) = (x \lhd u \rhd v) \lhd y \rhd (z \lhd u \rhd w). \tag{CL2}$$

Hence, axiom (CL1) represents the axiom that characterises conditional logic. In [5, La.3.1] it is shown that

$$\mathrm{CP}_{mem} \cup \{(\mathrm{CL1})\} \vdash (\mathrm{CL2}) \quad \text{and} \quad \mathrm{CP}_{mem} \cup \{(\mathrm{CL2})\} \vdash (\mathrm{CL1}),$$

and we repeat a proof of this fact in Appendix B. In this section, we define Guzmán-Squier U-evaluation trees and GSse^{U}-congruence, and we provide a complete axiomatisation of this congruence.

Definition 4.1. $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ *is the set of axioms* $\mathrm{CP}^{\mathsf{U}}_{mem} \cup \{(\mathrm{CL2})\}$*, and* $\mathrm{CP}_{\mathrm{GS}}$ *is the set* $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \setminus \{(\mathrm{CP\text{-}U})\}$.

In [5], Bergstra and Ponse introduced $\mathrm{CP}^{\mathsf{U}}_{cl}$, an axiomatic extention of $\mathrm{CP}^{\mathsf{U}}_{mem}$ based on Conditional Logic, hence the subscript *cl*. We later realised

that, in the context of proposition algebra, this use of the term 'conditional' could lead to confusion, which is why we now write $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ for this set of axioms.

Thus, in $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ the evaluation value of each atom in a conditional statement is memorised. Note that axiom (CL2) is self-dual, so the duality principle applies in $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$. Note also that if in a mem-basic form $P \lhd b \rhd Q$, P and Q have the same root atom, say a, then according to axiom (CL2), b and a can be swapped when their further subterms are also rearranged:

$$\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash (P_1 \lhd a \rhd P_2) \lhd b \rhd (Q_1 \lhd a \rhd Q_2) = (P_1 \lhd b \rhd Q_1) \lhd a \rhd (P_2 \lhd b \rhd Q_2).$$

Another simple and interesting derivation in $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ is the following:

$$\begin{aligned} \mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash \mathsf{U} \lhd x \rhd \mathsf{U} &= (y \lhd \mathsf{U} \rhd z) \lhd x \rhd (v \lhd \mathsf{U} \rhd w) \\ &= (y \lhd x \rhd v) \lhd \mathsf{U} \rhd (z \lhd x \rhd w) \\ &= \mathsf{U}. \end{aligned}$$

As a consequence, if neither T nor F appears in a mem-basic form P, then $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = \mathsf{U}$. From now on, we will refer to such a term as a mem-basic form with *only* U*-leaves*.

We use two ingredients to reason with axiom (CL2) and to define basic forms for $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$:
1) A strict total ordering $(A, <)$ on the set A of atoms, and
2) The *shared alphabet* $\widetilde{\alpha}(P)$ of a term $P \in MBF^{\mathsf{U}}_A$.

For mem-basic forms that do not contain U, a definition of their shared alphabet, i.e., the atoms that are *always* evaluated, is straightforward:

$$\widetilde{\alpha}(\mathsf{T}) = \widetilde{\alpha}(\mathsf{F}) = \emptyset \quad \text{and} \quad \widetilde{\alpha}(P \lhd a \rhd Q) = \{a\} \cup (\widetilde{\alpha}(P) \cap \widetilde{\alpha}(Q)). \tag{19}$$

However, for a mem-basic form that does contain U, the definition of its shared alphabet is less straightforward, and in Lemma 5.1 we prove that (19) follows from the definition below, which is taken from [5]. We begin with an example (also taken from [5]):

$$\begin{aligned} \mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash (\mathsf{F} \lhd a \rhd \mathsf{U}) \lhd b \rhd (\mathsf{T} \lhd a \rhd \mathsf{U}) &= (\mathsf{F} \lhd b \rhd \mathsf{T}) \lhd a \rhd (\mathsf{U} \lhd b \rhd \mathsf{U}) \\ &= (\mathsf{F} \lhd b \rhd \mathsf{T}) \lhd a \rhd \mathsf{U}. \end{aligned}$$

It will be seen that $(\mathsf{F} \lhd b \rhd \mathsf{T}) \lhd a \rhd (\mathsf{U} \lhd b \rhd \mathsf{U})$ is the basic form according to the ordering $a < b$, while this is $(\mathsf{F} \lhd a \rhd \mathsf{U}) \lhd b \rhd (\mathsf{T} \lhd a \rhd \mathsf{U})$ if $b < a$. Thus, the shared alphabet of *each* of the three terms in this example must be $\{a, b\}$.

Definition 4.2 (Shared alphabet of a mem-basic form)**.** *For a mem-basic form P, its* ***alphabet*** *$\alpha(P)$ is the set of all its atoms, i.e. those that occur in at least one evaluation path:*

$$\alpha(B) = \emptyset \;\; \textit{for } B \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, \textit{ and } \;\; \alpha(P_1 \lhd a \rhd P_2) = \{a\} \cup \alpha(P_1) \cup \alpha(P_2).$$

For a mem-basic form P, its ***shared alphabet*** *$\widetilde{\alpha}(P)$ is the following set of atoms, defined with help of the auxiliary function $h : MBF_A^{\mathsf{U}} \times 2^A \to 2^A$:*

$$\begin{aligned}
&\widetilde{\alpha}(P) = h(P, \alpha(P)),\\
&h(\mathsf{U}, X) = X,\\
&h(\mathsf{T}, X) = h(\mathsf{F}, X) = \emptyset,\\
&h(P_1 \lhd a \rhd P_2, X) = \{a\} \cup (h(P_1, X \setminus \{a\}) \cap h(P_2, X \setminus \{a\})).
\end{aligned}$$

As an example,

$$\widetilde{\alpha}((\mathsf{U} \lhd b \rhd \mathsf{U}) \lhd a \rhd (\mathsf{U} \lhd c \rhd \mathsf{F})) = \{a, c\},$$

and this is also the case if the left subterm $\mathsf{U} \lhd b \rhd \mathsf{U}$ is replaced by U or by another term that only has U-leaves.

To decide whether different mem-basic forms are provably equal in $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$, we define so-called GSU-basic forms in a recursive way. We first recall some technical notions from [5] that are used in the definition of these basic forms and introduce a 'representation function'.

Definition 4.3. *For A a set of atoms, A^s is the set of strings over A with the property that each $\sigma \in A^s$ contains no multiple occurrences of the same atom. We write ϵ for the empty string, thus $\epsilon \in A^s$. For $a \in A$, $\sigma \in A^s$ with length $|\sigma| = n$ and $P_0, ..., P_{2^n-1}$ mem-basic forms, define $F_\sigma(P_0, ..., P_{2^n-1})$ by*

$$\begin{aligned}
&\sigma = \epsilon : \quad F_\sigma(P_0) = P_0,\\
&\sigma = a\rho : \quad F_\sigma(P_0, ..., P_{2^n-1}) = F_\rho(P_0, ..., P_{2^{|\rho|}-1}) \lhd a \rhd F_\rho(P_{2^{|\rho|}}, ..., P_{2^n-1}).
\end{aligned}$$

A string $\sigma \in A^s$ ***agrees with*** *$(A, <)$ if σ's atoms occur in increasing order. For A' a finite set of atoms, the string $\sigma \in (A')^s$ that contains all atoms of A' in increasing order* ***represents*** *$(A', <)$, and conversely, the* ***representation function*** *$rf : 2^{|A|} \to A^s$ returns the string that represents $(A', <)$.*

The following definition is also taken from [5], except for the renaming $\mathrm{CP}^{\mathsf{U}}_{cl} \to \mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ and the addition "*if* $|\sigma| > 0$, *then*" in clause (i) (which is a correction).

Definition 4.4. *A mem-basic form P is a* ***GSU-basic form*** *if for $\sigma = rf(\widetilde{\alpha}(P))$,*

$$P = F_\sigma(P_0, ..., P_{2^{|\sigma|}-1})$$

and the following conditions are satisfied:

(i) *if $|\sigma| > 0$, then for some $j \in \{0, ..., 2^{|\sigma|} - 1\}$, $P_j \neq \mathsf{U}$, and*

(ii) *for all $i \in \{0, ..., 2^{|\sigma|} - 1\}$, P_i is a GSU-basic form.*

We will often write **GSUbf** instead of "GSU-basic form".[8] For each $\sigma \in A^s$, we write $F_\sigma(\vec{\mathsf{U}})$ for $F_\sigma(P_0, ..., P_{2^{|\sigma|}-1})$ if for all i, $P_i = \mathsf{U}$ (where $|\sigma|$ determines the length of $\vec{\mathsf{U}}$). Note that the above definition implies that $F_\rho(\vec{\mathsf{U}})$ is not a GSUbf for any $\rho \in A^s \setminus \{\epsilon\}$. More generally, a mem-basic form in MBF_A^{U} of depth larger than 0 with only U-leaves is not a GSUbf.

Lemma 4.5. *If P is a mem-basic form that has only U-leaves, then* (i) *$\widetilde{\alpha}(P) = \alpha(P)$ and* (ii) *for all $\sigma \in A^s$, $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_\sigma(\vec{\mathsf{U}}) = \mathsf{U}$.*

Proof. Both (i) and (ii) follow by structural induction on P. □

Observe that if in a mem-basic form P the axiom (CL2) cannot be applied (i.e., if $|\widetilde{\alpha}(Q)| \leq 1$ for all subterms Q of P), then P is a GSUbf. An example is $(\mathsf{T} \lhd b \rhd \mathsf{F}) \lhd a \rhd (\mathsf{T} \lhd c \rhd \mathsf{F})$. Also observe that if $P \lhd a \rhd Q$ is a GSUbf, it need not be the case that P and Q are GSUbfs. For example, if $a < b < c$ then

$$(\mathsf{T} \lhd c \rhd \mathsf{F}) \lhd a \rhd ((\mathsf{T} \lhd b \rhd \mathsf{F}) \lhd c \rhd (\mathsf{T} \lhd b \rhd \mathsf{F})) \tag{20}$$

is a GSUbf, but $(\mathsf{T} \lhd b \rhd \mathsf{F}) \lhd c \rhd (\mathsf{T} \lhd b \rhd \mathsf{F})$ is not.

For a mem-basic form P that differs from T, F and U, a consequence of axiom (CL2) concerns a provably equal representation ordered by $(\widetilde{\alpha}(P), <)$. The following lemma is identical to [5, La.4.8], except for the notations ℓ_a^{U} and r_a^{U} (see Definition 3.8), which occur in a lemma that supports its proof.

Lemma 4.6. *For each mem-basic form $P \in MBF_A^{\mathsf{U}}$ with $\sigma = rf(\widetilde{\alpha}(P))$, there exist mem-basic forms $P_0, ..., P_{2^{|\sigma|}-1}$ such that*

$$\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_\sigma(P_0, ..., P_{2^{|\sigma|}-1}).$$

[8] In [5], we introduced both *CL-basic forms* and *CLU-basic forms*, hence the U in its (abbreviated) name. We made this distinction in their naming and keep it in their renaming CL → GS because GS-basic forms (see Definition 5.2) share a property that is not preserved for GSU-basic forms containing U. Furthermore, neither GS-basic forms nor GSU-basic forms have the property that all their subterms of are of the same basic form, unlike the other basic forms. See Remark 6.1 for a further discussion.

Proof. By induction on the depth of P and based on [5, La.4.4], which reads

$$\text{If } P \in MBF_A^{\mathsf{U}} \text{ and } a \in \widetilde{\alpha}(P), \text{ then } \mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = \ell_a^{\mathsf{U}}(P) \lhd a \rhd r_a^{\mathsf{U}}(P). \tag{21}$$

Result (21) follows by structural induction on P. □

Example 4.7. *1. If $P = \mathsf{U} \lhd a \rhd (\mathsf{U} \lhd b \rhd \mathsf{U})$, then according to Lemma 4.5, $\widetilde{\alpha}(P) = \{a, b\}$. By Lemma 4.6 we find*

$$\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_{ab}(\vec{\mathsf{U}}) \text{ if } a < b, \text{ and } \mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_{ba}(\vec{\mathsf{U}}) \text{ if } b < a.$$

2. If $P = \mathsf{U} \lhd a \rhd (\mathsf{F} \lhd b \rhd \mathsf{U})$, then $\widetilde{\alpha}(P) = \{a, b\}$ and by Lemma 4.6,

$$\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_{ab}(\mathsf{U}, \mathsf{U}, \mathsf{F}, \mathsf{U}) \text{ if } a < b, \text{ and}$$
$$\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_{ba}(\mathsf{U}, \mathsf{F}, \mathsf{U}, \mathsf{U}) \text{ if } b < a.$$

Lemmas 4.5 and 4.6 lead directly to the following result, which is identical to [5, Thm.4.7], apart from the renaming $\mathrm{CP}_{cl}^{\mathsf{U}} \to \mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}}$ and a minor correction ($\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}}$ instead of $\mathrm{CP}_{\mathrm{GS}}$). We therefore provide a brief proof here.

Theorem 4.8. *For each mem-basic form $P \in MBF_A^{\mathsf{U}}$ there is a unique GSU-basic form Q such that $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = Q$.*

Proof. By induction on the depth of P. If $d(P) = 0$ then $P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ and we are done.

If $d(P) > 0$ then $|\widetilde{\alpha}(P)| = n$ for some $n \geq 1$. For $\sigma = rf(\widetilde{\alpha}(P))$, it follows by Lemma 4.6 that $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = F_\sigma(P_0, ..., P_{2^n-1})$. In the case that $F_\sigma(P_0, ..., P_{2^n-1}) = F_\sigma(\vec{\mathsf{U}})$, it follows by Lemma 4.5 that $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = \mathsf{U}$. In the other case, there are by induction unique GSUbfs Q_i such that $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P_i = Q_i$. Hence, $Q = F_\sigma(Q_0, ..., Q_{2^n-1})$ is the unique GSUbf that satisfies $\mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash P = Q$. □

Lemmas 4.5 and 4.6 can be used to define a normalisation function that yields GSUbfs. For a mem-basic form P with at least one leaf T or F, this can be done with an auxiliary function that transforms a mem-basic form P to depth $|\widetilde{\alpha}(P)|$ and is applied recursively to the arguments found.

Definition 4.9. *The **GSU-basic form function** $\mathrm{GS}bf^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to MBF_A^{\mathsf{U}}$ is defined with help of a case distinction and the auxiliary function $nf : MBF_A^{\mathsf{U}} \to MBF_A^{\mathsf{U}}$:*

$$\mathrm{GS}bf^{\mathsf{U}}(P) = \begin{cases} \mathsf{U} & \text{if } mbf^{\mathsf{U}}(P) \text{ has only } \mathsf{U}\text{-leaves, thus by} \\ & \quad \text{Lemma 4.5, } \mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \vdash mbf^{\mathsf{U}}(P) = \mathsf{U}, \\ nf(mbf^{\mathsf{U}}(P)) & \text{otherwise.} \end{cases}$$

The auxiliary function $nf : MBF_A^{\mathsf{U}} \to MBF_A^{\mathsf{U}}$ *is defined as follows:*

$$nf(P) = \begin{cases} P & \text{if } \widetilde{\alpha}(P) = \emptyset, \text{ thus } P \in \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}, \\ F_{rf(\widetilde{\alpha}(P))}(\ nf(P_0), & \text{if } |\widetilde{\alpha}(P)| = \ell > 0 \text{ and by Lemma 4.6,} \\ \quad \dots\ , & \quad P_0, ..., P_\ell \in MBF_A^{\mathsf{U}} \text{ are such that} \\ \quad nf(P_\ell)\ ) & \quad \mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = F_{rf(\widetilde{\alpha}(P))}(P_0, ..., P_\ell). \end{cases}$$

Proposition 4.10 ($\mathrm{GS}bf^{\mathsf{U}}$ **is a normalisation function).** *For all* $P \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{GS}bf^{\mathsf{U}}(P)$ *is a GSU-basic form, and for each GSU-basic form* P, $\mathrm{GS}bf^{\mathsf{U}}(P) = P$.

Proof. Let $P \in \mathcal{C}_A^{\mathsf{U}}$ be given and Q be the unique mem-basic form of P (cf. Proposition 3.11). By Theorem 4.8, let R be the unique GSUbf of Q. It suffices to prove that either $R = \mathsf{U}$ if Q only contains U-leaves, or that $R = nf(Q)$. The first case follows from Lemma 4.5, and we prove the second case by induction on the depth of Q. The base case $d(Q) = 0$ is trivial. If $d(Q) \geq 1$ then

$$\mathrm{GS}bf^{\mathsf{U}}(Q) = nf(Q) = F_{rf(\widetilde{\alpha}(Q))}(nf(Q_0), ..., nf(Q_{2^{|\widetilde{\alpha}(Q)|}-1})).$$

Because all Q_i have lesser depth than Q, we find by induction GSUbf's R_i such that $R_i = nf(Q_i)$. Hence, $R = F_{rf(\widetilde{\alpha}(Q))}(R_0, ..., R_{2^{|\widetilde{\alpha}(Q)|}-1})$ and by Definition 4.4, R is a GSUbf.

For a GSUbf P, $\mathrm{GS}bf^{\mathsf{U}}(P) = P$ follows from Theorem 4.8. □

Definition 4.11. *The binary relation* $=_{\mathrm{GS}bf^{\mathsf{U}}}$ *on* $\mathcal{C}_A^{\mathsf{U}}$ *is defined as follows:*

$$P =_{\mathrm{GS}bf^{\mathsf{U}}} Q \iff \mathrm{GS}bf^{\mathsf{U}}(P) = \mathrm{GS}bf^{\mathsf{U}}(Q).$$

Theorem 4.12. *For all* $P, Q \in \mathcal{C}_A^{\mathsf{U}}$, $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = Q \iff P =_{\mathrm{GS}bf^{\mathsf{U}}} Q$.

Proof. ($\Rightarrow$) Assume $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = Q$. According to Theorem 4.8, there is a unique GSUbf R that satisfies $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = R$, and thus $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash Q = R$. Hence, by Proposition 4.10, $R = \mathrm{GS}bf^{\mathsf{U}}(P) = \mathrm{GS}bf^{\mathsf{U}}(Q)$.

($\Leftarrow$) Assume $P =_{\mathrm{GS}bf^{\mathsf{U}}} Q$. According to ($\Rightarrow$), $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = \mathrm{GS}bf^{\mathsf{U}}(P)$, hence $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = Q$. □

Hence, the relation $=_{\mathrm{GS}bf^{\mathsf{U}}}$ is a congruence on $\mathcal{C}_A^{\mathsf{U}}$ that is axiomatised by $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$. Below we define Guzmán-Squier U-evaluation trees, GS U-evaluation trees for short, with help of the normalisation function $\mathrm{GS}bf^{\mathsf{U}}$ and prove that equality of two such trees characterises the congruence that is axiomatised by $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$. We comment on this way of defining GS U-evaluation trees in Remark 6.2.

Definition 4.13. *The unary* ***Guzmán-Squier U-evaluation function***

$$\textsc{gs}se^{\mathsf{U}} : \mathcal{C}_A^{\mathsf{U}} \to \mathcal{T}_A^{\mathsf{U}}$$

yields ***Guzmán-Squier U-evaluation trees*** *and is defined as follows:*

$$\textsc{gs}se^{\mathsf{U}}(P) = se^{\mathsf{U}}(\textsc{gs}bf^{\mathsf{U}}(P)),$$

where $\textsc{gs}bf^{\mathsf{U}}$ *is defined in Definition 4.9 and* se^{U} *in Definition 2.3.*

The close relation between the functions $\textsc{gs}se^{\mathsf{U}}$ and $\textsc{gs}bf^{\mathsf{U}}$ leads to the following definition and subsequent lemma, which implies that $\textsc{gs}se^{\mathsf{U}}$ defines a congruence on $\mathcal{C}_A^{\mathsf{U}}$.

Definition 4.14. ***Guzmán-Squier*** se^{U}***-congruence,*** $\textsc{gs}se^{\mathsf{U}}$*-congruence for short and with notation* $=_{\textsc{gs}se^{\mathsf{U}}}$*, is defined on* $\mathcal{C}_A^{\mathsf{U}}$ *as follows:*

$$P =_{\textsc{gs}se^{\mathsf{U}}} Q \iff \textsc{gs}se^{\mathsf{U}}(P) = \textsc{gs}se^{\mathsf{U}}(Q).$$

Lemma 4.15. *For all* $P, Q \in \mathcal{C}_A^{\mathsf{U}}$*,* $P =_{\textsc{gs}se^{\mathsf{U}}} Q \iff P =_{\textsc{gs}bf^{\mathsf{U}}} Q$*.*

Proof. By definition of the evaluation function $\textsc{gs}se^{\mathsf{U}}$ and Theorem 4.12. □

We find the following completeness result for $\textsc{gs}se^{\mathsf{U}}$-congruence.

Theorem 4.16 (Completeness of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$)**.** *For all* $P, Q \in \mathcal{C}_A^{\mathsf{U}}$*,*

$$\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = Q \iff P =_{\textsc{gs}se^{\mathsf{U}}} Q.$$

Moreover, $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ *is an independent axiomatisation.*

Proof. Combine Theorem 4.12 and Lemma 4.15.

The independence of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ follows with *Mace4* [19] and in Appendix A we prove this independence result in detail. □

Alternative GSU-basic forms. GSU-basic forms serve their purpose well, and their definition has a particular consequence for terms that contain U-leaves and at least one other leaf. The focus on shared alphabets and axiom (CL2) introduces a symmetry in GSU-basic forms that might be debatable. An obvious alternative would be to replace any subterm of a GSU-basic form that only has only U-leaves by U. Consider Example 4.7.2, thus $P = \mathsf{U} \lhd a \rhd (\mathsf{F} \lhd b \rhd \mathsf{U})$. If $a < b$, then P itself would be the candidate for such

an alternative basic form, instead of P's GSU-basic form $F_{ab}(\mathsf{U},\mathsf{U},\mathsf{F},\mathsf{U}) = (\mathsf{U} \lhd b \rhd \mathsf{U}) \lhd a \rhd (\mathsf{F} \lhd b \rhd \mathsf{U})$. If $b < a$, then by Lemma 4.6 and Lemma 4.5,

$$\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = (\mathsf{U} \lhd a \rhd \mathsf{F}) \lhd b \rhd \mathsf{U},$$

so $(\mathsf{U} \lhd a \rhd \mathsf{F}) \lhd b \rhd \mathsf{U}$ would be the candidate for P's alternative basic form, instead of P's GSU-basic form $F_{ba}(\mathsf{U},\mathsf{F},\mathsf{U},\mathsf{U}) = (\mathsf{U} \lhd a \rhd \mathsf{F}) \lhd b \rhd (\mathsf{U} \lhd a \rhd \mathsf{U})$. This leads to the following definition.

Definition 4.17. *An **Alternative GSU-basic form**, for short an **AGSU-basic form**, is obtained from a GSU-basic form by applying the auxiliary function $aclbf^{\mathsf{U}} : MBF^{\mathsf{U}}_A \to MBF^{\mathsf{U}}_A$:*

$$aclbf^{\mathsf{U}}(B) = B \quad \text{if } B \in \{\mathsf{T},\mathsf{F},\mathsf{U}\},$$

$$aclbf^{\mathsf{U}}(P \lhd a \rhd Q) = \begin{cases} \mathsf{U} & \text{if } P \lhd a \rhd Q \text{ has only } \mathsf{U}\text{-leaves}, \\ aclbf^{\mathsf{U}}(P) \lhd a \rhd aclbf^{\mathsf{U}}(Q) & \text{otherwise.} \end{cases}$$

Of course, this definition is intended to be applied to GSU-basic forms, and the following lemma implies that it is correct.

Lemma 4.18. *For all $P \in MBF^{\mathsf{U}}_A$, $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash P = aclbf^{\mathsf{U}}(P)$.*

Proof. By structural induction on P and Lemma 4.5. □

5 Two-Valued Guzmán-Squier Proposition Algebra

Guzmán-Squier proposition algebra has a two-valued version, which we introduce in this section and motivate in Section 6. In [5, Def.3.2], Bergstra and Ponse defined two-valued conditional short-circuit logic by means of the axiom system $\mathrm{CP}_{\mathrm{GS}}$, that is, $\mathrm{CP}_{mem} \cup \{(\mathrm{CL2})\}$ (see Definition 4.1). We note that a two-valued 'proposition algebra' is an algebra over the signature $\Sigma_{\mathrm{CP}}(A) = \{_ \lhd _ \rhd _, \mathsf{T}, \mathsf{F}, a \mid a \in A\}$ (thus without U) that satisfies the axioms of CP, cf. Definitions 2.5 and 2.6. Below, we define Guzmán-Squier evaluation trees and GS*se*-congruence for the two-valued case and it will appear that this case is a simple reduction of the three-valued case. Finally, we note that all other major two-valued se-congruences are defined in [9].

First, we define some notions (taken from [9]) that are obtained by leaving out U: $\mathcal{C}_A \subset \mathcal{C}^{\mathsf{U}}_A$ is the set of closed terms over the signature $\Sigma_{\mathrm{CP}}(A)$, $MBF_A \subset MBF^{\mathsf{U}}_A$ is the set of mem-basic forms that do not contain U, and the function $mbf : \mathcal{C}_A \to MBF_A$ is defined as the restriction of mbf^{U} to $\mathcal{C}_A$.

Furthermore, we write $\mathcal{T}_A$ for $\mathcal{T}_A^{\mathsf{U}}$ without U and leave out the underlining of atoms in $\mathcal{T}_A$.

Next, we show that the shared alphabet of a mem-basic form $P \in \mathit{MBF}_A$ is much simpler to determine than in the three-valued case.

Lemma 5.1. *For all* $(Q \lhd a \rhd R) \in \mathit{MBF}_A$*,*

$$\widetilde{\alpha}(Q \lhd a \rhd R) = \{a\} \cup (\widetilde{\alpha}(Q) \cap \widetilde{\alpha}(R)).$$

Proof. We first show that for all $P \in \mathit{MBF}_A$ and $X \in 2^A$, $h(P, X) = h(P, \emptyset)$. This follows by structural induction on P. If $P \in \{\mathsf{T}, \mathsf{F}\}$ this is trivial, and if $P = Q \lhd a \rhd R$ then

$$\begin{aligned}
&h(Q \lhd a \rhd R, X) \\
&= \{a\} \cup (h(Q, X \setminus \{a\}) \cap h(R, X \setminus \{a\})) && \text{by Def. 4.2} \\
&= \{a\} \cup (h(Q, \emptyset) \cap h(R, \emptyset)) && \text{by IH} \\
&= h(Q \lhd a \rhd R, \emptyset).
\end{aligned}$$

Hence, $\widetilde{\alpha}(Q \lhd a \rhd R) = h(Q \lhd a \rhd R, \alpha(Q \lhd a \rhd R)) = h(Q \lhd a \rhd R, \emptyset) = \{a\} \cup (h(Q, \emptyset) \cap h(R, \emptyset)) = \{a\} \cup (\widetilde{\alpha}(Q) \cap \widetilde{\alpha}(R))$. □

The definition of GS-basic forms is obtained by restricting that of GSU-basic forms (Definition 4.4) to MBF_A, where we assume that $(A, <)$ is given.

Definition 5.2. *A mem-basic form P is a* ***GS-basic form*** *if for $\sigma = \mathit{rf}(\widetilde{\alpha}(P))$,*

$$P = F_\sigma(P_0, ..., P_{2^{|\sigma|}-1})$$

and for all $i \in \{0, ..., 2^{|\sigma|} - 1\}$, P_i is a GS-basic form. For the functions rf and F_σ, see Definition 4.3.

We will sometimes write **GSbf** instead of "GS-basic form". For $P \in \mathit{MBF}_A$ different from T and F, a consequence of axiom (CL2) concerns a simplified version of Lemma 4.6 (where $\mathrm{CP}_{\mathrm{GS}} = \mathrm{CP}_{\mathrm{GS}}^{\mathsf{U}} \setminus \{\text{(CP-U)}\}$, see Definition 4.1).

Lemma 5.3. *For each mem-basic form $P \in \mathit{MBF}_A$ with $\sigma = \mathit{rf}(\widetilde{\alpha}(P))$, there exist mem-basic forms $P_0, ..., P_{2^{|\sigma|}-1}$ such that*

$$\mathrm{CP}_{\mathrm{GS}} \vdash P = F_\sigma(P_0, ..., P_{2^{|\sigma|}-1}).$$

The following result is a consequence of Theorem 4.8.

Theorem 5.4. *For each mem-basic form $P \in MBF_A$ there is a unique GS-basic form Q such that* $\mathrm{CP}_{\mathrm{GS}} \vdash P = Q$.

Lemma 5.3 can be used to define a normalisation function that yields GSbfs. For a mem-basic form P this function can be defined with an auxiliary function that transforms a mem-basic form P to depth $|\widetilde{\alpha}(P)|$ and is applied recursively to the arguments found.

Definition 5.5. *The* ***GS-basic form function*** $\mathrm{GS}bf : \mathcal{C}_A \to MBF_A$ *is defined with help of an auxiliary function* nf_r*:*

$$\mathrm{GS}bf(P) = nf_r(mbf(P)).$$

The auxiliary function $nf_r : MBF_A \to MBF_A$ *is defined as follows:*

$$nf_r(P) = F_{rf(\widetilde{\alpha}(P))}(nf_r(P_0), \ldots, nf_r(P_\ell))$$

if $|\widetilde{\alpha}(P)| = \ell$ *and by Lemma 5.3,* $P_0, \ldots, P_\ell \in MBF_A$ *are such that* $\mathrm{CP}_{\mathrm{GS}} \vdash P = F_{rf(\widetilde{\alpha}(P))}(P_0, \ldots, P_\ell)$.

Proposition 5.6 ($\mathrm{GS}bf$ is a normalisation function)**.** *For all $P \in \mathcal{C}_A$,* $\mathrm{GS}bf(P)$ *is a GS-basic form, and for each GS-basic form P,* $\mathrm{GS}bf(P) = P$.

Proof. This follows from the proof of Proposition 4.10. □

Definition 5.7. *The binary relation* $=_{\mathrm{GS}bf}$ *on* $\mathcal{C}_A$ *is defined as follows:*

$$P =_{\mathrm{GS}bf} Q \iff \mathrm{GS}bf(P) = \mathrm{GS}bf(Q).$$

Theorem 5.8. *For all* $P, Q \in \mathcal{C}_A$, $\mathrm{CP}_{\mathrm{GS}} \vdash P = Q \iff P =_{\mathrm{GS}bf} Q$.

Proof. ($\Rightarrow$) Assume $\mathrm{CP}_{\mathrm{GS}} \vdash P = Q$. According to Theorem 5.4, there is a unique GSbf R that satisfies $\mathrm{CP}_{\mathrm{GS}} \vdash P = R$, and thus $\mathrm{CP}_{\mathrm{GS}} \vdash Q = R$. Hence, by Proposition 5.6, $R = \mathrm{GS}bf(P) = \mathrm{GS}bf(Q)$.

($\Leftarrow$) Assume $P =_{\mathrm{GS}bf} Q$. According to ($\Rightarrow$), $\mathrm{CP}_{\mathrm{GS}} \vdash P = \mathrm{GS}bf(P)$, hence $\mathrm{CP}_{\mathrm{GS}} \vdash P = Q$. □

Hence, the relation $=_{\mathrm{GS}bf}$ is a congruence on $\mathcal{C}_A$ that is axiomatised by $\mathrm{CP}_{\mathrm{GS}}$. Below we define Guzmán-Squier evaluation trees, GS evaluation trees for short, with help of the normalisation function $\mathrm{GS}bf$ and prove that equality of two such trees characterises the congruence that is axiomatised by $\mathrm{CP}_{\mathrm{GS}}$.

Definition 5.9. *The unary* ***Guzmán-Squier evaluation function***

$$\textsc{gs}se : \mathcal{C}_A \to \mathcal{T}_A$$

yields ***Guzmán-Squier evaluation trees*** *and is defined as follows:*

$$\textsc{gs}se(P) = se(\textsc{gs}bf(P)),$$

where $\textsc{gs}bf$ *is defined in Definition 5.5 and* se *is the restriction of* se^{U} *(see Definition 2.3) to* $\Sigma_{\mathrm{CP}}(A)$*.*

As in the approach in Section 4, the close relationship between the functions $\textsc{gs}se$ and $\textsc{gs}bf$ leads to the following definition and the subsequent theorem, which implies that $\textsc{gs}se$ defines a congruence on $\mathcal{C}_A$.

Definition 5.10. ***Guzmán-Squier se-congruence****,* $\textsc{gs}se$*-congruence for short and with notation* $=_{\textsc{gs}se}$*, is defined on* $\mathcal{C}_A$ *as follows:*

$$P =_{\textsc{gs}se} Q \iff \textsc{gs}se(P) = \textsc{gs}se(Q).$$

Lemma 5.11. *For all* $P, Q \in \mathcal{C}_A$*,* $P =_{\textsc{gs}se} Q \iff P =_{\textsc{gs}bf} Q$*.*

Proof. By definition of the evaluation function $\textsc{gs}se$ and Theorem 5.8. □

We find the following completeness result for $\textsc{gs}se$-congruence.

Theorem 5.12 (Completeness of $\mathrm{CP}_{\mathrm{GS}}$)**.** *For all* $P, Q \in \mathcal{C}_A$*,*

$$\mathrm{CP}_{\mathrm{GS}} \vdash P = Q \iff P =_{\textsc{gs}se} Q.$$

Moreover, $\mathrm{CP}_{\mathrm{GS}}$ *is an independent axiomatisation.*

Proof. For completeness, combine Theorem 5.8 and Lemma 5.11. The independence of $\mathrm{CP}_{\mathrm{GS}}$ follows from Theorem 4.16. □

6 Conclusions

As mentioned in the Introduction, this paper is a follow-up to the paper *Conditional logic as a short-circuit logic* [5] (2025) by Bergstra and Ponse. In turn, [5] is based on the paper *The algebra of conditional logic* by Guzmán and Squier [14] (1990) and is motivated by the observation that conditional logic represents a relevant three-valued short-circuit logic that was lacking in our framework of 'three-valued short-circuit logics' as set out in [8]. In this paper we have extended the framework of 'three-valued proposition algebras'

that underlies the definitions of short-circuit logics, to include GS proposition algebra. In this section, we provide a brief overview of our work in the field of 'short-circuit logics' (SCLs) and a short discussion of related work. We end the paper with some conclusions and Remarks 6.1 and 6.2.

In the period 2010–2013, we introduced in [7] several two-valued SCLs, based on the use of Hoare's conditional connective $_ \lhd _ \rhd _$ and the observation that the sequential connectives **cand** and **cor** admit semantics that take side effects into account. We write $\mathbin{\owedge}$ and $\mathbin{\ovee}$ for these connectives, and refer to them as 'short-circuit conjunction' and 'short-circuit disjunction'. Using the constants T and F, these connectives and negation are defined by

$$x \mathbin{\owedge} y = y \lhd x \rhd \mathsf{F} \quad \text{and} \quad x \mathbin{\ovee} y = \mathsf{T} \lhd x \rhd y, \quad \text{and} \quad \neg x = \mathsf{F} \lhd x \rhd \mathsf{T}.$$

The work in [7] is based on the paper *Proposition algebra* [4] by Bergstra and Ponse. As an example, 'Free Short-Circuit Logic' (FSCL, the weakest SCL considered) is the equational theory in the signature $\{\neg, \owedge, \ovee, \mathsf{T}, \mathsf{F}, a \mid a \in A\}$ defined by $\mathrm{CP}(\neg, \owedge)$, that is, $\mathrm{CP} \cup \{\neg x = \mathsf{F} \lhd x \rhd \mathsf{T},\ x \owedge y = y \lhd x \rhd \mathsf{F}\}$, and $x \ovee y = \neg(\neg x \owedge \neg y)$. (See Definition 2.5 for CP.) In 2021, we introduced in [8] *three-valued* proposition algebra and defined the axiom systems CP^{U} and $\mathrm{CP}^{\mathsf{U}}_{mem} = \mathrm{CP}^{\mathsf{U}} \cup \{(\mathrm{CPmem})\}$, and the corresponding U-evaluation trees and short-circuit logics $\mathrm{FSCL}^{\mathsf{U}}$ and $\mathrm{MSCL}^{\mathsf{U}}$, which implies that $\mathrm{FSCL}^{\mathsf{U}}$ is a sublogic of $\mathrm{MSCL}^{\mathsf{U}}$. The completeness result for $\mathrm{CP}^{\mathsf{U}}_{mem}$ in this paper (Theorem 3.16) follows from the completeness of $\mathrm{MSCL}^{\mathsf{U}}$, since the conditional connective is expressible, namely: $\mathrm{CP}_{mem}(\neg, \owedge) \vdash x \lhd y \rhd z = (y \owedge x) \ovee (\neg y \owedge z)$. However, this would require a somewhat complicated and ad-hoc proof, which we did not provide in [8].[9] In [5], Bergstra and Ponse defined $\mathrm{CP}^{\mathsf{U}}_{cl}$ (here called $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$) and C$\ell$SCL, *conditional* SCL, which is axiomatised by seven simple axioms (most of which are taken from [14]). Because $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} = \mathrm{CP}^{\mathsf{U}}_{mem} \cup (\mathrm{CL2})$, $\mathrm{MSCL}^{\mathsf{U}}$ is a sublogic of CℓSCL. In [6], an application with CℓSCL is discussed which relates to a calculus that takes division by zero into account and employs a partial data type for arithmetic operations. A direction for future work, briefly discussed in [4], is to investigate whether $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ and CℓSCL can be usefully applied to other calculi that take division by zero into account.

In the following, we outline another application of the conditional connective in the field of many-valued logics. Two well-known, commutative three-valued logics with undefinedness are (strong) Kleene logic $\mathbf{K}_3$ (choose one of T and F whenever possible) and its weak variant $\mathbf{K}_3^w$ (choose U whenever possible; this logic is also known as Bochvar's logic). For $*$ representing

[9] In [8, App.A.8], we have, however, provided a proof of the completeness of CP^{U}.

the third truth value, negation is in both logics defined by $\neg * = *$ and the difference between $\mathbf{K}_3$ and $\mathbf{K}_3^w$ can be easily demonstrated by the following instances of their conjunction, written here as $\wedge_s$ and $\wedge_w$, respectively:

$$\mathsf{F} \wedge_s * = \mathsf{F} \quad \text{and} \quad \mathsf{F} \wedge_w * = *.$$

In [14], Guzmán and Squier briefly discuss these two logics. In the paper *Belnap's logic and conditional composition* [22], Ponse and Van der Zwaag consider the combination of these two readings of $*$ and characterise Belnap's four-valued logic [2], hereafter denoted by $\mathbf{B}_4$, using only the conditional connective and four constants. Furthermore, they argue that $\mathbf{B}_4$ arises naturally from $\mathbf{K}_3$ (§.7.1). The truth value *undefined* is in $\mathbf{B}_4$ represented by the constant N (*none*), and the fourth truth value "inconsistent information" is represented by the constant B (*both*). The main result in [22] is a complete axiomatisation of the representation of $\mathbf{B}_4$ with the conditional that consists of the four axioms that define a proposition algebra and the six axioms (CL2),

$$x \lhd \mathsf{N} \rhd y = \mathsf{N}, \quad x \lhd \mathsf{B} \rhd y = y \lhd \mathsf{B} \rhd x, \quad x \lhd \mathsf{B} \rhd \mathsf{N} = x, \quad \mathsf{B} \lhd \mathsf{B} \rhd x = \mathsf{B},$$

and

$$(x_1 \lhd y \rhd x_2) \lhd y \rhd x_3 = x_1 \lhd y \rhd (x_2 \lhd y \rhd x_3),$$

where the latter one paves the way to 'symmetric contraction'. All in all, it is argued that the exclusive use of the conditional connective provides an intuitive characterisation of $\mathbf{B}_4$, because its central condition must be evaluated first. We here note that the characterisation of $\mathbf{K}_3$ with only the conditional is axiomatised by omitting all axioms containing N. This characterisation follows easily with the theorem prover *Prover9* [19]: first, from the remaining eight axioms above and the defining axioms $\neg x = \mathsf{F} \lhd x \rhd \mathsf{T}$, $x \wedge_s y = (y \lhd x \rhd \mathsf{F}) \lhd \mathsf{B} \rhd (x \lhd y \rhd \mathsf{F})$ and the dual $x \vee_s y = (\mathsf{T} \lhd x \rhd y) \lhd \mathsf{B} \rhd (\mathsf{T} \lhd x \rhd y)$, an axiomatisation of $\mathbf{K}_3$ with also the constants T and F (see, e.g., [3, Tab.1]) is derivable. Secondly, from the mentioned axiomatisation of $\mathbf{K}_3$ and the defining axiom $x \lhd y \rhd z = (y \wedge_s x) \vee_s (\neg y \wedge_s z)$, these eight axioms are derivable. Next, we show that $\mathbf{K}_3^w$ is a sublogic of CℓSCL, the short-circuit logic defined by $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$. This follows from the fact that the equations in [5, Tab.5] that axiomatise $\mathbf{K}_3^w$ (in that paper named Bochvar's logic) are derivable from $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ and the defining axioms $\neg x = \mathsf{F} \lhd x \rhd \mathsf{T}$, $x \wedge_w y = y \lhd x \rhd (\mathsf{F} \lhd y \rhd \mathsf{F})$ and $x \vee_w y = (\mathsf{T} \lhd y \rhd \mathsf{T}) \lhd x \rhd y$ (*Prover9* is useful here). Because U is fully absorbing in $\mathbf{K}_3^w$ (any term with an occurence of U is provably equal to U), it cannot be the case that $\mathbf{K}_3^w$ can be characterised with only the conditional connective and the constants T, F and U. For suppose otherwise, then $\mathsf{T} \lhd \mathsf{T} \rhd \mathsf{U} = \mathsf{U}$, which contradicts the CP-axiom $x \lhd \mathsf{T} \rhd y = x$.

In their paper *Classical implication for three-valued logic* [16] (2025), Li and Naumov introduce a variant of three-valued 'short-circuit logic' which makes use of short-circuit conjunction, short-circuit disjunction and a different form of negation, so-called 'classical negation', say $\neg_c$, defined by

$$\neg_c \mathsf{T} = \mathsf{F}, \quad \neg_c \mathsf{F} = \mathsf{T} \quad \text{and} \quad \neg_c \mathsf{U} = \mathsf{T}.$$

The intuition is that U stands for 'uncertain' and that $\neg_c P$ means that "P does not have value T". This contradicts basic identities that hold in $\mathrm{CP}^{\mathsf{U}}(\neg, \owedge)$, such as $\neg\mathsf{U} = \mathsf{U}$ and $\neg\neg x = x$. It is therefore questionable whether [16] can be regarded as related to our work, apart from the use of the terms 'short-circuit evaluation' and 'short-circuit logic'.

We conclude that in this paper we have given full attention to 'Guzmán-Squier proposition algebra', the proposition algebra axiomatised by $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ which underlies the definition of conditional short-circuit logic. We have focused on the completeness of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$, on $\mathrm{GS}se^{\mathsf{U}}$-congruence and on the normalisation of three-valued conditional statements. Also, as a stepping stone, we have given a full and systematic account of three-valued 'memorising proposition algebra' and 'memorising se^{U}-congruence'. We believe that 'GS proposition algebra' is, in a general sense, the strongest possible three-valued proposition algebra that satisfies $x \lhd \mathsf{U} \rhd y = \mathsf{U}$. Finally, we feel that 'two-valued GS proposition algebra' should also be distinguished separately, as it induces an se-congruence, $=_{\mathrm{GS}se}$, which properly includes (two-valued) memorising se-congruence (cf. [9]) and which has not previously been described. So-called *static* se-congruence characterises propositional logic and is axiomatised by $\mathrm{CP}_{mem} \cup \{\mathsf{F} \lhd x \rhd \mathsf{F} = \mathsf{F}\}$ (cf. Definition 3.1). Naturally, axiom (CL2) is valid in $=_{\mathrm{GS}se}$, in contrast to $\mathsf{F} \lhd x \rhd \mathsf{F} = \mathsf{F}$.

We end the paper with a remark on terminology and another on the way we have defined 'GS U-evaluation trees'.

Remark 6.1 ('GS-basic forms' and 'GSU-basic forms'). In $\mathrm{CP}_{\mathrm{GS}}$, the shared alphabet of mem-basic forms is preserved under provable equality:

$$\mathrm{CP}_{\mathrm{GS}} \vdash P = Q \;\Rightarrow \widetilde{\alpha}(P) = \widetilde{\alpha}(Q) \quad \text{(this is [5, La.3.6])}.$$

This is not the case for mem-basic forms that contain U: $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash \mathsf{U} = \mathsf{U} \lhd a \rhd \mathsf{U}$, but $\widetilde{\alpha}(\mathsf{U}) = \emptyset \neq \{a\} = \widetilde{\alpha}(\mathsf{U} \lhd a \rhd \mathsf{U})$. Together with the 'alternative GSU-basic forms' as discussed in Section 4, this motivates our choice for the different names 'GS-basic form' and 'GSU-basic form'.

Furthermore, the subterms of a GS-basic form are not necessarily GS-basic forms, as is demonstrated by the GS-basic form in the example in (20):

if $a < b < c$ then

$$(\mathsf{T} \lhd c \rhd \mathsf{F}) \lhd a \rhd ((\mathsf{T} \lhd b \rhd \mathsf{F}) \lhd c \rhd (\mathsf{T} \lhd b \rhd \mathsf{F}))$$

is a GS-basic form, but $(\mathsf{T} \lhd b \rhd \mathsf{F}) \lhd c \rhd (\mathsf{T} \lhd b \rhd \mathsf{F})$ is not. By definition, this property also applies to GSU-basic forms, but is not shared with the ordinary basic forms (Definition 2.7) and mem-basic forms (Definition 3.2).

Remark 6.2 (On GS U-evaluation trees and GS evaluation trees)**.** Given an ordering $(A, <)$ and a conditional statement P, we have defined the 'GS U-evaluation function' by $\mathrm{GS}se^{\mathsf{U}}(P) = se^{\mathsf{U}}(\mathrm{GS}bf^{\mathsf{U}}(P))$, and $\mathrm{GS}bf^{\mathsf{U}}$ with help of the function mbf^{U} (Definitions 4.13 and 4.9). This seems a reasonable approach: thanks to Theorem 3.13 and Corollary 3.15, we know that mem-basic forms and memorising U-evaluation trees are 'isomorphic', and it does not seem reasonable and/or desirable to define a complicated transformation on memorising U-evaluation trees that mimics the function $\mathrm{GS}bf^{\mathsf{U}}$.

For the two-valued case and Definition 5.9, the same argument applies. Moreover, in [9, Def.28] we did something similar for the 'static evaluation function' sse_σ, which characterises propositional logic and where $\sigma \in A^s$ represents $(A', <)$ for a finite set A' of atoms. For $P \in \mathcal{C}_{A'}$, we defined $sse_\sigma(P) = se(sbf_\sigma(P))$, and $sbf_\sigma(P) = mbf(\mathsf{T} \lhd E_\sigma \rhd P)$, where $E_\epsilon = \mathsf{F}$ and $E_{a\rho} = E_\rho \lhd a \rhd E_\rho$, so that after the evaluation of E_σ that of P is enforced. For example,

$$sse_{ab}(\mathsf{F} \lhd a \rhd b) = (\mathsf{F} \trianglelefteq b \trianglerighteq \mathsf{F}) \trianglelefteq a \trianglerighteq (\mathsf{T} \trianglelefteq b \trianglerighteq \mathsf{F}),$$
$$sse_{ba}(\mathsf{F} \lhd a \rhd b) = (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{T}) \trianglelefteq b \trianglerighteq (\mathsf{F} \trianglelefteq a \trianglerighteq \mathsf{F}).$$

One more example: $sse_\sigma(\mathsf{F} \lhd (a \lhd b \rhd \mathsf{T}) \rhd \mathsf{T}) = sse_\sigma(\mathsf{F} \lhd a \rhd b)$ for all appropriate σ. In [15], Hoare defined similar 'normal forms'.

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A Independence results

We prove that $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ is independent, and since every other axiomatisation considered is a subset of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$, this implies that it is also independent.

First, it is immediately clear that the axiom (CL2) is independent from those of $\mathrm{CP}^{\mathsf{U}}_{mem}$: the memorising U-evaluation trees of U and $\mathsf{U} \lhd a \rhd \mathsf{U}$ are different, while $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \vdash \mathsf{U} = \mathsf{U} \lhd a \rhd \mathsf{U}$. Furthermore, U does not occur in $\mathrm{CP}_{\mathrm{GS}}$, hence the axiom (CP-U) is independent from those of $\mathrm{CP}_{\mathrm{GS}}$.

Below, we provide for each other axiom E of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}}$ a model of $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \setminus \{E\}$ that refutes E, while requiring at most only one atom $a \in A$. Each of these models is generated by the tool *Mace4* [19] by specifying the axioms in $\mathrm{CP}^{\mathsf{U}}_{\mathrm{GS}} \setminus \{E\}$ as assumptions and E as a goal to be refuted. For $B \in \{\mathsf{T}, \mathsf{F}\}$ and each model below, the interpretation $[\![B]\!]$ satisfies $[\![B]\!] = B$.

Independence of axiom (CP1): consider the model with domain $\{\mathsf{T}, \mathsf{F}\}$, interpretation $[\![\mathsf{U}]\!] = \mathsf{T}$ and all remaining cases defined by the following truth table (split into two parts):

x	y	z	$x \lhd y \rhd z$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	F

x	y	z	$x \lhd y \rhd z$
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

This model refutes the (CP1)-instances $\mathsf{F} \lhd \mathsf{T} \rhd \mathsf{T} = \mathsf{F}$ and $\mathsf{F} \lhd \mathsf{T} \rhd \mathsf{F} = \mathsf{F}$.

Independence of axiom (CP2): consider the 'dual model' of the one above, with domain $\{\mathsf{T}, \mathsf{F}\}$, and with interpretation $[\![\mathsf{U}]\!] = \mathsf{F}$ and the following truth table (obtained by dualisation of the one above):

x	y	z	$x \lhd y \rhd z$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F

x	y	z	$x \lhd y \rhd z$
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

In this model, the (CP2)-instances $\mathsf{F} \lhd \mathsf{F} \rhd \mathsf{T} = \mathsf{T}$ and $\mathsf{T} \lhd \mathsf{F} \rhd \mathsf{T} = \mathsf{T}$ are refuted (the duals of the (CP1)-instances above).

Independence of axiom (CP3): let $a \in A$ and consider the model with domain $\{\mathsf{T}, \mathsf{F}, \mathsf{U}, \mathsf{C}\}$, and with $[\![\mathsf{U}]\!] = \mathsf{U}$, $[\![a]\!] = \mathsf{C}$ and $x \lhd y \rhd z$ defined by

$$x \lhd \mathsf{T} \rhd y = x, \quad x \lhd \mathsf{F} \rhd y = y, \quad x \lhd \mathsf{U} \rhd y = \mathsf{U}, \quad x \lhd \mathsf{C} \rhd z = z.$$

Then $\mathsf{T} \lhd \mathsf{C} \rhd \mathsf{F} = \mathsf{F}$, which refutes (CP3).

Independence of axiom (CP4): let $a \in A$ and consider the model with the domain as above, and with $[\![\mathsf{U}]\!] = \mathsf{U}$, $[\![a]\!] = \mathsf{C}$ and $x \lhd y \rhd z$ defined by

$$x \lhd \mathsf{T} \rhd y = x, \quad x \lhd \mathsf{F} \rhd y = y, \quad x \lhd \mathsf{U} \rhd y = \mathsf{U},$$
$$\mathsf{T} \lhd \mathsf{C} \rhd x = \mathsf{C}, \quad \mathsf{U} \lhd \mathsf{C} \rhd x = \mathsf{U}, \quad \mathsf{F} \lhd \mathsf{C} \rhd x = \mathsf{T}, \quad \mathsf{C} \lhd \mathsf{C} \rhd x = \mathsf{F}.$$

In this model, $\mathsf{F} \lhd (\mathsf{F} \lhd \mathsf{C} \rhd \mathsf{F}) \rhd \mathsf{F} = \mathsf{F}$ and $(\mathsf{F} \lhd \mathsf{F} \rhd \mathsf{F}) \lhd \mathsf{C} \rhd (\mathsf{F} \lhd \mathsf{F} \rhd \mathsf{F}) = \mathsf{T}$, which refutes (CP4).

Independence of axiom (CPmem): let $a \in A$ and consider the model with the domain as above, and with $[\![\mathsf{U}]\!] = \mathsf{U}$, $[\![a]\!] = \mathsf{C}$, and $x \lhd y \rhd z$ defined by

$$x \lhd \mathsf{T} \rhd y = x, \quad x \lhd \mathsf{F} \rhd y = y, \quad x \lhd \mathsf{U} \rhd y = \mathsf{U},$$
$$\mathsf{T} \lhd \mathsf{C} \rhd \mathsf{T} = \mathsf{T}, \quad \mathsf{T} \lhd \mathsf{C} \rhd \mathsf{F} = \mathsf{C}, \quad \mathsf{T} \lhd \mathsf{C} \rhd \mathsf{U} = \mathsf{T}, \quad \mathsf{T} \lhd \mathsf{C} \rhd \mathsf{C} = \mathsf{C},$$
$$\mathsf{F} \lhd \mathsf{C} \rhd \mathsf{T} = \mathsf{C}, \quad \mathsf{F} \lhd \mathsf{C} \rhd \mathsf{F} = \mathsf{F}, \quad \mathsf{F} \lhd \mathsf{C} \rhd \mathsf{U} = \mathsf{F}, \quad \mathsf{F} \lhd \mathsf{C} \rhd \mathsf{C} = \mathsf{C},$$
$$\mathsf{U} \lhd \mathsf{C} \rhd x = x, \quad \mathsf{C} \lhd \mathsf{C} \rhd x = x.$$

In this model, $\mathsf{F} \lhd \mathsf{C} \rhd (\mathsf{F} \lhd \mathsf{F} \rhd (\mathsf{T} \lhd \mathsf{C} \rhd \mathsf{F})) = \mathsf{C}$, and $\mathsf{F} \lhd \mathsf{C} \rhd (\mathsf{F} \lhd \mathsf{F} \rhd \mathsf{F}) = \mathsf{F}$, which refutes (CPmem).

B The proof of Lemma 3.1 from [5]

Lemma B.1 (Lemma 3.1 from [5])**.** (i) $\mathrm{CP}_{mem} \cup \{(\mathrm{CL1})\} \vdash (\mathrm{CL2})$ *and* (ii) $\mathrm{CP}_{mem} \cup \{(\mathrm{CL2})\} \vdash (\mathrm{CL1})$.

Proof. (i) Derive

$$\begin{aligned} x \lhd (\mathsf{T} \lhd (y \lhd u \rhd \mathsf{F}) \rhd (u \lhd y \rhd \mathsf{F})) \rhd v & \qquad (22)\\ = x \lhd (y \lhd u \rhd \mathsf{F}) \rhd (x \lhd (u \lhd y \rhd \mathsf{F}) \rhd v)\\ = (x \lhd y \rhd (x \lhd (u \lhd y \rhd \mathsf{F}) \rhd v)) \lhd u \rhd (x \lhd (u \lhd y \rhd \mathsf{F}) \rhd v)\\ = (x \lhd y \rhd ((x \lhd u \rhd v) \lhd y \rhd v)) \lhd u \rhd ((x \lhd u \rhd v) \lhd y \rhd v)\\ = (x \lhd y \rhd v) \lhd u \rhd (v \lhd y \rhd v), & \qquad (23) \end{aligned}$$

and apply axiom (CL1) in (22), so exchange u and y. By (23),

$$(x \lhd y \rhd v) \lhd u \rhd (v \lhd y \rhd v) = (x \lhd u \rhd v) \lhd y \rhd (v \lhd u \rhd v).$$

Now replace v by $v \lhd y \rhd (z \lhd u \rhd w)$ and apply memorisation:

$$(x \lhd y \rhd z) \lhd u \rhd (v \lhd y \rhd w) = (x \lhd u \rhd v) \lhd y \rhd (z \lhd u \rhd w).$$

(ii) Derive $\mathsf{T} \lhd (y \lhd x \rhd \mathsf{F}) \rhd (x \lhd y \rhd \mathsf{F}) = (\mathsf{T} \lhd y \rhd \mathsf{F}) \lhd x \rhd (\mathsf{F} \lhd y \rhd \mathsf{F})$ and apply (CL2). □