

Cancellation Meadows: A Generic Basis Theorem and Some Applications[†]

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Let \mathbb{Q}_0 denote the rational numbers expanded to a ‘meadow’, that is, after taking its zero-totalized form ($0^{-1} = 0$) as the preferred interpretation. In this paper, we consider ‘cancellation meadows’, i.e. meadows without proper zero divisors, such as \mathbb{Q}_0 and prove a generic completeness result. We apply this result to cancellation meadows expanded with differentiation operators, the sign function, and with floor, ceiling and a signed variant of the square root, respectively. We give an equational axiomatization of these operators and thus obtain a finite basis for various expanded cancellation meadows.

Keywords: meadow; Von Neumann regular ring; zero-totalized field

Received 20 December 2011; revised 20 December 2011

Handling editor: Jens Blanck

1. INTRODUCTION

This paper contributes to the algebraic specification theory of number systems. Advantages and disadvantages of algebraic specification of abstract data types have been amply discussed in the computer science literature and we do not wish to add anything to those matters here and refer to Wirsing [1], the seminal 1977-paper of Goguen *et al.* [2], the overview in Bjørner and Henson [3], and the ASF+SDF meta-environment of Klint and colleagues [4].

Our focus will be on a particular loose algebraic specification for fields called *meadows*, using the terminology of Broy and Wirsing [5] who first wrote about loose specifications—i.e. the semantic approach not restricted to the isomorphism class of initial algebras. The theory of algebraic specifications is based on theories of universal algebras. Some references to universal algebra are, e.g. Wechler [6] and Graetzer [7].

The equational specification of the variety of meadows has been proposed by Bergstra and Tucker [8] and it has subsequently been elaborated with more systematic detail in [9]. Starting from the signature of fields, one obtains the signature of meadows by adding a unary inverse operator. At the basis of

meadows, now, lies the design decision to turn the inverse (or division if one prefers a binary notation for pragmatic reasons) into a total operator by means of the assumption that $0^{-1} = 0$. By doing so, the investigation of number systems as abstract data types can be carried out within the original framework of algebraic specifications without taking any precautions for partial functions.

Following [8], we write \mathbb{Q}_0 for the rational numbers expanded to a meadow, that is after taking its zero-totalized form as the preferred interpretation. The main result of Bergstra and Tucker [8] consists of obtaining an equational initial algebra specification of \mathbb{Q}_0 . The specification takes the form of a general loose specification, valid in all fields equipped with a totalized inverse, to which an equation L_4 specifically designed for the case of rational numbers is taken in addition: the equation L_4 is based on Lagrange’s theorem that every natural number can be represented as the sum of 4 squares and reads

$$\frac{1 + x^2 + y^2 + z^2 + u^2}{1 + x^2 + y^2 + z^2 + u^2} = 1.$$

So L_4 expresses that, for a large collection of numbers q , it holds that $q \cdot q^{-1} = 1$ (in particular, those q which can be written as 1 plus the sum of four squares). Recently, Yoram Hirshfeld has proved that

$$\frac{1 + x^2 + y^2}{1 + x^2 + y^2} = 1$$

suffices (for a proof see [10]).

[†]This paper is devoted to the occasion of John Tucker’s 60th birthday. The authors acknowledge his broad scholarly work on algebraic methods in computing. In addition Jan Bergstra expresses his great appreciation for over 35 years of joint work with John, often unexpectedly emerging from our continuous stream of discussions about the field in general.

In [11] meadows without proper zero divisors are termed *cancellation meadows*. Recently, we found in [12] that meadows were already introduced by Komori [13] in a report from 1975, where they go by the name of *desirable pseudo-fields*. In [9] it is shown that meadows are precisely the Von Neumann regular rings expanded with an inverse operator $_^{-1}$ and that the equational theory of cancellation meadows (there called zero-totalized fields) has a finite basis. In this paper, we will extend that result to a generic form. This enables its application to extended signatures. In particular, we will examine the case of differential meadows—i.e. meadows equipped with differentiation operators. A second extension is obtained by adding a sign function that provides one of several mutually interchangeable ways in which the presence of an ordering can be equationally specified. The importance of the latter extension follows from the fact that most uses of rational numbers in computer science theory exploit their ordering.

We note that the proof of the generic basis theorem is an elaboration of the proof used for the case of closed terms that has been dealt with in [8]. The proof of the finite basis theorem in [9] uses the existence of maximal ideals. Although shorter and simpler, the proof via ideals seems not to generalize in the way our proof below does.

Bethke *et al.* [14] demonstrate that finite meadows are products of fields, thus strengthening the result in [9] (for the finite case) that establishes that each meadow can be embedded in a product of fields, a result that was named the *embedding theorem for meadows*. We note that the basis theorem for meadows, but not its generic form, is an immediate consequence of the embedding theorem.

The paper is structured as follows: in the next section we recall the axioms for meadows and introduce a representation result. Then, in Section 3 we present our main result, the generic basis theorem. In Section 4 we introduce differential meadows. Then, in Section 5 we extend cancellation meadows with the sign function. We discuss a further extension with floor and ceiling functions and with a square root in Section 6. We end the paper in Section 7 with some conclusions.

This paper is compiled from our earlier work as reported in [15–17].

2. MEADOWS: PRELIMINARIES AND REPRESENTATION

In this section, we introduce cancellation meadows in detail and we discuss a representation result that will be used in Section 4.

In [8], *meadows* were defined as the members of a variety specified by 12 equations. However, in [9] it was established that the 10 equations in Table 1 imply those used in [8]. Summarizing, a meadow is a commutative ring with unit equipped with a total unary operation $(_)^{-1}$ named inverse that

TABLE 1. The set Md of axioms for meadows.

$(x + y) + z = x + (y + z)$
$x + y = y + x$
$x + 0 = x$
$x + (-x) = 0$
$(x \cdot y) \cdot z = x \cdot (y \cdot z)$
$x \cdot y = y \cdot x$
$1 \cdot x = x$
$x \cdot (y + z) = x \cdot y + x \cdot z$
$(x^{-1})^{-1} = x$
$x \cdot (x \cdot x^{-1}) = x$

satisfies the two equations

$$(x^{-1})^{-1} = x, \\ x \cdot (x \cdot x^{-1}) = x, \quad (\text{RIL})$$

and in which $0^{-1} = 0$. Here RIL abbreviates *Restricted Inverse Law*. We write Md for the set of axioms in Table 1.

From the axioms in Md the following identities are derivable:

$$(0)^{-1} = 0, \\ (-x)^{-1} = -(x^{-1}), \\ (x \cdot y)^{-1} = x^{-1} \cdot y^{-1}, \\ 0 \cdot x = 0, \\ x \cdot -y = -(x \cdot y), \\ -(-x) = x.$$

The term *cancellation meadow* is introduced in [11] for a zero-totalized field that satisfies the so-called ‘cancellation axiom’

$$x \neq 0 \ \& \ x \cdot y = x \cdot z \ \longrightarrow \ y = z.$$

An equivalent version of the cancellation axiom that we shall further use in this paper is the *Inverse Law* (IL), i.e. the conditional axiom

$$x \neq 0 \ \longrightarrow \ x \cdot x^{-1} = 1. \quad (\text{IL})$$

So IL states that there are no proper zero divisors. (Another equivalent formulation of the cancellation property is $x \cdot y = 0 \ \longrightarrow \ x = 0$ or $y = 0$.)

We write $\Sigma_m = (0, 1, +, \cdot, -, ^{-1})$ for the signature of (cancellation) meadows and we shall often write $1/t$ or

$$\frac{1}{t}$$

for t^{-1} , tu for $t \cdot u$, t/u for $t \cdot 1/u$, $t - u$ for $t + (-u)$, and freely use numerals and exponentiation with constant integer

exponents. We shall further write

$$1_x \text{ for } \frac{x}{x} \quad \text{and} \quad 0_x \text{ for } 1 - 1_x,$$

so $0_0 = 1_1 = 1$, $0_1 = 1_0 = 0$, and for all terms t ,

$$0_t + 1_t = 1.$$

With the axioms in Table 1 we find by RIL that

$$\begin{aligned} 1_t \cdot t &= t, \\ 1_t \cdot 1/t &= 1/t, \\ (1_t)^2 &= 1_t, \end{aligned} \tag{1}$$

and we derive the following useful identities:

$$\begin{aligned} 1_t \cdot 0_t &= 0 \quad (\text{by } 1_t \cdot 0_t = 1_t(1 - 1_t) = 1_t - 1_t = 0), \\ 0_t \cdot t &= 0 \quad (\text{by } (1 - 1_t)t = t - t = 0), \\ 0_t \cdot 1/t &= 0 \quad (\text{by } (1 - 1_t)1/t = 1/t - 1/t = 0), \\ (0_t)^2 &= 0_t \quad (\text{by } (1 - 1_t)^2 \\ &= 1 - 2 \cdot 1_t + (1_t)^2 = 1 - 1_t = 0_t). \end{aligned} \tag{2}$$

In the remainder of this section we discuss a particular standard representation for meadow terms. We will use this representation in Section 4 in order to prove an expressiveness result.

DEFINITION 2.1. *A term P over Σ_m is a Standard Meadow Form (SMF) if, for some $n \in \mathbb{N}$, P is an SMF of level n . SMFs of level n are defined as follows:*

SMF of level 0: *each expression of the form s/t with s and t ranging over polynomials (i.e. expressions over Σ_m without inverse operator),*

SMF of level $n + 1$: *each expression of the form*

$$0_t \cdot P + 1_t \cdot Q$$

with t ranging over polynomials and P and Q over SMFs of level n .

Observe that if P is an SMF of level n , then it is also of level $n + k$ for all $k \in \mathbb{N}$.

LEMMA 2.1. *If P and Q are SMFs, then in Md, $P + Q$, $P \cdot Q$, $-P$, and $1/P$ are provably equal to an SMF having the same variables.*

Proof. By natural induction on level height n . We spell out the proof in which RIL is often used. The mentioned property of having the same variables follows trivially.

Case $n = 0$. Let s, t, u, v be polynomials, and $P = s/t$ and $Q = u/v$.

First observe that $0_t \cdot s/t = 0_t \cdot 1/t \cdot s = 0$. We derive

$$\begin{aligned} P + Q &= 0_t \cdot (P + Q) + 1_t \cdot (P + Q) \\ &= 0_t \cdot (s/t + u/v) + 1_t \cdot (s/t + u/v) \\ &= 0_t \cdot u/v + 1_t \cdot (s/t + 1_t \cdot u/v), \end{aligned}$$

so it suffices to show that $R = s/t + 1_t \cdot u/v$ is equal to an SMF of level 1:

$$\begin{aligned} R &= 0_v \cdot (s/t + 1_t \cdot u/v) + 1_v \cdot (s/t + 1_t \cdot u/v) \\ &= 0_v \cdot s/t + 1_v \cdot (s/t \cdot 1_v + 1_t \cdot u/v) \\ &= 0_v \cdot s/t + 1_v \cdot \left(\frac{sv + tu}{tv} \right). \end{aligned}$$

The remaining cases are trivial:

$$P \cdot Q = su/tv, \quad -P = -s/t, \quad \text{and} \quad 1/P = t/s.$$

Case $n + 1$. Let $P = 0_t \cdot S + 1_t \cdot T$ and $Q = 0_s \cdot U + 1_s \cdot V$ with S, T, U, V all SMFs of level n .

We first derive

$$\begin{aligned} P + Q &= 0_t \cdot P + 1_t \cdot P + Q \\ &= 0_t \cdot (S + Q) + 1_t \cdot (T + Q) \\ &= 0_t \cdot (0_s \cdot (S + U) + 1_s \cdot (S + V)) \\ &\quad + 1_t \cdot (0_s \cdot (T + U) + 1_s \cdot (T + V)) \end{aligned}$$

and, by induction, each of the pairwise sums of S, T, U, V equals some SMF.

Next, we derive

$$\begin{aligned} P \cdot Q &= 0_s \cdot P \cdot U + 1_s \cdot P \cdot V \\ &= 0_s \cdot (0_t \cdot S \cdot U + 1_t \cdot T \cdot U) \\ &\quad + 1_s \cdot (0_t \cdot S \cdot V + 1_t \cdot T \cdot V) \end{aligned}$$

and by induction each of the pairwise products of S, T, U, V equals some SMF.

Furthermore, $-P = 0_t \cdot (-S) + 1_t \cdot (-T)$, which, by induction, is provably equal to an SMF.

Finally, $1/P = 0_t \cdot (1/P) + 1_t \cdot (1/P)$, hence

$$\begin{aligned} 1/P &= 0_t \cdot \frac{1}{0_t \cdot S + 1_t \cdot T} + 1_t \cdot \frac{1}{0_t \cdot S + 1_t \cdot T} \\ &= 0_t \cdot \frac{0_t}{0_t \cdot (0_t \cdot S + 1_t \cdot T)} + 1_t \cdot \frac{1_t}{1_t \cdot (0_t \cdot S + 1_t \cdot T)} \\ &= 0_t \cdot \frac{0_t}{0_t \cdot S} + 1_t \cdot \frac{1_t}{1_t \cdot T} \\ &= 0_t \cdot 1/S + 1_t \cdot 1/T \end{aligned}$$

and, by induction, there exist SMFs S' and T' such that $S' = 1/S$ and $T' = 1/T$, hence $1/P = 0_t \cdot S' + 1_t \cdot T'$. \square

THEOREM 2.1. *Each term over Σ_m can be represented by an SMF with the same variables.*

Proof. By structural induction. Let P be a term over Σ_m . If $P = 0$ or $P = 1$ or $P = x$, then $P = P/1$, and the latter is an SMF of level 0. The other cases follow immediately from Lemma 2.1. \square

3. A GENERIC BASIS THEOREM

In this section, we prove a finite basis result for the equational theory of cancellation meadows. This result is formulated in a generic way so that it can be used for any expansion of a meadow that satisfies the propagation properties defined below.

DEFINITION 3.1. *Let Σ be an extension of $\Sigma_m = (0, 1, +, \cdot, -, {}^{-1})$, the signature of meadows. Let $E \supseteq \text{Md}$ (with Md the set of axioms for meadows given in Table 1).*

- (1) (Σ, E) has the propagation property for pseudo units *if, for each pair of Σ -terms t, r and context $C[\]$,*

$$E \vdash 1_t \cdot C[r] = 1_t \cdot C[1_t \cdot r].$$

- (2) (Σ, E) has the propagation property for pseudo zeros *if, for each pair of Σ -terms t, r and context $C[\]$,*

$$E \vdash 0_t \cdot C[r] = 0_t \cdot C[0_t \cdot r].$$

Preservation of these propagation properties admits the following nice result:

THEOREM 3.1 *Generic Basis Theorem for Cancellation Meadows.* *If $\Sigma \supseteq \Sigma_m$, $E \supseteq \text{Md}$ and (Σ, E) has the pseudo unit propagation property and the pseudo zero propagation property, then E is a basis (a complete axiomatization) of $\text{Mod}_\Sigma(E \cup \text{IL})$.*

The structure of our proof of this theorem is as follows: let $r = r(\bar{x})$ and $s = s(\bar{x})$ be Σ -terms and let \bar{c} be a series of fresh constants. We write $\Sigma(\bar{c})$ for the signature extended with these constants. Then

$$E \cup \text{IL} \models r = s \quad \text{in } \Sigma$$

$$\iff E \cup \text{ILC} \models r(\bar{c}) = s(\bar{c}) \quad \text{in } \Sigma(\bar{c}) \quad (3)$$

$$\iff E \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c}) \quad (4)$$

$$\iff E \vdash r = s \quad \text{in } \Sigma. \quad (5)$$

Here provability (\vdash) refers to equational logic; the notation further used means the following:

- **ILC**, the *Inverse Law for Closed terms* is the set $\{t = 0 \vee 1_t = 1 \mid t \in T(\Sigma(\bar{c}))\}$, where $T(\Sigma(\bar{c}))$ denotes the set of closed terms over $\Sigma(\bar{c})$.
- **IR** is the *Inverse Rule*: $E \vdash_{\text{IR}} r = s$ means that $\exists k \in \mathbb{N}$ s.t. $E \vdash_{\text{IR}}^k r = s$, and $E \vdash_{\text{IR}}^k r = s$ means that $E \vdash r = s$ provided that the rule

$$\text{IR} \quad \frac{E \cup \{t = 0\} \vdash r = s \quad E \cup \{1_t = 1\} \vdash r = s}{E \vdash r = s}$$

with t ranging over $T(\Sigma(\bar{c}))$ may be used k times.

Before we prove Theorem 3.1—i.e. equivalences (3)–(5)—we establish the following preliminary result.

PROPOSITION 3.1. *Assume $\Sigma \supseteq \Sigma_m$, $E \supseteq \text{Md}$ and (Σ, E) has the propagation property for pseudo units and for pseudo zeros. Then, for $t, r, s \in T(\Sigma)$,*

$$E \cup \{t = 0\} \vdash_{\text{IR}} r = s \implies E \vdash 0_t \cdot r = 0_t \cdot s, \quad (6)$$

$$E \cup \{1_t = 1\} \vdash_{\text{IR}} r = s \implies E \vdash 1_t \cdot r = 1_t \cdot s. \quad (7)$$

Proof. We prove

$$E \cup \{t = 0\} \vdash_{\text{IR}}^k r = s \implies E \vdash 0_t \cdot r = 0_t \cdot s, \quad (8)$$

$$E \cup \{1_t = 1\} \vdash_{\text{IR}}^k r = s \implies E \vdash 1_t \cdot r = 1_t \cdot s \quad (9)$$

simultaneously by induction on k . We use the symbol \equiv to denote syntactic equivalence.

Case $k = 0$. By induction on proof lengths. For (8) the only interesting case is $(r = s) \equiv (t = 0)$, so we have to show that $E \vdash 0_t \cdot t = 0_t \cdot 0$. This follows directly from $E \supseteq \text{Md}$.

For (9) the only interesting case is $(r = s) \equiv (1_t = 1)$, and also $E \vdash (1_t)^2 = 1_t \cdot 1$ follows directly from $E \supseteq \text{Md}$.

Case $k + 1$. By induction on the length of the proofs of $E \cup \{t = 0\} \vdash_{\text{IR}}^{k+1} r = s$ and $E \cup \{1_t = 1\} \vdash_{\text{IR}}^{k+1} r = s$. There are three interesting cases for each of (8) and (9):

- (1) The \vdash_{IR}^{k+1} results follow from the assumption $(r = s) \equiv (t = 0)$ or $(r = s) \equiv (1_t = 1)$, respectively. These results follow in the same way as above.

- (2) The \vdash_{IR}^{k+1} results follow from the context rule, so $r \equiv C[v]$, $s \equiv C[w]$ and

(8) $E \cup \{t = 0\} \vdash_{\text{IR}}^{k+1} v = w$. By induction, $E \vdash 0_t \cdot v = 0_t \cdot w$. Hence, $E \vdash 0_t \cdot C[0_t \cdot v] = 0_t \cdot C[0_t \cdot w]$, and by (Σ, E) having the propagation property for pseudo zeros, $E \vdash 0_t \cdot C[v] = 0_t \cdot C[w]$.

(9) $E \cup \{1_t = 1\} \vdash_{\text{IR}}^{k+1} v = w$. By induction, $E \vdash 1_t \cdot v = 1_t \cdot w$. Hence, $E \vdash 1_t \cdot C[1_t \cdot v] = 1_t \cdot C[1_t \cdot w]$, and by (Σ, E) having the propagation property for pseudo units, $E \vdash 1_t \cdot C[v] = 1_t \cdot C[w]$.

- (3) The \vdash_{IR}^{k+1} results follow from the IR rule, that is

(8) $E \cup \{t = 0\} \cup \{h = 0\} \vdash_{\text{IR}}^k r = s$ and $E \cup \{t = 0\} \cup \{1_h = 1\} \vdash_{\text{IR}}^k r = s$. By induction, $E \cup \{h = 0\} \vdash 0_t \cdot r = 0_t \cdot s$ and $E \cup \{1_h = 1\} \vdash 0_t \cdot r = 0_t \cdot s$. Again applying induction (\vdash derivability implies \vdash_{IR}^k derivability) yields

$$E \vdash 0_h \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot s,$$

$$E \vdash 1_h \cdot 0_t \cdot r = 1_h \cdot 0_t \cdot s.$$

$$\text{We derive } 0_t \cdot r = (0_h + 1_h) \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot r + 1_h \cdot 0_t \cdot r = 0_h \cdot 0_t \cdot s + 1_h \cdot 0_t \cdot s = 0_t \cdot s.$$

- (9) $E \cup \{1_t = 1\} \cup \{h = 0\} \vdash_{\text{IR}}^k r = s$ and $E \cup \{1_t = 1\} \cup \{1_h = 1\} \vdash_{\text{IR}}^k r = s$. Similar. \square

Proof of Theorem 3.1. We now give a detailed proof of equivalences (3)–(5), using Proposition 3.1. For model theoretic details, we refer to [18].

- (3) (\implies) Assume $E \cup \text{IL} \models r = s$. Let \mathbb{M} be a model of $E \cup \text{ILC}$ (over $\Sigma(\bar{c})$). Then $\mathbb{M} \models r(\bar{c}) = s(\bar{c})$ if and only if $\mathbb{M}' \models r(\bar{c}) = s(\bar{c})$ for \mathbb{M}' the minimal submodel of \mathbb{M} . Now \mathbb{M}' is also a model for IL because ILC concerns all closed terms and each value in the domain of \mathbb{M}' is the interpretation of a closed term. So, by assumption $\mathbb{M}' \models r = s$, and, in particular (by substitution), $\mathbb{M}' \models r(\bar{c}) = s(\bar{c})$.

(\impliedby) Assume $E \cup \text{ILC} \models r(\bar{c}) = s(\bar{c})$. Let \mathbb{M} be a model of $E \cup \text{IL}$ (over Σ). We have to show that $\mathbb{M} \models r(\bar{x}) = s(\bar{x})$, or, stated differently, that for $\bar{a} = a_1, \dots, a_n$ a series of values from \mathbb{M} 's domain, $(\mathbb{M}, x_i \mapsto a_i) \models r = s$ where $x_i \mapsto a_i$ represents the assignment of a_i to x_i . Extend Σ with a fresh constant c_i for each a_i and let $\mathbb{M}(\bar{c})$ be the expansion of \mathbb{M} in which each constant c_i is interpreted as a_i . Then $\mathbb{M}(\bar{c})$ satisfies ILC because \mathbb{M} satisfies IL, and so, by assumption, $\mathbb{M}(\bar{c}) \models r(\bar{c}) = s(\bar{c})$, and therefore $(\mathbb{M}(\bar{c}), x_i \mapsto a_i) \models r = s$ and thus also $(\mathbb{M}, x_i \mapsto a_i) \models r = s$, as was to be shown.

- (4) (\implies) Let E^C be the set of all closed instances over the extended signature $\Sigma(\bar{c})$; then

$$E^C \cup \text{ILC} \models r(\bar{c}) = s(\bar{c}).$$

By compactness there is a finite set $F \subseteq E^C \cup \text{ILC}$ such that $F \models r(\bar{c}) = s(\bar{c})$.

Now apply induction on the number of elements from ILC in F , say k .

Case $k = 0$. By completeness we find $E \vdash r(\bar{c}) = s(\bar{c})$, and thus $E \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c})$.

Case $k + 1$. Assume $(t = 0 \vee 1_t = 1) \in F$ and let $F' = F \setminus \{t = 0 \vee 1_t = 1\}$. Reasoning with propositional logic, we find

$$F' \models (t = 0 \vee 1_t = 1) \rightarrow r(\bar{c}) = s(\bar{c})$$

and thus

$$F' \models (t = 0 \rightarrow r(\bar{c}) = s(\bar{c})) \wedge (1_t = 1 \rightarrow r(\bar{c}) = s(\bar{c})),$$

which in turn is equivalent with

$$F' \cup \{t = 0\} \models r(\bar{c}) = s(\bar{c}), \\ F' \cup \{1_t = 1\} \models r(\bar{c}) = s(\bar{c}).$$

By induction, $E \cup \{t = 0\} \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c})$ and $E \cup \{1_t = 1\} \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c})$, and thus by IR,

$$E \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c}).$$

(\impliedby) This follows from the soundness of IR with respect to ILC. That is, if $E \vdash u = v$ because $E \cup \{t = 0\} \vdash u = v$ and $E \cup \{1_t = 1\} \vdash u = v$, then $E \cup \{t = 0 \vee 1_t = 1\} \models u = v$, so $E \cup \text{ILC} \models u = v$.

- (5) (\implies) By induction on the length of the proof, using Proposition 3.1: if $E \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c})$ follows from IR (the only interesting case), then

$$E \cup \{t = 0\} \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c}), \\ E \cup \{1_t = 1\} \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c}),$$

so $E \vdash 0_t \cdot r(\bar{c}) = 0_t \cdot s(\bar{c})$ by (6) and $E \vdash 1_t \cdot r(\bar{c}) = 1_t \cdot s(\bar{c})$ by (7). Thus

$$E \vdash r(\bar{c}) = (0_t + 1_t) \cdot r(\bar{c}) = 0_t \cdot r(\bar{c}) + 1_t \cdot r(\bar{c}) \\ = 0_t \cdot s(\bar{c}) + 1_t \cdot s(\bar{c}) \\ = s(\bar{c}).$$

A similar proof result is obtained by replacing $r(\bar{c})$ by r and $s(\bar{c})$ by s .

(\impliedby) Trivial: if $E \vdash r = s$, then $E \vdash r(\bar{c}) = s(\bar{c})$ in the extended signature $\Sigma(\bar{c})$. So, $E \vdash_{\text{IR}} r(\bar{c}) = s(\bar{c})$. \square

A first application of Theorem 3.1 concerns the equational theory of cancellation meadows:

COROLLARY 3.1. *The set of axioms Md (see Table 1) is a finite basis (a complete axiomatization) of $\text{Mod}_{\Sigma_m}(\text{Md} \cup \text{IL})$.*

Proof. It remains to be shown that the propagation properties for pseudo units and for pseudo zeros hold in Md. This follows easily by case distinction on the forms that $C[r]$ may take and the various identities on 1_t and 0_t . As an example consider the case $C[_] \equiv _ + u$. Then

$$1_t \cdot C[r] = 1_t \cdot (r + u) \\ = 1_t \cdot r + 1_t \cdot u \\ = 1_t \cdot 1_t \cdot r + 1_t \cdot u \\ = 1_t \cdot C[1_t \cdot r].$$

The remaining cases can be proved in a similar way. \square

4. DIFFERENTIAL MEADOWS

In this section, we provide an elegant equational axiomatization of differential operators and, with the generic basis theorem, we obtain a finite basis for differential cancellation meadows.

4.1. Differential meadows

Given some $n \geq 1$, we extend the signature Σ_m of meadows with differentiation operators and constants X_1, \dots, X_n to model

TABLE 2. The set DE of axioms for differentiation.

$$\frac{\partial}{\partial X_i}(x + y) = \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}(y) \quad (10)$$

$$\frac{\partial}{\partial X_i}(x \cdot y) = \frac{\partial}{\partial X_i}(x) \cdot y + x \cdot \frac{\partial}{\partial X_i}(y) \quad (11)$$

$$\frac{\partial}{\partial X_i}(x \cdot x^{-1}) = 0 \quad (12)$$

$$\frac{\partial}{\partial X_i}(X_i) = 1 \quad (13)$$

$$\frac{\partial}{\partial X_i}(X_j) = 0 \quad \text{if } i \neq j \quad (14)$$

functions to be differentiated:

$$\frac{\partial}{\partial X_i} : \mathbb{M} \rightarrow \mathbb{M}$$

for $i = 1, \dots, n$ and some meadow \mathbb{M} . We write Σ_{md} for this extended signature. Equational axioms for $\partial/\partial X_i$ are given in Table 2, where (13) and (14) define n^2 equational axioms. Observe that the Md axioms together with Axiom (12) imply

$$\frac{\partial}{\partial X_i}(0) = 0.$$

Furthermore, using Axiom (10), one easily proves

$$\frac{\partial}{\partial X_i}(-x) = -\frac{\partial}{\partial X_i}(x).$$

First, we establish the expected corollary of Theorem 3.1:

COROLLARY 4.1. *The set of axioms $\text{Md} \cup \text{DE}$ (see Tables 1 and 2) is a complete axiomatization of $\text{Mod}_{\Sigma_{\text{md}}}(\text{Md} \cup \text{DE} \cup \text{IL})$.*

Proof. The pseudo unit propagation property requires a check for $\frac{\partial}{\partial X_i}(_)$ only:

$$\frac{\partial}{\partial X_i}(1_t \cdot r) = \frac{\partial}{\partial X_i}(1_t) \cdot r + 1_t \cdot \frac{\partial}{\partial X_i}(r) = 1_t \cdot \frac{\partial}{\partial X_i}(r). \quad (15)$$

Multiplication with 1_t now yields the property. From (15) we obtain

$$\begin{aligned} 0_t \cdot \frac{\partial}{\partial X_i}(r) &= \frac{\partial}{\partial X_i}(r) - 1_t \cdot \frac{\partial}{\partial X_i}(r) \\ &\stackrel{(15)}{=} \frac{\partial}{\partial X_i}(r) - \frac{\partial}{\partial X_i}(1_t \cdot r) = \frac{\partial}{\partial X_i}(0_t \cdot r), \end{aligned}$$

and multiplication with 0_t then yields the pseudo zero propagation property. \square

A *differential meadow* is a meadow equipped with formal variables X_1, \dots, X_n and differentiation operators $\partial/\partial X_i(_)$ that satisfies the axioms in DE.

We conclude this section with an elegant consequence of the fact that we are working in the setting of meadows, namely the consequence that the differential of an inverse follows from the DE axioms.

PROPOSITION 4.1.

$$\text{Md} \cup \text{DE} \vdash \frac{\partial}{\partial X_i}(1/x) = -(1/x^2) \cdot \frac{\partial}{\partial X_i}(x).$$

Proof. By Axioms (12) and (11),

$$0 = \frac{\partial}{\partial X_i}(x/x) = \frac{\partial}{\partial X_i}(x) \cdot 1/x + x \cdot \frac{\partial}{\partial X_i}(1/x),$$

so

$$0 = 0 \cdot (1/x) = \frac{\partial}{\partial X_i}(x/x) \cdot (1/x)$$

$$= \frac{\partial}{\partial X_i}(x) \cdot 1/x^2 + (x/x) \cdot \frac{\partial}{\partial X_i}(1/x)$$

$$\stackrel{(15)}{=} 1/x^2 \cdot \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}((x/x) \cdot (1/x))$$

$$\stackrel{\text{RIL}}{=} 1/x^2 \cdot \frac{\partial}{\partial X_i}(x) + \frac{\partial}{\partial X_i}(1/x),$$

and hence

$$\frac{\partial}{\partial X_i}(1/x) = -(1/x^2) \cdot \frac{\partial}{\partial X_i}(x). \quad \square$$

4.2. Existence of differential meadows

In this section, we show the *existence* of differential meadows with formal variables X_1, \dots, X_n for arbitrary finite $n > 0$. First, we define a particular cancellation meadow, and then we expand this meadow to a differential cancellation meadow by adding formal differentiation.

The Zariski topology congruence over \mathbb{C}_0^n . We will use some terminology from algebraic geometry; in particular we will use the Zariski topology [19, 20]. Open (closed) sets in this topology will be indicated as Z-open (Z-closed). Recall that complements of Z-closed sets are Z-open and complements of Z-open sets are Z-closed, finite unions of Z-closed sets are Z-closed and intersections of Z-closed sets are Z-closed. Let \mathbb{C}_0 denote the zero-totalized expansion of the complex numbers. We will make use of the following facts:

- (1) The solutions of a set of polynomial equations (with n or less variables) within \mathbb{C}_0^n constitute a Z-closed subset of \mathbb{C}_0^n . Here ‘polynomial’ has the conventional meaning, not involving division. Taking equations $1 = 0$ and $0 = 0$, respectively, it follows that both \emptyset and \mathbb{C}_0^n are Z-closed (and Z-open as well).
- (2) Intersections of non-empty Z-open sets are non-empty.

In the following we consider terms

$$t(\vec{X}) = t(X_1, \dots, X_n)$$

with $t = t(\vec{x})$ a Σ_m -term and we write $T(\Sigma_m(\vec{X}))$ for the set of these terms. For $V \subseteq \mathbb{C}_0^n$ we define the equivalence

$$\equiv_{\mathbb{C}_0^n}^V$$

on $T(\Sigma_m(\bar{X}))$ by $t(\bar{X}) \equiv_{\mathbb{C}_0^V}^V r(\bar{X})$ if each assignment $\bar{X} \mapsto V$ evaluates both sides to equal values in \mathbb{C}_0 . It follows immediately that, for each $V \subseteq \mathbb{C}_0^n$, $T(\Sigma_m(\bar{X}))/\equiv_{\mathbb{C}_0^V}^V$ is a meadow. In particular, if $V = \emptyset$, one obtains the trivial meadow ($0 = 1$) as both 0 and 1 satisfy any universal quantification over an empty set. If V is a singleton, this quotient is a cancellation meadow. In other cases the meadow may not satisfy the cancellation property. Indeed, suppose that $n = 1$ and $V = \{0, 1\}$ and let $t(X) = X$. Now $t(1) \neq 0$. Thus $t(X) \neq 0$ in $T(\Sigma_m(X))/\equiv_{\mathbb{C}_0^V}^V$. If that is assumed to be a cancellation meadow, however, one has $1_{t(X)} = 1$, but $1_{t(0)} = 0$, thus refuting $1_{t(X)} = 1$.

We now define the relation \equiv_{ZTC} (Zariski Topology Congruence over \mathbb{C}_0^n) by

$$t \equiv_{ZTC} r \iff \exists V (V \text{ is Z-open, } V \neq \emptyset \text{ and } t \equiv_{\mathbb{C}_0^V}^V r).$$

The relation \equiv_{ZTC} is indeed a congruence for all meadow operators: the equivalence properties follow easily; for $0 \equiv_{ZTC} 0$ and $1 \equiv_{ZTC} 1$, take $V = \mathbb{C}_0^n$, and if $P \equiv_{ZTC} P'$ and $Q \equiv_{ZTC} Q'$, witnessed respectively by V and V' , then

$$P + P' \equiv_{ZTC} Q + Q' \quad \text{and} \quad P \cdot P' \equiv_{ZTC} Q \cdot Q'$$

are witnessed by $V \cap V'$ which is Z-open and non-empty because of fact 2 above. Finally $-P \equiv_{ZTC} -P'$ and $(P)^{-1} \equiv_{ZTC} (P')^{-1}$ are both witnessed by V .

Theorem 2.1, i.e., the (SMF) representation result for meadow terms implies for $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$ that each term can be represented by 0 or by p/q with p and q polynomials not equal to 0. We note that it is decidable whether or not a polynomial equals the 0-polynomial by taking all corresponding products of powers of the X_1, \dots, X_n together and then checking that all coefficients vanish.

As an example, let P be the SMF of level 1 defined by

$$P = 0_{1-X_1} \cdot \frac{2X_1}{X_2} + 1_{1-X_1} \cdot \frac{1 + X_2 - 2X_1X_3}{8 - X_1X_3^2}.$$

Now in $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$, the polynomial $1 - X_1$ is on some Z-open non-empty set V not equal to 0 (see fact 1 above), thus $1_{1-X_1} \equiv_{\mathbb{C}_0^V}^V 1$ and $0_{1-X_1} \equiv_{\mathbb{C}_0^V}^V 0$, and hence

$$P \equiv_{ZTC} \frac{1 + X_2 - 2X_1X_3}{8 - X_1X_3^2}.$$

So, in $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$, the SMF level-hierarchy collapses and terms can be represented by either 0 or by p/q with both p and q polynomials not equal to 0. In the second case $1_{p/q} = 1$ and therefore it is a cancellation meadow. Furthermore, equality is decidable in this model. Indeed to check that $1_p = 1$ (and $0_p = 0$) for a polynomial p , it suffices to check that p is not 0 over the complex numbers. Using the SMF representation, all closed terms are either 0 or take the form p/q with p and q

non-zero polynomials. For q and q' non-zero polynomials we find that

$$p/q \equiv_{ZTC} p'/q' \iff p \cdot q' - p' \cdot q = 0,$$

which we have already found to be decidable.

Constructing a differential cancellation meadow. In $T(\Sigma_m(\bar{X}))/\equiv_{ZTC}$ the differential operators can be defined as follows:

$$\frac{\partial}{\partial X_i}(0) = 0$$

and, using the fact that differentials on polynomials are known,

$$\frac{\partial}{\partial X_i} \left(\frac{p}{q} \right) = \frac{(\partial/\partial X_i)(p) \cdot q - p \cdot (\partial/\partial X_i)(q)}{q^2}.$$

Let V be the set of 0-points of q and let $U = \sim V$, the complement of V . Then p/q is differentiable on U and the derivative coincides with the formal derivative used in the definition. This definition is representation independent: consider $p'/q' \equiv_{ZTC} p/q$ with V' the 0-points of q' and $U' = \sim V'$. Then there is some non-empty and Z-open W such that $p/q \equiv_{\mathbb{C}_0^W}^W p'/q'$. Now $W \cap U \cap U'$ is non-empty and Z-open, and on this set,

$$\frac{\partial}{\partial X_i} \left(\frac{p}{q} \right) = \frac{\partial}{\partial X_i} \left(\frac{p'}{q'} \right).$$

So, formal differentiation $\partial/\partial X_i$ preserves the congruence properties. Finally, we check the soundness of the DE axioms.

Axiom (10): Consider $t + t'$. In the case that t or t' equals 0, axiom D1 is obviously sound. In the remaining case, $t = p/q$ and $t' = p'/q'$ with all polynomials not equal to 0 and

$$t + t' = \frac{pq' + p'q}{qq'}.$$

Using ordinary differentiation on polynomials, we derive

$$\begin{aligned} & \frac{\partial}{\partial X_i}(t + t') \\ &= \frac{(\partial/\partial X_i)(pq' + p'q) \cdot qq' - (pq' + p'q) \cdot (\partial/\partial X_i)(qq')}{(qq')^2} \\ &= \frac{(\partial/\partial X_i)(p) \cdot q \cdot (q')^2 + (\partial/\partial X_i)(p') \cdot q^2 \cdot q' +}{(qq')^2} \\ & \quad - \frac{p \cdot (\partial/\partial X_i)(q) \cdot (q')^2 - p' \cdot (\partial/\partial X_i)(q') \cdot q^2}{(qq')^2} \\ &= \frac{\partial}{\partial X_i} \left(\frac{p}{q} \right) \cdot 1_{(q')^2} + \frac{\partial}{\partial X_i} \left(\frac{p'}{q'} \right) \cdot 1_{q^2} \\ &= \frac{\partial}{\partial X_i}(t) + \frac{\partial}{\partial X_i}(t'). \end{aligned}$$

Axiom (11): Similar.

TABLE 3. The set *Signs* of axioms for the sign function.

$\mathbf{s}(1_x) = 1_x$	(16)
$\mathbf{s}(0_x) = 0_x$	(17)
$\mathbf{s}(-1) = -1$	(18)
$\mathbf{s}(x^{-1}) = \mathbf{s}(x)$	(19)
$\mathbf{s}(x \cdot y) = \mathbf{s}(x) \cdot \mathbf{s}(y)$	(20)
$0_{\mathbf{s}(x)-\mathbf{s}(y)} \cdot (\mathbf{s}(x+y) - \mathbf{s}(x)) = 0$	(21)

Axiom (12): Consider t ; then either $t = 0$ or $t/t = 1$, and in both cases $(\partial/\partial X_i)(t/t) = 0$.

Axiom schemes (13) and (14): We derive

$$\frac{\partial}{\partial X_i}(X_j) = \frac{\partial}{\partial X_i} \begin{pmatrix} X_j \\ 1 \end{pmatrix} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, by adding formal differentiation to $T(\Sigma_m(\bar{X}))$, we constructed a differential cancellation meadow.

5. SIGNED MEADOWS

In this section, we consider *signed meadows*: we extend the signature $\Sigma_m = (0, 1, +, \cdot, -, {}^{-1})$ of meadows with the unary sign (or signum) function $\mathbf{s}(x)$. We write Σ_{ms} for this extended signature, so $\Sigma_{ms} = (0, 1, +, \cdot, -, {}^{-1}, \mathbf{s})$. The sign function $\mathbf{s}(x)$ presupposes an ordering on its domain and is defined by

$$\mathbf{s}(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

We define the sign function in an equational manner by the set *Signs* of axioms given in Table 3. First, note that, by Md and axiom (16) (or axiom (17)), we find

$$\mathbf{s}(0) = 0 \quad \text{and} \quad \mathbf{s}(1) = 1.$$

Then, observe that in combination with the inverse law IL, axiom (21) is an equational representation of the conditional equational axiom

$$\mathbf{s}(x) = \mathbf{s}(y) \longrightarrow \mathbf{s}(x+y) = \mathbf{s}(x).$$

From Md and axioms (18)–(21) one can easily compute $\mathbf{s}(t)$ for any closed term t .

Some more consequences of the $\text{Md} \cup \text{Signs}$ axioms are the following:

$$\mathbf{s}(x^2) = 1_x, \quad (22)$$

$$\mathbf{s}(x^3) = \mathbf{s}(x), \quad (23)$$

$$1_x \cdot \mathbf{s}(x) = \mathbf{s}(x), \quad (24)$$

$$\mathbf{s}(x)^{-1} = \mathbf{s}(x). \quad (25)$$

Here (22) follows from $\mathbf{s}(x^2) = \mathbf{s}(x) \cdot \mathbf{s}(x) = \mathbf{s}(x) \cdot \mathbf{s}(x^{-1}) = \mathbf{s}(1_x) = 1_x$, (23) from $\mathbf{s}(x^3) = \mathbf{s}(x) \cdot \mathbf{s}(x) \cdot \mathbf{s}(x^{-1}) = \mathbf{s}(x \cdot (x \cdot x^{-1})) = \mathbf{s}(x)$, (24) from $1_x \cdot \mathbf{s}(x) = \mathbf{s}(x^2) \cdot \mathbf{s}(x) = \mathbf{s}(x^3) = \mathbf{s}(x)$, and (25) from

$$\begin{aligned} \mathbf{s}(x)^{-1} &= (\mathbf{s}(x)^2 \cdot \mathbf{s}(x)^{-1})^{-1} = (\mathbf{s}(x^2) \cdot \mathbf{s}(x)^{-1})^{-1} \\ &= (1_x \cdot \mathbf{s}(x)^{-1})^{-1} = 1_x \cdot \mathbf{s}(x) = \mathbf{s}(x). \end{aligned}$$

So, $0 = \mathbf{s}(x) - \mathbf{s}(x) = \mathbf{s}(x) - \mathbf{s}(x)^3 = \mathbf{s}(x)(1 - \mathbf{s}(x)^2)$ and hence

$$\mathbf{s}(x) \cdot (1 - \mathbf{s}(x)) \cdot (1 + \mathbf{s}(x)) = 0. \quad (26)$$

Identity (26) implies with IL that, for any closed term t , $\mathbf{s}(t) \in \{-1, 0, 1\}$, and thus also that $\mathbf{s}(\mathbf{s}(t)) = \mathbf{s}(t)$. However, with some effort we can derive $\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$, which, of course, is an interesting consequence.

PROPOSITION 5.1. $\text{Md} \cup \text{Signs} \vdash \mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$.

Before giving a proof of the idempotency of $\mathbf{s}(x)$, we explain how we found one, as there seems not to be an obvious proof for this identity—at the same time this explanation illustrates the proof of Theorem 3.1. Consider a fresh constant c and let e abbreviate the equation $\mathbf{s}(\mathbf{s}(c)) = \mathbf{s}(c)$; then

$$\begin{aligned} \text{Md} \cup \text{Signs} \cup \{\mathbf{s}(c) = 0\} &\vdash_{\text{IR}} e, \\ \text{Md} \cup \text{Signs} \cup \{1_{\mathbf{s}(c)} = 1, 1 - \mathbf{s}(c) = 0\} &\vdash_{\text{IR}} e, \\ \text{Md} \cup \text{Signs} \cup \{1_{\mathbf{s}(c)} = 1, 1_{1-\mathbf{s}(c)} = 1\} &\vdash_{\text{IR}} e. \end{aligned}$$

The first two derivabilities are trivial; the third one is obtained from (26) after multiplication with $1/\mathbf{s}(c) \cdot 1/(1 - \mathbf{s}(c))$ (thus yielding $\mathbf{s}(c) = -1 = \mathbf{s}(\mathbf{s}(c))$). The proof transformations that underly the proof of Theorem 3.1 dictate how to eliminate the IR rule in this particular case. The proof below shows the slightly polished result.

Proof of Proposition 5.1. Recall $0_t + 1_t = 1$. The result $\mathbf{s}(\mathbf{s}(x)) = \mathbf{s}(x)$ follows from

$$\begin{aligned} \mathbf{s}(\mathbf{s}(x)) &= (0_{\mathbf{s}(x)} + 1_{\mathbf{s}(x)}) \cdot \mathbf{s}(\mathbf{s}(x)), \\ \mathbf{s}(x) &= (0_{\mathbf{s}(x)} + 1_{\mathbf{s}(x)}) \cdot \mathbf{s}(x), \end{aligned}$$

and (27) and (28):

$$0_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(x), \quad (27)$$

$$1_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad (28)$$

Identity (27) follows from $0 = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(x)$ by $0 = \mathbf{s}(0) = \mathbf{s}(0_{\mathbf{s}(x)} \cdot \mathbf{s}(x)) = 0_{\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x))$, and (28) follows from combining

(29) and (30):

$$1_{\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot 0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x), \quad (29)$$

$$1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad (30)$$

Identity (29) follows simply: $0_{1-\mathbf{s}(x)} \cdot (1 - \mathbf{s}(x)) = 0$, so $0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x) = 0_{1-\mathbf{s}(x)}$ and thus

$$\begin{aligned} 0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) &= \mathbf{s}(0_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x)) \\ &= \mathbf{s}(0_{1-\mathbf{s}(x)}) \\ &= 0_{1-\mathbf{s}(x)} \\ &= 0_{1-\mathbf{s}(x)} \mathbf{s}(x). \end{aligned}$$

Identity (30) can be derived as follows: from (26) infer

$$1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot (1 + \mathbf{s}(x)) = 0,$$

thus $1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x) = 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot -1$, and thus with $\mathbf{s}(-1) = -1$,

$$\begin{aligned} 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(\mathbf{s}(x)) &= \mathbf{s}(1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x)) \\ &= 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot -1 \\ &= 1_{\mathbf{s}(x)} \cdot 1_{1-\mathbf{s}(x)} \cdot \mathbf{s}(x). \quad \square \end{aligned}$$

Next we establish the expected corollary of Theorem 3.1.

COROLLARY 5.1. *The set of axioms $\text{Md} \cup \text{Signs}$ (see Tables 1 and 3) is a finite basis (a complete axiomatization) of $\text{Mod}_{\Sigma_{ms}}(\text{Md} \cup \text{Signs} \cup \text{IL})$.*

Proof. It suffices to show that the propagation properties are satisfied for $\mathbf{s}(_)$.

Pseudo units: $1_x \cdot \mathbf{s}(y) = (1_x)^2 \cdot \mathbf{s}(y) = 1_x \cdot \mathbf{s}(1_x) \cdot \mathbf{s}(y) = 1_x \cdot \mathbf{s}(1_x \cdot y)$.

Pseudo zeros: $0_x \cdot \mathbf{s}(y) = (0_x)^2 \cdot \mathbf{s}(y) = 0_x \cdot \mathbf{s}(0_x) \cdot \mathbf{s}(y) = 0_x \cdot \mathbf{s}(0_x \cdot y)$. \square

We note that the initial algebra of $\text{Md} \cup \text{Signs}$ equals \mathbb{Q}_0 as introduced in [8] expanded with the sign function (a proof follows immediately from the techniques used in that paper). It remains to be shown that the *Signs* axioms (in combination with those of *Md*) are independent. We leave this as an open question.

In the following we show that the sign function is not definable in \mathbb{Q}_0 , the zero-totalized field of rational numbers as discussed in [8]. We say that $q, q' \in T(\mathbb{Q}_0)$ are *different* if $1_{q-q'} = 1$. Let $r = r(x)$ and $s = s(x)$ and let $T(\mathbb{Q}_0[x])$ be the set of terms that are either closed or have x as the only variable, so $r, s \in T(\mathbb{Q}_0[x])$. We define

$$\begin{aligned} r \equiv_{\infty} s &\iff r(q) = s(q) && \text{for infinitely many} \\ &&& \text{different } q \text{ in } T(\mathbb{Q}_0), \\ r \equiv_{\text{ae}} s &\iff r(q) \neq s(q) && \text{for finitely many} \\ &&& \text{different } q \text{ in } T(\mathbb{Q}_0). \end{aligned}$$

We call these relations *infinite equivalence* and *almost equivalence*, respectively. Observe that both these relations are congruences over $T(\mathbb{Q}_0[x])$.

THEOREM 5.1. *Let $r = r(x)$ and $s = s(x)$. If $r \equiv_{\infty} s$, then $r \equiv_{\text{ae}} s$.*

Proof. By Theorem 2.1 it suffices to prove this for SMFs, say $P = P(x)$ and $Q = Q(x)$. Because $P - Q$ is then provably equal to an SMF, we further assume without loss of generality that $Q = 0$.

So let $P \equiv_{\infty} 0$. We prove $P \equiv_{\text{ae}} 0$ by induction on the level n of P .

Case $n = 0$. Then $P = s/t$ for polynomials $s = s(x)$ and $t = t(x)$. Because $P \equiv_{\infty} 0$, at least one of $s \equiv_{\infty} 0$ and $t \equiv_{\infty} 0$ holds. Because polynomials always have a finite number of zero points, at least one of $s \equiv_{\text{ae}} 0$ and $t \equiv_{\text{ae}} 0$ holds. Thus $P \equiv_{\text{ae}} 0$.

Case $n + 1$. Then $P = 0_t \cdot S + 1_t \cdot T$.

- If $t \equiv_{\text{ae}} 0$, then $0_t \equiv_{\text{ae}} 1$ and $1_t \cdot T \equiv_{\text{ae}} 0$, so $S \equiv_{\infty} 0$. By induction, $S \equiv_{\text{ae}} 0$, and thus $0_t \cdot S \equiv_{\text{ae}} 0$ and hence $P \equiv_{\text{ae}} 0$.
- If $t \not\equiv_{\text{ae}} 0$, then $1_t \equiv_{\infty} 1$, so $1_t \equiv_{\text{ae}} 1$ and $0_t \cdot S \equiv_{\text{ae}} 0$, so $T \equiv_{\infty} 0$. By induction, $T \equiv_{\text{ae}} 0$, and thus $1_t \cdot T \equiv_{\text{ae}} 0$, and hence $P \equiv_{\text{ae}} 0$. \square

An immediate consequence of Theorem 5.1 is the following corollary.

COROLLARY 5.2. *The sign function is not definable in \mathbb{Q}_0 .*

Proof. Suppose otherwise. Then there is a term $t \in T(\mathbb{Q}_0[x])$ with $\mathbf{s}(x) = t(x)$. So

$$t(x) \equiv_{\infty} 1$$

(because of all positive rationals). But then $t(x) \equiv_{\text{ae}} 1$ by Theorem 5.1, which contradicts $t(x) = -1$ for all negative rationals. \square

Furthermore, we note that with the sign function $\mathbf{s}(x)$, the functions $\max(x, y)$ and $\min(x, y)$ have a simple equational specification:

$$\begin{aligned} \max(x, y) &= \max(x - y, 0) + y, \\ \max(x, 0) &= (\mathbf{s}(x) + 1) \cdot x/2, \end{aligned}$$

and, of course, $\min(x, y) = -\max(-x, -y)$.

Finally, the existence of non-trivial differential cancellation meadows with sign function is not an obvious matter and requires a modification of the existence proof given in Section 4.2.

6. FLOOR, CEILING AND SQUARE ROOT

In this section, we consider extensions of signed meadows with floor, ceiling and square root.

TABLE 4. The set FC of axioms for the floor and ceiling functions.

$$1_x \cdot \lfloor y \rfloor = 1_x \cdot \lfloor 1_x \cdot y \rfloor \quad (31)$$

$$0_x \cdot \lfloor y \rfloor = 0_x \cdot \lfloor 0_x \cdot y \rfloor \quad (32)$$

$$\lfloor x - 1 \rfloor = \lfloor x \rfloor - 1 \quad (33)$$

$$\lfloor x + 1 \rfloor = \lfloor x \rfloor + 1 \quad (34)$$

$$\lfloor 0 \rfloor = 0 \quad (35)$$

$$(0_{1-s(x)} \cdot 0_{1-s(1-x)}) \cdot \lfloor x \rfloor = 0 \quad (36)$$

$$\lceil x \rceil = -\lfloor -x \rfloor \quad (37)$$

6.1. Signed meadows with floor and ceiling

We briefly discuss the extension of signed meadows with the *floor* function $\lfloor x \rfloor$ and the *ceiling* function $\lceil x \rceil$. These functions are defined by

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leq x\}.$$

and

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}.$$

We define these functions in an equational manner by the axioms in Table 4.

Some comments on these axioms: first, (31) and (32) guarantee the propagation properties. Then, consider $0_{1-s(x)} \cdot 0_{1-s(1-x)}$, which equals 1 if both $x > 0$ and $1 - x > 0$, and 0 otherwise. So, axiom (36) states that $\lfloor x \rfloor = 0$ whenever $0 < x < 1$. With (33)–(35) this is sufficient to compute $\lfloor t \rfloor$ for any closed t . Axiom (37), defining the ceiling function $\lceil x \rceil$, is totally standard.

Let Σ_{msfc} be the signature of this extension. As before, we have an immediate corollary of Theorem 3.1.

COROLLARY 6.1. *The set of axioms $\text{Md} \cup \text{Signs} \cup \text{FC}$ (see Tables 1–4) is a finite basis (a complete axiomatization) of $\text{Mod}_{\Sigma_{msfc}}(\text{Md} \cup \text{Signs} \cup \text{FC} \cup \text{IL})$.*

Proof. For the floor, the propagation properties for pseudo units and for pseudo zeros are directly axiomatized by axioms (31) and (32), and those for the ceiling follow easily. So, the corollary follows immediately from Theorem 3.1 and the proof of Corollary 5.1. \square

We note that the initial algebra of $\text{Md} \cup \text{Signs} \cup \text{FC}$ is \mathbb{Q}_0 extended with the sign function $s(x)$ and the floor and ceiling functions $\lfloor x \rfloor$ and $\lceil x \rceil$, respectively. It remains to be shown that the FC axioms (in combination with those of $\text{Md} \cup \text{Signs}$) are independent. We leave this as an open question.

We continue this section by proving that in $\mathbb{Q}_0(\mathbf{s})$, i.e. the rational numbers viewed as a signed meadow, a definition of ceiling and floor cannot be given. To this end, we first prove a general property of unary functions definable in $\mathbb{Q}_0(\mathbf{s})$.

THEOREM 6.1. *For any function $h(x)$ definable in $\mathbb{Q}_0(\mathbf{s})$, there exist $r \in T(\mathbb{Q}_0)$ and a function $g(x)$ definable in $\mathbb{Q}_0[x]$ such that*

$$x > r \implies h(x) = g(x).$$

Proof. By structural induction on the form that $h(x)$ may take.

If $h(x) \in \{0, 1, x\}$, we are done. For $h(x) = -f(x)$ or $h(x) = 1/f(x)$ or $h(x) = f_1(x) + f_2(x)$ or $h(x) = f_1(x) \cdot f_2(x)$, the result also follows immediately (in the latter cases take $r = \max(r_1, r_2)$ for r_i satisfying the property for $f_i(x)$).

In the remaining case, $h(x) = s(f(x))$. Let $g(x) \in T(\mathbb{Q}_0[x])$ be such that $f(x) = g(x)$ for $x > r$. By induction on the form that $g(x)$ may take, it follows that an r' exists such that, for $x > r'$, $s(g(x))$ is constant. This proves that, for $x > \max(r, r')$, $h(x) = s(f(x)) = s(g(x))$ is constant. \square

COROLLARY 6.2. *The floor function $\lfloor x \rfloor$ is not definable in $\mathbb{Q}_0(\mathbf{s})$.*

Proof. Consider

$$h(x) = \frac{x - \lfloor x \rfloor}{x - \lfloor x \rfloor}.$$

If $h(x)$ were definable in $\mathbb{Q}_0(\mathbf{s})$, then, by the preceding result, there exist r and a function $g(x)$ definable in $\mathbb{Q}_0[x]$ such that $h(x) = g(x)$ for $x > r$. But then $g(x) \equiv_{\infty} 0$ (for all integers above r) and $g(x) \equiv_{\infty} 1$ (for all non-integers above r), and this contradicts Theorem 5.1. \square

We finally note that for each term $t(x)$ one can add an induction rule:

$$\frac{t(0) = 0, \quad 0_{1-s(x)} \cdot 0_{t(\lfloor x \rfloor)} \cdot t(\lfloor x \rfloor + 1) = 0, \quad 0_{1+s(x)} \cdot 0_{t(\lceil x \rceil)} \cdot t(\lceil x \rceil - 1) = 0}{t(\lfloor x \rfloor) = 0, \quad t(\lceil x \rceil) = 0}$$

thus

$$\frac{t(0) = 0, \quad (x > 0 \ \& \ t(\lfloor x \rfloor) = 0) \implies t(\lfloor x \rfloor + 1) = 0, \quad (x < 0 \ \& \ t(\lceil x \rceil) = 0) \implies t(\lceil x \rceil - 1) = 0}{t(\lfloor x \rfloor) = 0, \quad t(\lceil x \rceil) = 0}.$$

With this particular induction rule, the idempotency of $\lfloor x \rfloor$ can be easily proved (take $t(x) = x - \lfloor x \rfloor$), as well as the idempotency of ceiling. With a little more effort one can prove $\lfloor x - \lfloor x \rfloor \rfloor = 0$: first prove $\lfloor -\lfloor x \rfloor \rfloor = -\lfloor x \rfloor$ by induction on x , and then $\lfloor x + \lfloor y \rfloor \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ by induction on y . As a consequence, $\lfloor x - \lfloor x \rfloor \rfloor = \lfloor x \rfloor + \lfloor -\lfloor x \rfloor \rfloor = \lfloor x \rfloor + -\lfloor x \rfloor = 0$. In general, if using IL the premises can be proved (from some extension of Md that satisfies the propagation properties), then this can also be proved without IL, and therefore this also is the case for the conclusion.

TABLE 5. The set *SquareRoots* of axioms for the square root.

$$\sqrt{x^{-1}} = (\sqrt{x})^{-1} \quad (38)$$

$$\sqrt{x \cdot y} = \sqrt{x} \cdot \sqrt{y} \quad (39)$$

$$\sqrt{x \cdot x \cdot \mathbf{s}(x)} = x \quad (40)$$

$$\mathbf{s}(\sqrt{x} - \sqrt{y}) = \mathbf{s}(x - y) \quad (41)$$

6.2. Signed meadows with square root

A plausible way to totalize the square root operation is to postulate $\sqrt{-1} = i$ and to abandon the domain of signed fields in favor of the complex numbers. Here we choose a different approach by stipulating $\sqrt{x} = -\sqrt{-x}$ for $x < 0$. In order to avoid confusion with the principal square root function, we deviate from the standard notation and introduce the unary operation $\sqrt{-}$ called *signed square root*. We write Σ_{mss} for this extended signature, so $\Sigma_{\text{mss}} = (0, 1, +, \cdot, -,^{-1}, \mathbf{s}, \sqrt{-})$, and define the signed square root operation in an equational manner by the set *SquareRoots* of axioms given in Table 5.

Some additional consequences of the $\text{Md} \cup \text{Signs} \cup \text{SquareRoots}$ axioms are the following:

$$\sqrt{\mathbf{s}(x)} = \mathbf{s}(x), \quad (42)$$

$$\sqrt{1_x} = 1_x, \quad (43)$$

$$\sqrt{0_x} = 0_x, \quad (44)$$

$$\sqrt{-x} = -\sqrt{x}, \quad (45)$$

$$\sqrt{x^2} = x \cdot \mathbf{s}(x). \quad (46)$$

Here identity (42) follows from

$$\begin{aligned} \sqrt{\mathbf{s}(x)} &= \sqrt{\mathbf{s}(x x x^{-1})} \\ &= \sqrt{\mathbf{s}(x) \mathbf{s}(x) \mathbf{s}(x^{-1})} \\ &= \sqrt{\mathbf{s}(x) \mathbf{s}(x) \mathbf{s}(x)} \\ &= \sqrt{\mathbf{s}(x) \mathbf{s}(x) \mathbf{s}(\mathbf{s}(x))} = \mathbf{s}(x), \end{aligned}$$

and identity (43) from $\sqrt{1_x} = \sqrt{\mathbf{s}(1_x)} = \mathbf{s}(1_x) = 1_x$; identity (44) is proved similarly. Identity (45) follows from

$$\begin{aligned} \sqrt{-x} &= \sqrt{-1 \cdot x} = \sqrt{-1} \cdot \sqrt{x} = \sqrt{\mathbf{s}(-1)} \cdot \sqrt{x} \\ &= \mathbf{s}(-1) \cdot \sqrt{x} = -1 \cdot \sqrt{x} = -\sqrt{x}, \end{aligned}$$

and (46) from

$$\begin{aligned} \sqrt{x^2} &= \sqrt{x^2 \cdot 1_x} = \sqrt{x^2} \cdot 1_x = \sqrt{x^2} \cdot \mathbf{s}(1_x) \\ &= \sqrt{x^2} \cdot \mathbf{s}(x)^2 = \sqrt{x^2} \cdot \sqrt{\mathbf{s}(x)} \cdot \mathbf{s}(x) \\ &= \sqrt{x^2 \mathbf{s}(x)} \cdot \mathbf{s}(x) = x \cdot \mathbf{s}(x). \end{aligned}$$

Since $(\Sigma_{\text{mss}}, \text{Md} \cup \text{Signs} \cup \text{SquareRoots})$ satisfies both propagation properties, we can apply Theorem 3.1.

TABLE 6. The signed square root for differential meadows.

$$\frac{\partial}{\partial X_i} \mathbf{s}(y) = 0 \quad (47)$$

$$\frac{\partial}{\partial X_i} \sqrt{y} = \frac{\mathbf{s}(y)}{2} (\sqrt{y})^{-1} \cdot \frac{\partial}{\partial X_i} y \quad (48)$$

COROLLARY 6.3. *The set of axioms $\text{Md} \cup \text{Signs} \cup \text{SquareRoots}$ is a complete axiomatization of $\text{Mod}_{\Sigma_{\text{mss}}}(\text{Md} \cup \text{Signs} \cup \text{SquareRoots} \cup \text{IL})$.*

Proof. We have to prove that the propagation properties for pseudo units and pseudo zeros hold in $\text{Md} \cup \text{Signs} \cup \text{SquareRoots}$. This follows easily by a case distinction on the forms that $C[r]$ may take. As an example we consider here the case $C[_] \equiv \sqrt{-}$. Then

$$1_t \cdot \sqrt{r} = 1_t^2 \cdot \sqrt{r} = 1_t \cdot \sqrt{1_t} \cdot \sqrt{r} = 1_t \cdot \sqrt{1_t \cdot r}$$

by (1) and (43). The propagation property for pseudo zeros is proved in a similar way applying (2) and (44). \square

We denote by $\mathbb{Q}_0(\mathbf{s}, \sqrt{-})$ the zero-totalized signed prime field that contains \mathbb{Q} and is closed under $\sqrt{-}$. Note that $\mathbb{Q}_0(\mathbf{s}, \sqrt{-})$ is a computable data type (see, e.g. Bergstra and Tucker [21]). This statement still requires an efficient and readable proof.

Finally, differential meadows can be equipped with a signed square root operator by the axioms given in Table 6. Axiom (48) can actually be derived from Axiom (47) and the equational axiomatization of differential meadows as follows:

$$\begin{aligned} 2 \cdot \sqrt{y} \cdot \frac{\partial}{\partial X_i} (\sqrt{y}) &= \sqrt{y} \cdot \frac{\partial}{\partial X_i} (\sqrt{y}) + \sqrt{y} \cdot \frac{\partial}{\partial X_i} (\sqrt{y}) \\ &\stackrel{(11)}{=} \frac{\partial}{\partial X_i} (\sqrt{y} \cdot \sqrt{y}) \\ &\stackrel{(39)}{=} \frac{\partial}{\partial X_i} (\sqrt{y^2}) \\ &\stackrel{(46)}{=} \frac{\partial}{\partial X_i} (y \cdot \mathbf{s}(y)) \\ &\stackrel{(11)}{=} \mathbf{s}(y) \cdot \frac{\partial}{\partial X_i} (y) + y \cdot \frac{\partial}{\partial X_i} (\mathbf{s}(y)) \\ &\stackrel{(47)}{=} \mathbf{s}(y) \cdot \frac{\partial}{\partial X_i} (y). \end{aligned}$$

Moreover, by identity (43), $1_y = 1_{\sqrt{y}}$, and thus

$$\sqrt{y} = \sqrt{1_y \cdot y} = 1_y \cdot \sqrt{y}.$$

Hence

$$\begin{aligned} \frac{\partial}{\partial X_i} (\sqrt{y}) &= \frac{\partial}{\partial X_i} (1_y \cdot \sqrt{y}) \\ &\stackrel{(11)}{=} \sqrt{y} \cdot \frac{\partial}{\partial X_i} (1_y) + 1_y \cdot \frac{\partial}{\partial X_i} (\sqrt{y}) \\ &\stackrel{(12)}{=} 1_y \cdot \frac{\partial}{\partial X_i} (\sqrt{y}) \end{aligned}$$

$$\begin{aligned} &\stackrel{(43)}{=} 1_{\sqrt{y}} \cdot \frac{\partial}{\partial X_i}(\sqrt{y}) \\ &= \frac{s(y)}{2}(\sqrt{y})^{-1} \cdot \frac{\partial}{\partial X_i}y. \end{aligned}$$

So, the existence of non-trivial differential cancellation meadows with signed square roots depends heavily on the existence of an appropriate interpretation of the sign function.

7. CONCLUSIONS

The main result of this paper is a generic basis theorem for cancellation meadows. We have applied this result to various expansions of meadows. The first expansion concerns differential fields. It appears that the interaction between differential operators and equations for meadows is entirely unproblematic. The propagation properties follow immediately from well-known axioms for differential fields.

As stated before, most uses of rational numbers in computer science exploit their ordering. We include this ordering by extending the initial algebraic specification of \mathbb{Q}_0 with an equational specification of the sign function, resulting in a finite basis for what we called $\mathbb{Q}_0(\mathbf{s})$ and we provided a non-trivial proof of the idempotency of the sign function in $\mathbb{Q}_0(\mathbf{s})$. However, the question whether our particular axioms for $\mathbf{s}(x)$ are independent is left open.

As a further example we added the floor function $\lfloor x \rfloor$, the ceiling function $\lceil x \rceil$ and the signed square root to $\mathbb{Q}_0(\mathbf{s})$ and showed that the resulting equational specification is a finite basis. Again, we did not investigate the independency of these axioms.

In [21], it is shown that computable algebras can be specified by means of a complete term rewrite system, provided auxiliary functions can be used. Useful candidates for auxiliary operators in the case of rational numbers can be found in Moss [22] and Calkin and Wilf [23]. In [8], the existence of an equational specification of \mathbb{Q}_0 which is confluent and terminating as a rewrite system has been formulated as an open question. To that question we now add the corresponding question in the presence of the sign operator.

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