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Normal forms for $\mathrm{FSCL}^{\mathsf{U}}$, three-valued Free Short-Circuit Logic

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Abstract

Short-circuit evaluation only evaluates the second argument of an expression if the first argument is sufficient to determine the value of an expression. In free short-circuit logic (FSCL), the evaluation of atoms can cause side effects and atoms' evaluation will not be memorised. An atom that is evaluated multiple times in a compound statement could have different truth values. In three-valued FSCL an evaluation can be True, False and Undefined. This thesis provides us with the normal forms, and the normalisation function that can potentially be used to prove the completeness for three-valued free short-circuit with Undefined.

Contents

CHAPTER 1

Introduction

Short-circuit logic is a variation of logic that evaluates the second argument of an expression if and only if the first argument is not enough to determine the value of an expression. To differentiate between a short-circuit conjunction or disjunction and a not short-circuit conjunction or disjunction, programming languages such as C and Java use different notations for short-circuit logic expressions. C has the short-circuit conjunction && , the bitwise conjunction & , the short-circuit disjunction || and the bitwise disjunction |. It is important to differentiate between short-circuit and bitwise in computer science, because atoms can have side effects. For example in C the Listing 1.1 will print "a is 0 , b is 1" where the code in Listing 1.2 will print "a is 0 , b is 0".

Listing 1.1: bitwise disjunction

```
int a = 0;
int b = 0;
if(a==0 | (b+=1)){}

printf("a is %d ,",a);
printf("b is %d ",b);
```

Listing 1.2: short-circuit disjunction

```
int a = 0;
int b = 0;
if(a==0 || (b+=1)){}

printf("a is %d ,",a);
printf("b is %d ",b);
```

In Listing 1.2 $a == 0$ is true so $b{+}{=}\ 1$ will not have any influence on the evaluation result of the expression and is for that reason not evaluated. In this paper, we represent short-circuit conjunction with the symbol ${}^{\circ}\!\wedge$ and short-circuit disjuntion with the symbol ${}^{\circ}\!\vee$. This will follow notation set by [1].

In FSCL atomic side effects that happen during the evaluation of atoms will be taken into account [3]. Two statements in FSCL are only equivalent if they have always the same evaluation result. In FSCL $a \,{}^{\circ}\!\vee\, a$ and a are not equivalent because it is possible that the evaluation results differ. For example, if $x = 0$ and $a = (x{+} = 1 \,{}^{\circ}\!\wedge\, x == 2)$ the second a of $a \,{}^{\circ}\!\vee\, a$ evaluates as T (True) due to the increment. Hence, the statement $a \,{}^{\circ}\!\vee\, a$ will be evaluated as T, while the statement a will be evaluated as F (False).

In FSCL atoms can only evaluate as T or F. Hence, an expression such as 1.1 is not possible in FSCL. In the case of $y = 0$ this statement is neither T, nor F because division by zero is undefined.

$$(x/y = 5) \tag{1.1}$$

For this expression to evaluate only as T or F this expression needs to be rewritten to $y = 0 \triangledown (x/y = 5)$ [4]. When y is equal to zero, the evaluation of the left side of the disjunct is T, so the right side of the disjunct will not be evaluated. In FSCL, expressions such as $\mathsf{T} \triangledown x$ can be rewritten to T by the use of the axioms from EqFSCL, which we will explain later. These axioms have been proven complete for closed terms [7].

In this thesis project, we will talk about FSCL^{U}, three-valued FSCL with U, where the evaluation of expressions can be T, F and U (undefined) and provide normal forms for FSCL^{U} and the normalisation function. These normal forms can potentially be used to prove the completeness of $\text{EqFSCL}^{\mathsf{U}}$ (the axioms of FSCL^{U}) for closed terms because for the completeness proof of EqFSCL normal forms have been used. The normal forms for EqFSCL are not sufficient for $\text{EqFSCL}^{\mathsf{U}}$, because trees with U leaves can not be represented by these normal forms.

1.1 Outline

The structure of this thesis is as follows: in Chapter 2, we will summarize the definition of evaluation trees for short-circuit logic with undefinedness given by [2] and [7]. In Chapter 3, we will define the normal forms, and normalisation function for FSCL^{U} will be defined. This normalisation function will be defined and proven correct with the use of auxiliary functions. In Appendix A we will prove the correctness of this normalisation function by using these auxiliary functions. We will end this thesis by discussing the use of these normal forms for a completeness proof.

This thesis extends [7], so considerable parts of the text in the approaching sections stem from [7].

1.2 Ethical considerations

This paper aims to contribute to proving the completeness of $\text{EqFSCL}^{\mathsf{U}}$, by defining the normal forms that could, potentially, be used in the proof. Its scope falls entirely within theoretical computer science, the formal foundation of computer science based on logic and mathematics. This research is therefore not directly linked to the real world and it is difficult to know what kind of repercussions (if any) this line of work could have.

What it could, and hopefully will do is to contribute to better education and better understanding of the logic that underlies computer programs.

CHAPTER 2

Evaluation trees and axioms, for short-circuit logic with undefinedness

This chapter will discuss the relevant work and the theoretical background needed to define the normal forms, by first introducing the normal forms used in the completeness proof of EqFSCL and finally extending the axioms of EqFSCL to $\text{EqFSCL}^{\mathsf{U}}$.

2.1 Axioms and normal forms FSCL

To define correct normal forms and prove the correctness of the normalisation function, we can build on normal form, and normalisation function from *An independent axiomatisation for free short-circuit logic*[7]. In this section, we will discuss the given normal forms and axiomatisation of FSCL.

$$\mathsf{F} = \neg\mathsf{T} \tag{F1}$$

$$x \mathbin{{}^{\circ}\!\vee} y = \neg(\neg x \mathbin{{}_{\circ}\!\wedge} \neg y) \tag{F2}$$

$$\neg\neg x = x \tag{F3}$$

$$\mathsf{T} \mathbin{{}_{\circ}\!\wedge} x = x \tag{F4}$$

$$x \mathbin{{}^{\circ}\!\vee} \mathsf{F} = x \tag{F5}$$

$$\mathsf{F} \mathbin{{}_{\circ}\!\wedge} x = \mathsf{F} \tag{F6}$$

$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}_{\circ}\!\wedge} z = x \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}_{\circ}\!\wedge} z) \tag{F7}$$

$$\neg x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} = x \mathbin{{}_{\circ}\!\wedge} \mathsf{F} \tag{F8}$$

$$(x \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) \mathbin{{}^{\circ}\!\vee} y = (x \mathbin{{}^{\circ}\!\vee} \mathsf{T}) \mathbin{{}_{\circ}\!\wedge} y \tag{F9}$$

$$(x \mathbin{{}_{\circ}\!\wedge} y) \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F}) = (x \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \mathbin{{}_{\circ}\!\wedge} (y \mathbin{{}^{\circ}\!\vee} (z \mathbin{{}_{\circ}\!\wedge} \mathsf{F})) \tag{F10}$$

Table 2.1: EqFSCL, a set of axioms for FSCL

$\mathcal{S}_A$ is a defined set of closed (sequential) propositional statements over A by the following grammar:

$$P ::= \mathrm{a}|\mathsf{T}|\mathsf{F}|\neg P|P \mathbin{{}_{\circ}\!\wedge} P|P \mathbin{{}^{\circ}\!\vee} P$$

where $\mathrm{a} \in \mathrm{A}$. T is a constant for true, F for false and $\neg$ is negation.

The unary short-circuit evaluation function (se) is used to define $\mathcal{S}_A$ functions as evaluation trees ($\mathcal{T}_A$). For the proof for the completeness of EqFSCL for closed terms, normal forms and a normalisation function have been introduced. The normalisation function is used to rewrite $\mathcal{S}_A$ to SNF. Normal forms for FSCL are:

Definition 2.1.1. *A term $P \in \mathcal{S}_A$ is said to be in* **SCL** ***(short-circuit logic) Normal Form*** *(**SNF**) if it is generated by the following grammar:*

$$
\begin{aligned}
P &::= P^{\mathsf{T}} \mid P^{\mathsf{F}} \mid P^{\mathsf{T}} \leftand P^* && (\textit{SNF-terms})\\
P^{\mathsf{T}} &::= \mathsf{T} \mid (a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{T}} && (\mathsf{T}\textit{-terms})\\
P^{\mathsf{F}} &::= \mathsf{F} \mid (a \leftor P^{\mathsf{F}}) \leftand P^{\mathsf{F}} && (\mathsf{F}\textit{-terms})\\
P^* &::= P^c \mid P^d && (*\textit{-terms})\\
P^c &::= P^\ell \mid P^* \leftand P^d\\
P^d &::= P^\ell \mid P^* \leftor P^c\\
P^\ell &::= (a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{F}} \mid (\neg a \leftand P^{\mathsf{T}}) \leftor P^{\mathsf{F}} && (\ell\textit{-terms})
\end{aligned}
$$

2.2 Axioms and evaluation trees $\mathrm{FSCL}^{\mathsf{U}}$

To extend to normal form for FSCL to the normal forms for $\mathrm{FSCL}^{\mathsf{U}}$ we first need to define $\mathrm{FSCL}^{\mathsf{U}}$ and its axioms. In this section we will discuss the definition given in *Non-commutative propositional logic with short-circuit evaluation* [2] for $\mathrm{FSCL}^{\mathsf{U}}$.

The definition we will use for $\mathrm{FSCL}^{\mathsf{U}}$ is FSCL extended by the axiom CP-U.

$$x \triangleleft \mathsf{U} \triangleright y = \mathsf{U} \qquad \text{(CP-U)}$$

CP-U implies $\mathsf{U} \leftand x = \mathsf{U} \leftor x = \neg\mathsf{U} = \mathsf{U}$, and also $\mathsf{F} \leftand \mathsf{U} = \mathsf{F}$. In CP-U Hoare's conditional is used [5]. Notation using Hoare's conditional should be read from the middle to the corner. For example, the term $x \triangleleft y \triangleright z$ should be read as **"if** y **then** x **else** z**"**. This implies that first y is evaluated, in case of $y = \mathsf{T}$ x is evaluated and in case of $y = \mathsf{F}$ z is evaluated.

Definition 2.2.1. *The set of $\mathcal{T}_A^{\mathsf{U}}$ of* U***-evaluation trees*** *over A with leaves in* {T,F,U} *is defined inductively by*

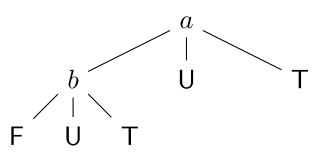
$$\mathsf{T} \in \mathcal{T}_A^{\mathsf{U}}, F \in \mathcal{T}_A^{\mathsf{U}}, \mathsf{U} \in \mathcal{T}_A^{\mathsf{U}}, (X \trianglelefteq \underline{a} \trianglerighteq Y) \in \mathcal{T}_A^{\mathsf{U}} \text{ for any } X, Y \in \mathcal{T}_A^{\mathsf{U}} \text{ and } a \in A$$

The operator $\_ \trianglelefteq \underline{a} \trianglerighteq \_$ is called U***-tree composition over*** *a. In the evaluation tree $X \trianglelefteq \underline{a} \trianglerighteq Y$, the root represented by a, the left branch by X, the right branch by Y and the underlining of the a represents a middle branch to the leaf* U*.*

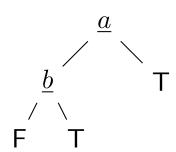
A pictorial representation is an alternative for a formal notation of evaluation trees. For example, the tree

$$(\mathsf{F} \trianglelefteq \underline{b} \trianglerighteq \mathsf{T}) \trianglelefteq \underline{a} \trianglerighteq \mathsf{T}$$

can be represented as follows, where $\trianglelefteq$ gives a left branch and $\trianglerighteq$ a right branch:



Replacing the middle branches of these nodes with an underlined nodes is an alternative representation of U-evaluation trees and the representation of U-evaluation trees that will be used in this thesis. The tree above can also be represented as follows:



The set S_A can be extended to ${\mathcal{S}_A}^U$ of closed (sequential) propositional statements over A with U using the following grammar:

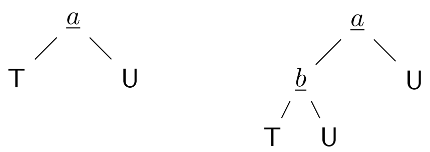
$$P ::= \text{ a } \mid \mathsf{T} | \mathsf{F} | \mathsf{U} | \neg P | P \mathbin{{}^{\circ}\!\!\wedge} P | P \mathbin{{}^{\circ}\!\!\vee} P$$

where a $\in$ A. T is a constant for true, F for false, U for undefined and $\neg$ is negation. The signature of ${\mathcal{S}_A}^U$ is $\sum_{SCL}(A) = \{\mathbin{{}^{\circ}\!\!\wedge}, \mathbin{{}^{\circ}\!\!\vee}, \neg, \mathsf{T}, \mathsf{F}, \mathsf{U}, a \mid a \in A\}$.

Definition 2.2.2. *The unary* ***short-circuit evaluation function*** se^u*:* ${\mathcal{S}_A}^U \to \mathcal{T}_A^{\mathsf{U}}$ *is defined as follows, where* $a \in A$*:*

$$\begin{aligned}
se^u(\mathsf{T}) &= \mathsf{T} & se^u(\neg P) &= se^u(P)[\mathsf{T} \mapsto \mathsf{F}, \mathsf{F} \mapsto \mathsf{T}] \\
se^u(\mathsf{F}) &= \mathsf{F} & se^u(P \mathbin{{}^{\circ}\!\!\wedge} Q) &= se^u(P)[\mathsf{T} \mapsto se^u(Q)] \\
se^u(\mathsf{U}) &= \mathsf{U} & se^u(P \mathbin{{}^{\circ}\!\!\vee} Q) &= se^u(P)[\mathsf{F} \mapsto se^u(Q)] \\
se^u(a) &= \mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{F} & &
\end{aligned}$$

Examples: $se^u(P \mathbin{{}^{\circ}\!\!\vee} Q)$ where $P = a$ and $Q = \mathsf{U}$. This can be rewritten to $se^u(a)[\mathsf{F} \mapsto se^u(\mathsf{U})] = \mathsf{T} \trianglelefteq \underline{a} \trianglerighteq \mathsf{U}$ and $se^u((a \mathbin{{}^{\circ}\!\!\wedge} b) \mathbin{{}^{\circ}\!\!\vee} \mathsf{U}) = (\mathsf{T} \trianglelefteq \underline{b} \trianglerighteq \mathsf{U}) \trianglelefteq \underline{a} \trianglerighteq \mathsf{U}$ the pictural representation for this is:



For duality of EqFSCL$^{\mathsf{U}}$ we cite [2, p.22]: *"Defining* $\mathsf{U}^{dl} = \mathsf{U}$ *implies that EqMSCL*$^{\mathsf{U}}$ *satisfies the duality principle"*. This also implies that EqFSCL$^{\mathsf{U}}$ satisfies the duality principle.

2.3 Axioms, for short-circuit logic with undefinedness

In this thesis we have chosen to extend EqFSCL by three axioms:

$$\neg \mathsf{U} = \mathsf{U} \tag{FU1}$$
$$\mathsf{U} \mathbin{\circ\!\!\wedge} x = \mathsf{U} \tag{FU2}$$
$$\mathsf{U} \mathbin{\circ\!\!\vee} x = \mathsf{U} \tag{FU3}$$

Table 2.2: Additional axioms for EqFSCL

While CP-U also implies $\mathsf{F} \mathbin{\circ\!\!\wedge} \mathsf{U} = \mathsf{F}$ we have chosen not to extend the axioms with this axiom, because (F6) already implies that $\mathsf{F} \mathbin{\circ\!\!\wedge} x = \mathsf{F}$. It is also possible to rewrite the additional axioms to the single axiom $\mathsf{U} \mathbin{\circ\!\!\wedge} x = \neg\mathsf{U}$. We will use (FU1), (FU2) and (FU3) a number of times in this thesis, for that reason we have chosen to include all three of these axioms. Because we have chosen to include these three axioms instead of one the extra axioms will not be independent. Together with the axioms of EqFSCL these axioms form the axioms for $\mathrm{FSCL}^{\mathsf{U}}$:

$$\mathsf{F} = \neg\mathsf{T} \tag{F1}$$
$$x \mathbin{\circ\!\!\vee} y = \neg(\neg x \mathbin{\circ\!\!\wedge} \neg y) \tag{F2}$$
$$\neg\neg x = x \tag{F3}$$
$$\mathsf{T} \mathbin{\circ\!\!\wedge} x = x \tag{F4}$$
$$x \mathbin{\circ\!\!\vee} \mathsf{F} = x \tag{F5}$$
$$\mathsf{F} \mathbin{\circ\!\!\wedge} x = \mathsf{F} \tag{F6}$$
$$(x \mathbin{\circ\!\!\wedge} y) \mathbin{\circ\!\!\wedge} z = x \mathbin{\circ\!\!\wedge} (y \mathbin{\circ\!\!\wedge} z) \tag{F7}$$
$$\neg x \mathbin{\circ\!\!\wedge} \mathsf{F} = x \mathbin{\circ\!\!\wedge} \mathsf{F} \tag{F8}$$
$$(x \mathbin{\circ\!\!\wedge} \mathsf{F}) \mathbin{\circ\!\!\vee} y = (x \mathbin{\circ\!\!\vee} \mathsf{T}) \mathbin{\circ\!\!\wedge} y \tag{F9}$$
$$(x \mathbin{\circ\!\!\wedge} y) \mathbin{\circ\!\!\vee} (z \mathbin{\circ\!\!\wedge} \mathsf{F}) = (x \mathbin{\circ\!\!\vee} (z \mathbin{\circ\!\!\wedge} \mathsf{F})) \mathbin{\circ\!\!\wedge} (y \mathbin{\circ\!\!\vee} (z \mathbin{\circ\!\!\wedge} \mathsf{F})) \tag{F10}$$

$$\neg \mathsf{U} = \mathsf{U} \tag{FU1}$$
$$\mathsf{U} \mathbin{\circ\!\!\wedge} x = \mathsf{U} \tag{FU2}$$
$$\mathsf{U} \mathbin{\circ\!\!\vee} x = \mathsf{U} \tag{FU3}$$

Table 2.3: $\mathrm{EqFSCL}^{\mathsf{U}}$, a set of axioms for $\mathrm{FSCL}^{\mathsf{U}}$

CHAPTER 3

Normal forms for three-valued logic

In this chapter, we define the SCL Normal Forms and the normalisation function. The normalisation function will be defined with the use of auxiliary functions.

3.1 Normal forms

When considering a tree from ${\mathcal{S}_A}^U$, we note that some trees only have T-leaves, some only F-leaves, some only U-leaves and some a combination of these.

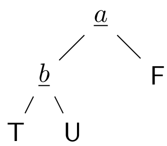
Definition 3.1.1. *A term $P \in {\mathcal{S}_A}^U$ is said to be in* **SCL *Normal Form* (*SNF*)** *if it is generated by the following grammar:*

$$
\begin{array}{rll}
P & ::= P^{\mathsf{T}} \ \mid \ P^{\mathsf{F}} \ \mid P^{\mathsf{U}} \ \mid P^{\mathsf{T}} \mathbin{{}^{\circ}\!\!\wedge} P^{a} & (\textit{SNF-terms}) \\
P^{\mathsf{T}} & ::= \mathsf{T} \ \mid \ (a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{T}} & (\mathsf{T}\textit{-terms}) \\
P^{\mathsf{F}} & ::= \mathsf{F} \ \mid \ (a \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{F}}) \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{F}} & (\mathsf{F}\textit{-terms}) \\
P^{\mathsf{U}} & ::= \mathsf{U} \ \mid \ (a \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{U}}) \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{U}} & (\mathsf{U}\textit{-terms}) \\
P^{a} & ::= P^{*} \ \mid \ P^{n*} \ \mid \ P^{w*} \ \mid \ P^{b*} & (a\textit{-terms}) \\
P^{*} & ::= P^{c} \ \mid \ P^{d} & (*\textit{-terms}) \\
P^{c} & ::= P^{\ell} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\wedge} \ P^{d} & \\
P^{d} & ::= P^{\ell} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\vee} P^{c} & \\
P^{\ell} & ::= (a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{F}} \ \mid \ (\neg a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{F}} & (\ell\textit{-terms}) \\
P^{n*} & ::= P^{n\ell} \ \mid \ P^{w*} \mathbin{{}^{\circ}\!\!\wedge} P^{n*} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\wedge} P^{n*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\wedge} P^{n*} & (n*\textit{-terms}) \\
P^{n\ell} & ::= (a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{U}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{F}} \ \mid \ (\neg a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{U}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{F}} & (n\ell\textit{-terms}) \\
P^{w*} & ::= P^{w\ell} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\vee} P^{w*} \ \mid \ P^{n*} \mathbin{{}^{\circ}\!\!\vee} P^{w*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\vee} P^{w*} & (w*\textit{-terms}) \\
P^{w\ell} & ::= (a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{U}} \ \mid \ (\neg a \mathbin{{}^{\circ}\!\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\!\vee} P^{\mathsf{U}} & (w\ell\textit{-terms}) \\
P^{b*} & ::= P^{b\ell} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\wedge} P^{w*} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\wedge} P^{b*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\wedge} P^{*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\wedge} P^{b*} \ \mid & (b*\textit{-terms}) \\
 & \quad P^{w*} \mathbin{{}^{\circ}\!\!\wedge} P^{*} \ \mid \ P^{w*} \mathbin{{}^{\circ}\!\!\wedge} P^{b*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\wedge} P^{w*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\vee} P^{*} \ \mid & \\
 & \quad P^{*} \mathbin{{}^{\circ}\!\!\vee} P^{n*} \ \mid \ P^{*} \mathbin{{}^{\circ}\!\!\vee} P^{b*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\vee} P^{n*} \ \mid \ P^{b*} \mathbin{{}^{\circ}\!\!\vee} P^{b*} & \\
P^{b\ell} & ::= P^{\ell} \mathbin{{}^{\circ}\!\!\wedge} P^{w\ell} \ \mid \ P^{w\ell} \mathbin{{}^{\circ}\!\!\wedge} P^{\ell} \ \mid \ P^{\ell} \mathbin{{}^{\circ}\!\!\vee} P^{n\ell} & (b\ell\textit{-terms})
\end{array}
$$

where $a \in A$. We refer to P^{T}-forms as T-terms, to P^{F}-forms as F-terms, to P^{U}-forms as U-terms, to P^{ℓ}-forms as ℓ-terms (the name refers to literal terms), to P^{*}-forms as $*$-terms, to $P^{n\ell}$-forms as $n\ell$-terms, to P^{n*} as $n*$-terms, to $P^{w\ell}$-forms as $w\ell$-terms, to P^{w*} as $w*$-terms, *to $P^{b\ell}$-forms as $b\ell$-terms, to P^{b*} as $b*$-terms and to P^{a}-forms as a-terms. A term of the form $P^{\mathsf{T}} \leftand P^{a}$ is referred to as a T-a-term. A term of the form $P^{\mathsf{T}} \leftand P^{*}$ is referred to as a T-$*$-term. A term of the form $P^{\mathsf{T}} \leftand P^{n*}$ is referred to as a T-$n*$-term. A term of the form $P^{\mathsf{T}} \leftand P^{w*}$ is referred to as a T-$w*$-term. A term of the form $P^{\mathsf{T}} \leftand P^{b*}$ is referred to as a T-$b*$-term.*

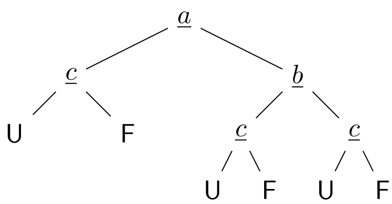
An ℓ-term has one node (its root) that has paths to both T and F. An $n\ell$-term has one node (its root) that has paths to both U and F. A $w\ell$-term has one node (its root) that has paths to both T and U. An $b\ell$-term has one node (its root) that has paths to an $n\ell$-term and an ℓ-term, $w\ell$-term and an ℓ-term, or an $n\ell$-term and a $w\ell$-term.

The term $a \leftand (b \leftor \mathsf{U})$ has the se^{u}-image:



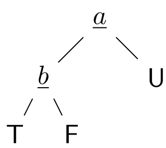
This se^{u}-image can evaluate as T, F or U, and can only be represented by one term from the normal forms: the $b\ell$-term $P^{\ell} \leftand Q^{w\ell}$, this term can be written as $((a \leftand \mathsf{T}) \leftor \mathsf{F}) \leftand ((b \leftand \mathsf{T}) \leftor \mathsf{U})$.

The term $(a \leftor (b \leftor \mathsf{T})) \leftand (c \leftand \mathsf{U})$ has the se^{u}-image:



This se^{u}-image is represented by a T-$n*$-term, the term $((a \leftand \mathsf{T}) \leftor ((b \leftand \mathsf{T}) \leftor \mathsf{T})) \leftand ((c \leftand \mathsf{U}) \leftor \mathsf{F})$. The atoms a and b are not relevant for the evaluation results, while their side effects can be relevant for subsequent atomic evaluations. For this reason we need the T's in a T-$n*$-term. For T-$*$-term, T-$b*$-term and T-$w*$-term we use T's for the same reasons.

The term $(a \leftor \mathsf{U}) \leftand b$ where the left side of the conjunct can evaluate as T and U and the right side of the conjunct can evaluate as T and F has the se^{u}-image:



We chose to represent this se^{u}-image by the $b\ell$-term $P^{w\ell} \leftand P^{\ell}$, which can be written as $((a \leftand \mathsf{T}) \leftor \mathsf{U}) \leftand ((b \leftand \mathsf{T}) \leftor \mathsf{F})$, while it is also possible to represent this se^{u}-image by the term $P^{n\ell} \leftor P^{\ell}$, which can be written as $((\neg a \leftand \mathsf{U}) \leftor \mathsf{F}) \leftor ((b \leftand \mathsf{T}) \leftor \mathsf{F})$, because in [7] the T-terms and conjunction for T-$*$-terms are used instead of F-terms and disjunction.

3.2 Defining normal forms using recursive functions

In this section, we will define the normalisation function f and the auxiliary functions to define the normalisation function recursively. We will use these auxiliary functions to prove the correctness of the normalisation function.

$$f : {\mathcal{S}_A}^U \to \mathit{SNF}.$$

f is defined recursively using the functions

$$f^n : \mathit{SNF} \to \mathit{SNF} \quad \text{and} \quad f^c : \mathit{SNF} \times \mathit{SNF} \to \mathit{SNF}.$$

The function f^n will be used to rewrite *SNF*-terms that are negated to *SNF*-terms. The function f^c will be used to rewrite a conjunction of two *SNF*-terms. We have no need for a function dedicated to the disjunction of two *SNF*-terms, because this can be rewritten to a disjunction with (F2) and f^n.

$$f(a) = \mathsf{T} \mathbin{^\circ\!\wedge} ((a \mathbin{^\circ\!\wedge} \mathsf{T}) \mathbin{^\circ\!\vee} \mathsf{F}) \tag{3.1}$$
$$f(\mathsf{T}) = \mathsf{T} \tag{3.2}$$
$$f(\mathsf{F}) = \mathsf{F} \tag{3.3}$$
$$f(\mathsf{U}) = \mathsf{U} \tag{3.4}$$
$$f(\neg P) = f^n(f(P)) \tag{3.5}$$
$$f(P \mathbin{^\circ\!\wedge} Q) = f^c(f(P), f(Q)) \tag{3.6}$$
$$f(P \mathbin{^\circ\!\vee} Q) = f^n(f^c(f^n(f(P)), f^n(f(Q)))). \tag{3.7}$$

3.2.1 Definition of f using f^n

Analysing the behaviour of T-terms and F-terms together with the definition of se^u on negations, it becomes clear that f^n must turn T-terms into F-terms and vice versa. Analysing the behaviour of U-terms with the definition of se^u on negations, it becomes clear that f^n must turn U-terms into U-terms. Any *SNF*-term is in exactly one of the grammatical categories identified in Definition 3.1.1.

$$f^n(\mathsf{T}) = \mathsf{F} \tag{3.8}$$
$$f^n((a \mathbin{^\circ\!\wedge} P^{\mathsf{T}}) \mathbin{^\circ\!\vee} Q^{\mathsf{T}}) = (a \mathbin{^\circ\!\vee} f^n(Q^{\mathsf{T}})) \mathbin{^\circ\!\wedge} f^n(P^{\mathsf{T}}) \tag{3.9}$$

$$f^n(\mathsf{F}) = \mathsf{T} \tag{3.10}$$
$$f^n((a \mathbin{^\circ\!\vee} P^{\mathsf{F}}) \mathbin{^\circ\!\wedge} Q^{\mathsf{F}}) = (a \mathbin{^\circ\!\wedge} f^n(Q^{\mathsf{F}})) \mathbin{^\circ\!\vee} f^n(P^{\mathsf{F}}) \tag{3.11}$$

$$f^n(\mathsf{U}) = \mathsf{U} \tag{3.12}$$
$$f^n((a \mathbin{^\circ\!\vee} P^{\mathsf{U}}) \mathbin{^\circ\!\wedge} Q^{\mathsf{U}}) = (a \mathbin{^\circ\!\vee} P^{\mathsf{U}}) \mathbin{^\circ\!\wedge} Q^{\mathsf{U}} \tag{3.13}$$

To define f^n we have to distinguish between the following cases:

(1) The f^n of $*$-terms
(2) The f^n of $n*$-terms
(3) The f^n of $w*$-terms
(4) The f^n of $b*$-terms

In case (1) we proceed by defining f^n for $*$-terms. We define $f^n : \mathit{SNF} \to \mathit{SNF}$ as follows, using the auxiliary function $f_1^n : P^* \to P^*$ to push in the negation symbols when negating a T-$*$-term.

$$f^n(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^*) = P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} f_1^n(Q^*) \tag{3.14}$$

$$f_1^n((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{F}}) = (\neg a \mathbin{{}_{\circ}\!\wedge} f^n(Q^{\mathsf{F}})) \mathbin{{}^{\circ}\!\vee} f^n(P^{\mathsf{T}}) \tag{3.15}$$

$$f_1^n((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{F}}) = (a \mathbin{{}_{\circ}\!\wedge} f^n(Q^{\mathsf{F}})) \mathbin{{}^{\circ}\!\vee} f^n(P^{\mathsf{T}}) \tag{3.16}$$

$$f_1^n(P^* \mathbin{{}_{\circ}\!\wedge} Q^d) = f_1^n(P^*) \mathbin{{}^{\circ}\!\vee} f_1^n(Q^d) \tag{3.17}$$

$$f_1^n(P^* \mathbin{{}^{\circ}\!\vee} Q^c) = f_1^n(P^*) \mathbin{{}_{\circ}\!\wedge} f_1^n(Q^c). \tag{3.18}$$

In case (2) we proceed by defining f^n for $n*$-terms. We note that the negation on an $n*$-term will provide us with a $w*$-term, because $f^n(\mathsf{U}) = \mathsf{U}$ and $f^n(\mathsf{F}) = \mathsf{T}$. We define $f^n : \mathit{SNF} \to \mathit{SNF}$ as follows, using the auxiliary function $f_2^n : P^{n*} \to P^{w*}$ to push in the negation symbols when negating a T-$n*$-term.

$$f^n(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^{n*}) = P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} f_2^n(Q^{n*}) \tag{3.19}$$

$$f_2^n(P^* \mathbin{{}_{\circ}\!\wedge} Q^{n*}) = f_1^n(P^*) \mathbin{{}^{\circ}\!\vee} f_2^n(Q^{n*}) \tag{3.20}$$

$$f_2^n(P^{w*} \mathbin{{}_{\circ}\!\wedge} Q^{n*}) = f_3^n(P^{w*}) \mathbin{{}^{\circ}\!\vee} f_2^n(Q^{n*}). \tag{3.21}$$

$$f_2^n(P^{b*} \mathbin{{}_{\circ}\!\wedge} Q^{n*}) = f_4^n(P^{b*}) \mathbin{{}^{\circ}\!\vee} f_2^n(Q^{n*}) \tag{3.22}$$

$$f_2^n((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{U}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{F}}) = (\neg a \mathbin{{}_{\circ}\!\wedge} f^n(Q^{\mathsf{F}})) \mathbin{{}^{\circ}\!\vee} P^{\mathsf{U}} \tag{3.23}$$

$$f_2^n((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{U}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{F}}) = (a \mathbin{{}_{\circ}\!\wedge} f^n(Q^{\mathsf{F}})) \mathbin{{}^{\circ}\!\vee} P^{\mathsf{U}} \tag{3.24}$$

In case (3) we proceed by defining f^n for $w*$-terms. We note that the negation on a $w*$-term will provide us with a $n*$-term, because $f^n(\mathsf{U}) = \mathsf{U}$ and $f^n(\mathsf{T}) = \mathsf{F}$. We define $f^n : \mathit{SNF} \to \mathit{SNF}$ as follows, using the auxiliary function $f_3^n : P^{w*} \to P^{n*}$ to push in the negation symbols when negating a T-$w*$-term.

$$f^n(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^{w*}) = P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} f_3^n(Q^{w*}) \tag{3.25}$$

$$f_3^n(P^* \mathbin{{}^{\circ}\!\vee} Q^{w*}) = f_1^n(P^*) \mathbin{{}_{\circ}\!\wedge} f_3^n(Q^{w*}) \tag{3.26}$$

$$f_3^n(P^{n*} \mathbin{{}^{\circ}\!\vee} Q^{w*}) = f_2^n(P^{n*}) \mathbin{{}_{\circ}\!\wedge} f_3^n(Q^{w*}). \tag{3.27}$$

$$f_3^n(P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{w*}) = f_4^n(P^{b*}) \mathbin{{}_{\circ}\!\wedge} f_3^n(Q^{w*}) \tag{3.28}$$

$$f_3^n((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) = (\neg a \mathbin{{}_{\circ}\!\wedge} Q^{\mathsf{U}}) \mathbin{{}^{\circ}\!\vee} f^n(P^{\mathsf{T}}) \tag{3.29}$$

$$f_3^n((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) = (a \mathbin{{}_{\circ}\!\wedge} Q^{\mathsf{U}}) \mathbin{{}^{\circ}\!\vee} f^n(P^{\mathsf{T}}) \tag{3.30}$$

In case (4) we proceed by defining f^n for $b*$-terms. We note that the negation on a $b*$-term will provide us with a $b*$-term. We define $f^n : \mathit{SNF} \to \mathit{SNF}$ as follows, using the auxiliary function $f_4^n : P^{b*} \to P^{b*}$ to push in the negation symbols when negating a T-$b*$-term.

$$
\begin{aligned}
f^n(P^{\mathsf{T}} \mathbin{\circ\!\!\wedge} Q^{b*}) &= P^{\mathsf{T}} \mathbin{\circ\!\!\wedge} f_4^n(Q^{b*}) && (3.31)\\
f_4^n(P^{*} \mathbin{\circ\!\!\wedge} Q^{w*}) &= f_1^n(P^{*}) \mathbin{\circ\!\!\vee} f_3^n(Q^{w*}) && (3.32)\\
f_4^n(P^{*} \mathbin{\circ\!\!\vee} Q^{n*}) &= f_1^n(P^{*}) \mathbin{\circ\!\!\wedge} f_2^n(Q^{n*}) && (3.33)\\
f_4^n(P^{w*} \mathbin{\circ\!\!\wedge} Q^{*}) &= P^{w*} \mathbin{\circ\!\!\wedge} f_1^n(Q^{*}) && (3.34)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\vee} Q^{*}) &= f_4^n(P^{b*}) \mathbin{\circ\!\!\wedge} f_1^n(Q^{*}) && (3.35)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\wedge} Q^{*}) &= f_4^n(P^{b*}) \mathbin{\circ\!\!\vee} f_1^n(Q^{*}) && (3.36)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\vee} Q^{n*}) &= f_4^n(P^{b*}) \mathbin{\circ\!\!\wedge} f_2^n(Q^{n*}) && (3.37)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\wedge} Q^{b*}) &= f_4^n(P^{b*}) \mathbin{\circ\!\!\vee} f_4^n(Q^{b*}) && (3.38)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\vee} Q^{b*}) &= f_4^n(P^{b*}) \mathbin{\circ\!\!\wedge} f_4^n(Q^{b*}) && (3.39)\\
f_4^n(P^{w*} \mathbin{\circ\!\!\wedge} Q^{b*}) &= P^{w*} \mathbin{\circ\!\!\wedge} f_4^n(Q^{b*}) && (3.40)\\
f_4^n(P^{*} \mathbin{\circ\!\!\vee} Q^{b*}) &= f_1^n(P^{*}) \mathbin{\circ\!\!\wedge} f_4^n(Q^{b*}) && (3.41)\\
f_4^n(P^{*} \mathbin{\circ\!\!\wedge} Q^{b*}) &= f_1^n(P^{*}) \mathbin{\circ\!\!\vee} f_4^n(Q^{b*}) && (3.42)\\
f_4^n(P^{b*} \mathbin{\circ\!\!\wedge} Q^{w*}) &= f_2^n(P^{b*}) \mathbin{\circ\!\!\vee} f_3^n(Q^{w*}) && (3.43)
\end{aligned}
$$

3.2.2 Definition of f using f^c

To define f^c we distinguish the following cases:

(1) $f^c(P^{\mathsf{T}}, Q)$
(2) $f^c(P^{\mathsf{F}}, Q)$
(3) $f^c(P^{\mathsf{U}}, Q)$
(4) $f^c(P^{\mathsf{T}} \mathbin{\circ\!\!\wedge} P^{a}, Q)$

For case (1)

$$
\begin{aligned}
f^c(\mathsf{T}, P) &= P && (3.44)\\
f^c((a \mathbin{\circ\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\vee} Q^{\mathsf{T}}, R^{\mathsf{T}}) &= (a \mathbin{\circ\!\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \mathbin{\circ\!\!\vee} f^c(Q^{\mathsf{T}}, R^{\mathsf{T}}) && (3.45)\\
f^c((a \mathbin{\circ\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\vee} Q^{\mathsf{T}}, R^{\mathsf{F}}) &= (a \mathbin{\circ\!\!\vee} f^c(Q^{\mathsf{T}}, R^{\mathsf{F}})) \mathbin{\circ\!\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{F}}) && (3.46)\\
f^c((a \mathbin{\circ\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\vee} Q^{\mathsf{T}}, R^{\mathsf{U}}) &= (a \mathbin{\circ\!\!\vee} f^c(Q^{\mathsf{T}}, R^{\mathsf{U}})) \mathbin{\circ\!\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{U}}) && (3.47)\\
f^c((a \mathbin{\circ\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\vee} Q^{\mathsf{T}}, R^{\mathsf{T}} \mathbin{\circ\!\!\wedge} S^{a}) &= f^c((a \mathbin{\circ\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\vee} Q^{\mathsf{T}}, R^{\mathsf{T}}) \mathbin{\circ\!\!\wedge} S^{a}. && (3.48)
\end{aligned}
$$

For case (2) we make use of (F6). This implies that the conjunction of F with another term yields the evaluation F. For this reason we can write:

$$f^c(P^{\mathsf{F}}, Q) = P^{\mathsf{F}} \tag{3.49}$$

For case (3) the only possible evaluation of the U-term is U. By making use of (FU2) we can write:

$$f^c(P^{\mathsf{U}}, Q) = P^{\mathsf{U}} \tag{3.50}$$

For the remaining case (4) (the first argument is a T-a-term) we distinguish four sub-cases:

(4.1) The a-term is a $*$-term,
(4.2) The a-term is an $n*$-term,
(4.3) The a-term is a $w*$-term,
(4.4) The a-term is a $b*$-term.

For case (4.1) (The a-term is a $*$-term) there are seven possible sub-cases for the second argument. We distinguish these seven sub-cases:

(4.1.a) The second argument is a T-term,

(4.1.b) The second argument is an F-term,
(4.1.c) The second argument is an U-term,
(4.1.d) The second argument is a T-$*$-term,
(4.1.e) The second argument is a T-$n*$-term,
(4.1.f) The second argument is a T-$w*$-term,
(4.1.g) The second argument is a T-$b*$-term

For case (4.1.a) (the second argument is a T-term), we will use an auxiliary function $f^c_{1a} : P^* \times P^{\mathsf{T}} \to P^*$ to turn conjunctions of a $*$-term with a T-term into $*$-terms. We define f^c_{1a} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{1a} are $*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}}) = P^{\mathsf{T}} \leftand f^c_{1a}(Q^*, R^{\mathsf{T}}) \tag{3.51}$$

$$f^c_{1a}((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{T}}) = (a \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \leftor Q^{\mathsf{F}} \tag{3.52}$$
$$f^c_{1a}((\neg a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{T}}) = (\neg a \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \leftor Q^{\mathsf{F}} \tag{3.53}$$
$$f^c_{1a}(P^* \leftand Q^d, R^{\mathsf{T}}) = P^* \leftand f^c_{1a}(Q^d, R^{\mathsf{T}}) \tag{3.54}$$
$$f^c_{1a}(P^* \leftor Q^c, R^{\mathsf{T}}) = f^c_{1a}(P^*, R^{\mathsf{T}}) \leftor f^c_{1a}(Q^c, R^{\mathsf{T}}). \tag{3.55}$$

For case (4.1.b) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{F}})$, which will be an F-term. Using (F7) we reduce this problem to converting Q^* to an F-term, for which we use the auxiliary function $f^c_{1b} : P^* \times P^{\mathsf{F}} \to P^{\mathsf{F}}$ that we define recursively by a case distinction on its first argument. Observe that the right-hand sides of the clauses defining f^c_{1b} are all F-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{F}}) = f^c(P^{\mathsf{T}}, f^c_{1b}(Q^*, R^{\mathsf{F}})) \tag{3.56}$$

$$f^c_{1b}((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{F}}) = (a \leftor Q^{\mathsf{F}}) \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{F}}) \tag{3.57}$$
$$f^c_{1b}((\neg a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{F}}) = (a \leftor f^c(P^{\mathsf{T}}, R^{\mathsf{F}})) \leftand Q^{\mathsf{F}} \tag{3.58}$$
$$f^c_{1b}(P^* \leftand Q^d, R^{\mathsf{F}}) = f^c_{1b}(P^*, f^c_{1b}(Q^d, R^{\mathsf{F}})) \tag{3.59}$$
$$f^c_{1b}(P^* \leftor Q^c, R^{\mathsf{F}}) = f^c_{1b}(f^n_1(f^c_{1a}(P^*, f^n(R^{\mathsf{F}}))), f^c_{1b}(Q^c, R^{\mathsf{F}})). \tag{3.60}$$

For case (4.1.c) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{U}})$, which will be an $n*$-term. Using (F7) we reduce this problem to converting Q^* to an $n*$-term, for which we use the auxiliary function $f^c_{1c} : P^* \times P^{\mathsf{U}} \to P^{n*}$ that we define recursively by a case distinction on its first argument. Observe that the right-hand sides of the clauses defining f^c_{1c} are all $n*$-term.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{U}}) = P^{\mathsf{T}} \leftand f^c_{1c}(Q^*, R^{\mathsf{U}}) \tag{3.61}$$

$$f^c_{1c}((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{U}}) = (a \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{U}})) \leftor Q^{\mathsf{F}} \tag{3.62}$$
$$f^c_{1c}((\neg a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}, R^{\mathsf{U}}) = (\neg a \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{U}})) \leftor Q^{\mathsf{F}} \tag{3.63}$$
$$f^c_{1c}(P^* \leftand Q^d, R^{\mathsf{U}}) = P^* \leftand f^c_{1c}(Q^d, R^{\mathsf{U}}) \tag{3.64}$$
$$f^c_{1c}(P^* \leftor Q^c, R^{\mathsf{U}}) = f^n_2(f^c_{1c}(P^*, R^{\mathsf{U}})) \leftand f^c_{1c}(Q^c, R^{\mathsf{U}}). \tag{3.65}$$

For case (4.1.d) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^*)$. We use the auxiliary function $f^c_{1d} : P^* \times (P^{\mathsf{T}} \leftand P^*) \to P^*$ to ensure that the result is a T-$*$-term, and we define f^c_{1d} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{1d} are all $*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^*) = P^{\mathsf{T}} \leftand f^c_{1d}(Q^*, R^{\mathsf{T}} \leftand S^*) \tag{3.66}$$

$$f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand R^\ell) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand R^\ell \tag{3.67}$$
$$f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftand S^d)) = f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand R^*) \leftand S^d \tag{3.68}$$
$$f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftor S^c)) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^* \leftor S^c). \tag{3.69}$$

For case (4.1.e) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{n*})$. We use the auxiliary function $f^c_{1e} : P^* \times (P^{\mathsf{T}} \leftand P^{n*}) \to P^{n*}$ to ensure that the result is a T-$n*$-term, and we define f^c_{1e} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{1e} are all $n*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{n*}) = P^{\mathsf{T}} \leftand f^c_{1e}(Q^*, R^{\mathsf{T}} \leftand S^{n*}) \tag{3.70}$$

$$f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand R^{n\ell}) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand R^{n\ell} \tag{3.71}$$
$$f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftand S^{n*})) = f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand R^*) \leftand S^{n*} \tag{3.72}$$
$$f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{n*})) = f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{n*} \tag{3.73}$$
$$f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{n*})) = f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{n*} \tag{3.74}$$

For case (4.1.f) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{w*})$. We use the auxiliary function $f^c_{1f} : P^* \times (P^{\mathsf{T}} \leftand P^{w*}) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{1f} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{1f} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{w*}) = P^{\mathsf{T}} \leftand f^c_{1f}(Q^*, R^{\mathsf{T}} \leftand S^{w*}) \tag{3.75}$$

$$f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand S^{w*}) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand S^{w*} \tag{3.76}$$

For case (4.1.g) we need to define $f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{b*})$. We use the auxiliary function $f^c_{1g} : P^* \times (P^{\mathsf{T}} \leftand P^{b*}) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{1g} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{1g} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^*, R^{\mathsf{T}} \leftand S^{b*}) = P^{\mathsf{T}} \leftand f^c_{1g}(Q^*, R^{\mathsf{T}} \leftand S^{b*}) \tag{3.77}$$

$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftand S^{w*})) = f^c_{1d}(P^*, Q^{\mathsf{T}} \leftand R^*) \leftand S^{w*} \tag{3.78}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftor S^{n*})) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^* \leftor S^{n*}) \tag{3.79}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^*)) = f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^* \tag{3.80}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^*)) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^*) \tag{3.81}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^*)) = f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^* \tag{3.82}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{n*})) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{n*}) \tag{3.83}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{b*})) = f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{b*} \tag{3.84}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{b*})) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{b*}) \tag{3.85}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{b*})) = f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{b*} \tag{3.86}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftor S^{b*})) = f^c_{1a}(P^*, Q^{\mathsf{T}}) \leftand (R^* \leftor S^{b*}) \tag{3.87}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^* \leftand S^{b*})) = f^c_{1d}(P^*, Q^{\mathsf{T}}, \leftand R^*) \leftand S^{b*} \tag{3.88}$$
$$f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{w*})) = f^c_{1g}(P^*, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{w*} \tag{3.89}$$

For case (4.2) (The a-term is a $n*$-term) there are only two evaluations of the $n*$-term. An U-term or an F-term. In case the evaluation is an F-term we can ignore the conjunction (3.49 and F6). In case the evaluation is an U-term we can ignore the conjunction as well (3.50, FU2). For this reason we define:

$$f^c_2(P^{n*}, Q) = P^{n*} \tag{3.90}$$

For case (4.3) (The a-term is a $w*$-term) there are seven possible sub-cases for the second argument. We distinguish these seven sub-cases:

(4.3.a) The second argument is a T-term,
(4.3.b) The second argument is an F-term,
(4.3.c) The second argument is an U-term,
(4.3.d) The second argument is a T-$*$-term,
(4.3.e) The second argument is a T-$n*$-term,
(4.3.f) The second argument is a T-$w*$-term,
(4.3.g) The second argument is a T-$b*$-term

For case (4.3.a) (the second argument is a T-term), we will use an auxiliary function $f^c_{3a} : P^{w*} \times P^{\mathsf{T}} \to P^{w*}$ to turn conjunctions of a $w*$-term with a T-term into $w*$-terms. We define f^c_{3a} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{3a} are $w*$-terms.

$$f^c(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^{w*}, R^{\mathsf{T}}) = P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} f^c_{3a}(Q^{w*}, R^{\mathsf{T}}) \tag{3.91}$$

$$f^c_{3a}((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{T}}) = (a \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}} \tag{3.92}$$

$$f^c_{3a}((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{T}}) = (\neg a \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}} \tag{3.93}$$

$$f^c_{3a}(P^{*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{T}}) = f^c_{1a}(P^{*}, R^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} f^c_{3a}(Q^{w*}, R^{\mathsf{T}}) \tag{3.94}$$

$$f^c_{3a}(P^{n*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{T}}) = P^{n*} \mathbin{{}^{\circ}\!\vee} f^c_{3a}(Q^{w*}, R^{\mathsf{T}}) \tag{3.95}$$

$$f^c_{3a}(P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{T}}) = f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} f^c_{3a}(Q^{w*}, R^{\mathsf{T}}) \tag{3.96}$$

For case (4.3.b) (the second argument is an F-term), we will use an auxiliary function $f^c_{3b} : P^{w*} \times P^{\mathsf{F}} \to P^{n*}$ to turn conjunctions of a $w*$-term with an F-term into $n*$-terms. We define f^c_{3b} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{3b} are $n*$-terms.

$$f^c(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^{w*}, R^{\mathsf{F}}) = P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) \tag{3.97}$$

$$f^c_{3b}((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{F}}) = (a \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{F}}) \tag{3.98}$$

$$f^c_{3b}((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{F}}) = (\neg a \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{F}})) \tag{3.99}$$

$$f^c_{3b}(P^{*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{F}}) = f^n_1(f^c_{1a}(P^{*}, f^n(R^{\mathsf{F}}))) \mathbin{{}_{\circ}\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) \tag{3.100}$$

$$f^c_{3b}(P^{n*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{F}}) = f^n_2(P^{n*}) \mathbin{{}_{\circ}\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) \tag{3.101}$$

$$f^c_{3b}(P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{F}}) = f^n_4(f^c_{4a}(P^{b*}, f^n(R^{\mathsf{F}}))) \mathbin{{}_{\circ}\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) \tag{3.102}$$

For case (4.3.c) (the second argument is an U-term), we will use an auxiliary function $f^c_{3c} : P^{w*} \times P^{\mathsf{U}} \to P^{\mathsf{U}}$ to turn conjunctions of a $w*$-term with a U-term into U-terms. We define f^c_{3c} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{3c} are U-terms.

$$f^c(P^{\mathsf{T}} \mathbin{{}_{\circ}\!\wedge} Q^{w*}, R^{\mathsf{U}}) = f^c(P^{\mathsf{T}}, f^c_{3c}(Q^{w*}, R^{\mathsf{U}})) \tag{3.103}$$

$$f^c_{3c}((a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{U}}) = (a \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{U}}) \tag{3.104}$$

$$f^c_{3c}((\neg a \mathbin{{}_{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{U}}) = (\neg a \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}_{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{U}}) \tag{3.105}$$

$$f^c_{3c}(P^{*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{U}}) = f^c_{3c}(f^n_2(f^c_{1c}(P^{*}, R^{\mathsf{U}})), f^c_{3c}(Q^{w*}, R^{\mathsf{U}})) \tag{3.106}$$

$$f^c_{3c}(P^{n*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{U}}) = f^c_{3c}(f^n_2(P^{n*}), f^c_{3c}(Q^{w*}, R^{\mathsf{U}})) \tag{3.107}$$

$$f^c_{3c}(P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{w*}, R^{\mathsf{U}}) = f^c_{3c}(f^n_2(f^c_{4c}(P^{b*}, R^{\mathsf{U}})), f^c_{3c}(Q^{w*}, R^{\mathsf{U}})) \tag{3.108}$$

For case (4.3.d) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^*)$. We use the auxiliary function $f^c_{3d} : P^{w*} \times (P^{\mathsf{T}} \leftand P^*) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{3d} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{3d} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^*) = P^{\mathsf{T}} \leftand f^c_{3d}(Q^{w*}, R^{\mathsf{T}} \leftand S^*) \tag{3.109}$$

$$f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand R^{\ell}) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand R^{\ell} \tag{3.110}$$

$$f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftand S^d)) = f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand R^*) \leftand S^d \tag{3.111}$$

$$f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftor S^c)) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^* \leftor S^c). \tag{3.112}$$

For case (4.3.e) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{n*})$. We use the auxiliary function $f^c_{3e} : P^{w*} \times (P^{\mathsf{T}} \leftand P^{n*}) \to P^{n*}$ to ensure that the result is a T-$n*$-term, and we define f^c_{3e} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{3e} are all $n*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{n*}) = P^{\mathsf{T}} \leftand f^c_{3e}(Q^{w*}, R^{\mathsf{T}} \leftand S^{n*}) \tag{3.113}$$

$$f^c_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand R^{n\ell}) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand R^{n\ell} \tag{3.114}$$

$$f^c_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftand S^{n*})) = f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand R^*) \leftand S^{n*} \tag{3.115}$$

$$f^c_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{n*})) = f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{n*} \tag{3.116}$$

$$f^c_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{n*})) = f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{n*} \tag{3.117}$$

For case (4.3.f) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{w*})$. We use the auxiliary function $f^c_{3f} : P^{w*} \times (P^{\mathsf{T}} \leftand P^{w*}) \to P^{w*}$ to ensure that the result is a T-$w*$-term, and we define f^c_{3f} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{3f} are all $w*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{w*}) = P^{\mathsf{T}} \leftand f^c_{3f}(Q^{w*}, R^{\mathsf{T}} \leftand S^{w*}) \tag{3.118}$$

$$f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \leftand R^{w*}) = f^n_3(f^c_{3a}(P^{w*}, Q^{\mathsf{T}})) \leftor R^{w*} \tag{3.119}$$

For case (4.3.g) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{b*})$. We use the auxiliary function $f^c_{3g} : P^{w*} \times (P^{\mathsf{T}} \leftand P^{b*}) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{3g} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{3g} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{w*}, R^{\mathsf{T}} \leftand S^{b*}) = P^{\mathsf{T}} \leftand f^c_{3g}(Q^{w*}, R^{\mathsf{T}} \leftand S^{b*}) \tag{3.120}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftand S^{w*})) = f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \leftand R^*) \leftand S^{w*} \tag{3.121}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftor S^{n*})) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^* \leftor S^{n*}) \tag{3.122}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^*)) = f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^* \tag{3.123}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^*)) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^*) \tag{3.124}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^*)) = f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^* \tag{3.125}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{n*})) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{n*}) \tag{3.126}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{b*})) = f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{b*} \tag{3.127}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{b*})) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{b*}) \tag{3.128}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{b*})) = f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{b*} \tag{3.129}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftor S^{b*})) = f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \leftand (R^* \leftor S^{b*}) \tag{3.130}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^* \leftand S^{b*})) = f^c_{3d}(P^{w*}, Q^{\mathsf{T}}, \leftand R^*) \leftand S^{b*} \tag{3.131}$$

$$f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{w*})) = f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{w*} \tag{3.132}$$

For case (4.4) (The a-term is a $b*$-term) there are seven possible sub-cases for the second argument. We distinguish these seven sub-cases:

(4.4.a) The second argument is a T-term,
(4.4.b) The second argument is an F-term,
(4.4.c) The second argument is an U-term,
(4.4.d) The second argument is a T-$*$-term,
(4.4.e) The second argument is a T-$n*$-term,
(4.4.f) The second argument is a T-$w*$-term,
(4.4.g) The second argument is a T-$b*$-term

For case (4.4.a) (the second argument is a T-term), we will use an auxiliary function $f^c_{4a} : P^{b*} \times P^{\mathsf{T}} \to P^{b*}$ to turn conjunctions of a $b*$-term with a T-term into $b*$-terms. We define f^c_{4a} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{4a} are $b*$-terms.

$$f^c(P^{\mathsf{T}} \mathbin{{}^\circ\!\wedge} Q^{b*}, R^{\mathsf{T}}) = P^{\mathsf{T}} \mathbin{{}^\circ\!\wedge} f^c_{4a}(Q^{b*}, R^{\mathsf{T}}) \tag{3.133}$$

$$f^c_{4a}(P^* \mathbin{{}^\circ\!\wedge} Q^{w*}, R^{\mathsf{T}}) = P^* \mathbin{{}^\circ\!\wedge} f^c_{3a}(Q^{w*}, R^{\mathsf{T}}) \tag{3.134}$$
$$f^c_{4a}(P^* \mathbin{{}^\circ\!\vee} Q^{n*}, R^{\mathsf{T}}) = f^c_{1a}(P^*, R^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} Q^{n*} \tag{3.135}$$
$$f^c_{4a}(P^{w*} \mathbin{{}^\circ\!\wedge} Q^*, R^{\mathsf{T}}) = P^{w*} \mathbin{{}^\circ\!\wedge} f^c_{1a}(Q^*, R^{\mathsf{T}}) \tag{3.136}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\vee} Q^*, R^{\mathsf{T}}) = f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} f^c_{1a}(Q^*, R^{\mathsf{T}}) \tag{3.137}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\wedge} Q^*, R^{\mathsf{T}}) = P^{b*} \mathbin{{}^\circ\!\wedge} f^c_{1a}(Q^*, R^{\mathsf{T}}) \tag{3.138}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\vee} Q^{n*}, R^{\mathsf{T}}) = f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} Q^{n*} \tag{3.139}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\wedge} Q^{b*}, R^{\mathsf{T}}) = P^{b*} \mathbin{{}^\circ\!\wedge} f^c_{4a}(Q^{b*}, R^{\mathsf{T}}) \tag{3.140}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\vee} Q^{b*}, R^{\mathsf{T}}) = f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \tag{3.141}$$
$$f^c_{4a}(P^{w*} \mathbin{{}^\circ\!\wedge} Q^{b*}, R^{\mathsf{T}}) = P^{w*} \mathbin{{}^\circ\!\wedge} f^c_{4a}(Q^{b*}, R^{\mathsf{T}}) \tag{3.142}$$
$$f^c_{4a}(P^* \mathbin{{}^\circ\!\vee} Q^{b*}, R^{\mathsf{T}}) = f^c_{1a}(P^*, R^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \tag{3.143}$$
$$f^c_{4a}(P^* \mathbin{{}^\circ\!\wedge} Q^{b*}, R^{\mathsf{T}}) = P^* \mathbin{{}^\circ\!\wedge} f^c_{4a}(P^{b*}, R^{\mathsf{T}}) \tag{3.144}$$
$$f^c_{4a}(P^{b*} \mathbin{{}^\circ\!\wedge} Q^{w*}, R^{\mathsf{T}}) = P^{b*} \mathbin{{}^\circ\!\wedge} f^c_{3a}(P^{w*}, R^{\mathsf{T}}) \tag{3.145}$$

For case (4.4.b) (the second argument is an F-term), we will use an auxiliary function $f^c_{4b} : P^{b*} \times P^{\mathsf{F}} \to P^{n*}$ to turn conjunctions of a $b*$-term with a T-term into $n*$-terms. We define f^c_{4b} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{4b} are $n*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{F}}) = P^{\mathsf{T}} \leftand f^c_{4b}(Q^{b*}, R^{\mathsf{F}}) \tag{3.146}$$

$$f^c_{4b}(P^* \leftand Q^{w*}, R^{\mathsf{F}}) = P^* \leftand f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) \tag{3.147}$$

$$f^c_{4b}(P^* \leftor Q^{n*}, R^{\mathsf{F}}) = f^n_1(f^c_{1a}(P^*, f^n(R^{\mathsf{F}}))) \leftand Q^{n*} \tag{3.148}$$

$$f^c_{4b}(P^{w*} \leftand Q^*, R^{\mathsf{F}}) = f^c_{3b}(P^{w*}, f^c_{1b}(Q^*, R^{\mathsf{F}})) \tag{3.149}$$

$$f^c_{4b}(P^{b*} \leftor Q^*, R^{\mathsf{F}}) = f^c_{4b}(f^n_4(f^c_{4a}(P^{b*}, f^n(R^{\mathsf{F}}))), f^c_{1b}(Q^*, R^{\mathsf{F}})) \tag{3.150}$$

$$f^c_{4b}(P^{b*} \leftand Q^*, R^{\mathsf{F}}) = f^c_{4b}(P^{b*}, f^c_{1a}(Q^*, R^{\mathsf{F}})) \tag{3.151}$$

$$f^c_{4b}(P^{b*} \leftor Q^{n*}, R^{\mathsf{F}}) = f^n_4(f^c_{4a}(P^{b*}, f^n(R^{\mathsf{F}}))) \leftand Q^{n*} \tag{3.152}$$

$$f^c_{4b}(P^{b*} \leftand Q^{b*}, R^{\mathsf{F}}) = P^{b*} \leftand f^c_{4b}(Q^{b*}, R^{\mathsf{F}}) \tag{3.153}$$

$$f^c_{4b}(P^{b*} \leftor Q^{b*}, R^{\mathsf{F}}) = f^n_4(f^c_{4a}(P^{b*}, f^n(R^{\mathsf{F}}))) \leftand f^c_{4b}(P^{b*}, R^{\mathsf{F}}) \tag{3.154}$$

$$f^c_{4b}(P^{w*} \leftand Q^{b*}, R^{\mathsf{F}}) = P^{w*} \leftand f^c_{4b}(Q^{b*}, R^{\mathsf{F}}) \tag{3.155}$$

$$f^c_{4b}(P^* \leftor Q^{b*}, R^{\mathsf{F}}) = f^n_1(f^c_{1a}(P^*, f^n(R^{\mathsf{F}}))) \leftand f^c_{4b}(P^{b*}, R^{\mathsf{F}}) \tag{3.156}$$

$$f^c_{4b}(P^* \leftand Q^{b*}, R^{\mathsf{F}}) = P^* \leftand f^c_{4b}(P^{b*}, R^{\mathsf{F}}) \tag{3.157}$$

$$f^c_{4b}(P^{b*} \leftand Q^{w*}, R^{\mathsf{F}}) = P^{b*} \leftand f^c_{3b}(P^{w*}, R^{\mathsf{F}}) \tag{3.158}$$

For case (4.4.c) (the second argument is an U-term), we will use an auxiliary function $f^c_{4c} : P^{b*} \times P^{\mathsf{U}} \to P^{n*}$ to turn conjunctions of a $b*$-term with an U-term into $n*$-terms. We define f^c_{4c} recursively by a case distinction on its first argument. Together with (F7) (associativity) this allows us to define f^c for this case. Observe that the right-hand sides of the clauses defining f^c_{4c} are $n*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{U}}) = P^{\mathsf{T}} \leftand f^c_{4c}(Q^{b*}, R^{\mathsf{U}}) \tag{3.159}$$

$$f^c_{4c}(P^* \leftand Q^{w*}, R^{\mathsf{U}}) = f^c_{1c}(P^*, f^c_{3c}(Q^{w*}, R^{\mathsf{U}})) \tag{3.160}$$

$$f^c_{4c}(P^* \leftor Q^{n*}, R^{\mathsf{U}}) = f^n_2(f^c_{1a}(P^*, R^{\mathsf{U}})) \leftand Q^{n*} \tag{3.161}$$

$$f^c_{4c}(P^{w*} \leftand Q^*, R^{\mathsf{U}}) = P^{w*} \leftand f^c_{1c}(Q^*, R^{\mathsf{U}}) \tag{3.162}$$

$$f^c_{4c}(P^{b*} \leftor Q^*, R^{\mathsf{U}}) = f^n_2(f^c_{4c}(P^{b*}, R^{\mathsf{U}})) \leftand f^c_{1c}(Q^*, R^{\mathsf{U}}) \tag{3.163}$$

$$f^c_{4c}(P^{b*} \leftand Q^*, R^{\mathsf{U}}) = P^{b*} \leftand f^c_{1c}(Q^*, R^{\mathsf{U}}) \tag{3.164}$$

$$f^c_{4c}(P^{b*} \leftor Q^{n*}, R^{\mathsf{U}}) = f^n_2(f^c_{4c}(P^{b*}, R^{\mathsf{U}})) \leftand Q^{n*} \tag{3.165}$$

$$f^c_{4c}(P^{b*} \leftand Q^{b*}, R^{\mathsf{U}}) = P^{b*} \leftand f^c_{4c}(Q^{b*}, R^{\mathsf{U}}) \tag{3.166}$$

$$f^c_{4c}(P^{b*} \leftor Q^{b*}, R^{\mathsf{U}}) = f^n_2(f^c_{4c}(P^{b*}, R^{\mathsf{U}})) \leftand f^c_{4c}(P^{b*}, R^{\mathsf{U}}) \tag{3.167}$$

$$f^c_{4c}(P^{w*} \leftand Q^{b*}, R^{\mathsf{U}}) = P^{w*} \leftand f^c_{4c}(Q^{b*}, R^{\mathsf{U}}) \tag{3.168}$$

$$f^c_{4c}(P^* \leftor Q^{b*}, R^{\mathsf{U}}) = f^n_2(f^c_{1c}(P^*, R^{\mathsf{U}})) \leftand f^c_{4c}(P^{b*}, R^{\mathsf{U}}) \tag{3.169}$$

$$f^c_{4c}(P^* \leftand Q^{b*}, R^{\mathsf{U}}) = P^* \leftand f^c_{4c}(P^{b*}, R^{\mathsf{U}}) \tag{3.170}$$

$$f^c_{4c}(P^{b*} \leftand Q^{w*}, R^{\mathsf{U}}) = f^c_{4c}(P^{b*}, f^c_{3c}(P^{w*}, R^{\mathsf{U}})) \tag{3.171}$$

For case (4.4.d) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^*)$. We use the auxiliary function $f^c_{4d} : P^{b*} \times (P^{\mathsf{T}} \leftand P^*) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{4d} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{4d} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^*) = P^{\mathsf{T}} \leftand f^c_{4d}(Q^{b*}, R^{\mathsf{T}} \leftand S^*) \tag{3.172}$$

$$f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand R^{\ell}) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand R^{\ell} \tag{3.173}$$

$$f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand (R^* \leftand S^d)) = f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand R^*) \leftand S^d \tag{3.174}$$

$$f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand (R^* \leftor S^c)) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^* \leftor S^c). \tag{3.175}$$

For case (4.4.e) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{n*})$. We use the auxiliary function $f^c_{4e} : P^{b*} \times (P^{\mathsf{T}} \leftand P^{n*}) \to P^{n*}$ to ensure that the result is a T-$n*$-term, and we define f^c_{4e} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{4e} are all $n*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{n*}) = P^{\mathsf{T}} \leftand f^c_{4e}(Q^{b*}, R^{\mathsf{T}} \leftand S^{n*}) \tag{3.176}$$

$$f^c_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand R^{n\ell}) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand R^{n\ell} \tag{3.177}$$
$$f^c_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{*} \leftand S^{n*})) = f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand R^{*}) \leftand S^{n*} \tag{3.178}$$
$$f^c_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{n*})) = f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{n*} \tag{3.179}$$
$$f^c_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{n*})) = f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{n*} \tag{3.180}$$

For case (4.4.f) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{w*})$. We use the auxiliary function $f^c_{4f} : P^{b*} \times (P^{\mathsf{T}} \leftand P^{w*}) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{4f} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{4f} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{w*}) = P^{\mathsf{T}} \leftand f^c_{4f}(Q^{b*}, R^{\mathsf{T}} \leftand S^{w*}) \tag{3.181}$$

$$f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand S^{w*}) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand S^{w*} \tag{3.182}$$
$$\tag{3.183}$$

For case (4.4.g) we need to define $f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{b*})$. We use the auxiliary function $f^c_{4g} : P^{b*} \times (P^{\mathsf{T}} \leftand P^{b*}) \to P^{b*}$ to ensure that the result is a T-$b*$-term, and we define f^c_{4g} by a case distinction on its second argument. Observe that the right-hand sides of the clauses defining f^c_{4g} are all $b*$-terms.

$$f^c(P^{\mathsf{T}} \leftand Q^{b*}, R^{\mathsf{T}} \leftand S^{b*}) = P^{\mathsf{T}} \leftand f^c_{4g}(Q^{b*}, R^{\mathsf{T}} \leftand S^{b*}) \tag{3.184}$$

$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{*} \leftand S^{w*})) = f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \leftand R^{*}) \leftand S^{w*} \tag{3.185}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{*} \leftor S^{n*})) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^{*} \leftor S^{n*}) \tag{3.186}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{*})) = f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{*} \tag{3.187}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{*})) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{*}) \tag{3.188}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{*})) = f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{*} \tag{3.189}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{n*})) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{n*}) \tag{3.190}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{b*})) = f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{b*} \tag{3.191}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftor S^{b*})) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^{b*} \leftor S^{b*}) \tag{3.192}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{w*} \leftand S^{b*})) = f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*}) \leftand S^{b*} \tag{3.193}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{*} \leftor S^{b*})) = f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand (R^{*} \leftor S^{b*}) \tag{3.194}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{*} \leftand S^{b*})) = f^c_{4d}(P^{b*}, Q^{\mathsf{T}}, \leftand R^{*}) \leftand S^{b*} \tag{3.195}$$
$$f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand (R^{b*} \leftand S^{w*})) = f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*}) \leftand S^{w*} \tag{3.196}$$

Theorem 3.2.1 (Normal forms)**.** *For any $P \in {\mathcal{S}_A}^U$, $f(P)$ terminates, $f(P) \in SNF$ and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f(P) = P.$$

Proof. See Appendix A.1. □

CHAPTER 4

Conclusion

4.1 Discussion

There are multiple ways to define the normal forms (a subset of *SNF* defined in Chapter 3, as explained in Chapter 2) for $\mathrm{FSCL}^{\mathsf{U}}$, as discussed in Chapter 3. This thesis project aimed to provide the minimal normal forms needed to represent all terms in $\mathrm{FSCL}^{\mathsf{U}}$. However, it could be potential that there is a more optimal definition of the normal forms.

A potential approach for the proof of completeness of $\mathrm{EqFSCL}^{\mathsf{U}}$ for closed terms is a generalization of the proof in [7, Section 3], which is based on normal forms. This proof is also the motivation for the research question of this thesis project.

Future studies need to reevaluate the use of normal forms for this proof, as the normal forms were more complex than we initially expected. Therefore, how the completeness of $\mathrm{EqFSCL}^{\mathsf{U}}$ for closed terms can be proven and whether the normal forms are helpful for the proof remains an open question.

4.2 Conclusion

We defined and used auxiliary functions for the negation and conjunction of terms to prove the correctness of the normalisation function.

Thus far, the completeness of $\mathrm{EqFSCL}^{\mathsf{U}}$ is unproven. This thesis does not change that. However, it does define normal forms that potentially can be used to prove that $\mathrm{EqFSCL}^{\mathsf{U}}$ is indeed complete.

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APPENDIX A

Correctness of the normalisation function

A.1 General results

Lemma A.1.1. *For all* $P \in P^{\mathsf{F}}$ *and* $Q \in P^{\mathsf{T}}$, $\mathrm{EqFSCL} \vdash P = P \leftand x$ *and* $\mathrm{EqFSCL} \vdash Q = Q \leftor x$.

Proof. See [7, La.A.2.1]. □

Lemma A.1.2. *The following equations can all be derived from* EqFSCL.

1. $(x \leftor (y \leftand \mathsf{F})) \leftand (z \leftand \mathsf{F}) = (\neg x \leftor (z \leftand \mathsf{F})) \leftand (y \leftand \mathsf{F})$,
2. $(x \leftand (y \leftor \mathsf{T})) \leftor (z \leftand \mathsf{F}) = (x \leftor (z \leftand \mathsf{F})) \leftand (y \leftor \mathsf{T})$,
3. $(x \leftor \mathsf{T}) \leftand \neg y = \neg((x \leftor \mathsf{T}) \leftand y)$,
4. $(x \leftand (y \leftand (z \leftor \mathsf{T}))) \leftor (w \leftand (z \leftor \mathsf{T})) = ((x \leftand y) \leftor w) \leftand (z \leftor \mathsf{T})$,
5. $(x \leftor ((y \leftor \mathsf{T}) \leftand (z \leftand \mathsf{F}))) \leftand ((w \leftor \mathsf{T}) \leftand (z \leftand \mathsf{F})) = ((x \leftand (w \leftor \mathsf{T})) \leftor (y \leftor \mathsf{T})) \leftand (z \leftand \mathsf{F})$,
6. $(x \leftor ((y \leftor \mathsf{T}) \leftand (z \leftand \mathsf{F}))) \leftand (w \leftand \mathsf{F}) = ((\neg x \leftand (y \leftor \mathsf{T})) \leftor (w \leftand \mathsf{F})) \leftand (z \leftand \mathsf{F})$.

Proof. These equations stem from [7] and have been checked with the theorem *Prover9* [6]. □

Lemma A.1.3. *For all* $P \in P^{\mathsf{U}}$, $\mathrm{EqFSCL}^{\mathsf{U}} \vdash P = P \leftand x$, $\mathrm{EqFSCL}^{\mathsf{U}} \vdash P = \neg P$ *and* $\mathrm{EqFSCL}^{\mathsf{U}} \vdash P = P \leftor x$.

Proof. We prove these claims simultaneously by induction. In the base case we have $\mathsf{U} \leftand x = \mathsf{U} \leftor x = \neg \mathsf{U} = \mathsf{U}$ by (FU1), (FU2) and (FU3). For the inductive step we assume the result holds for all U-terms with lesser complexity than $(a \leftor P^{\mathsf{U}}) \leftand Q^{\mathsf{U}}$.

For $P^{\mathsf{U}} \leftand x$ we get:

$$\begin{aligned} ((a \leftor P^{\mathsf{U}}) \leftand Q^{\mathsf{U}}) \leftand x &= (a \leftor P^{\mathsf{U}}) \leftand (Q^{\mathsf{U}} \leftand x) && \text{by (F7)} \\ &= (a \leftor P^{\mathsf{U}}) \leftand Q^{\mathsf{U}} && \text{by IH} \end{aligned}$$

For the inductive case for $\neg P^{\mathsf{U}}$ we get:

$$\begin{aligned} \neg((a \leftor P^{\mathsf{U}}) \leftand Q^{\mathsf{U}}) &= \neg(a \leftor (P^{\mathsf{U}} \leftor \mathsf{T})) \leftand Q^{\mathsf{U}} && \text{by IH} \\ &= \neg((a \leftor P^{\mathsf{U}}) \leftor \mathsf{T}) \leftand Q^{\mathsf{U}} && \text{by (F7)'} \\ &= ((a \leftor P^{\mathsf{U}}) \leftor \mathsf{T}) \leftand \neg Q^{\mathsf{U}} && \text{by La.A.1.2.3} \\ &= (a \leftor (P^{\mathsf{U}} \leftor \mathsf{T})) \leftand \neg Q^{\mathsf{U}} && \text{by (F7)'} \\ &= (a \leftor P^{\mathsf{U}}) \leftand Q^{\mathsf{U}} && \text{by IH} \end{aligned}$$

For the inductive case for $P^{\mathsf{U}} \leftor x$ we get:

$$
\begin{aligned}
((a \mathbin{{}^\circ\!\vee} P^{\mathsf{U}}) \mathbin{{}_\circ\!\wedge} Q^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} x &= ((a \mathbin{{}^\circ\!\vee} (P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} x && \text{by IH}\\
&= ((a \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} (P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}^\circ\!\vee} x && \text{by La.A.1.2.2}\\
&= (a \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} ((P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} \mathsf{F}) \mathbin{{}^\circ\!\vee} x) && \text{by (F7)'}\\
&= (a \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} (P^{\mathsf{U}} \mathbin{{}^\circ\!\vee} x) && \text{by IH}\\
&= (a \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} P^{\mathsf{U}} && \text{by IH}\\
&= (a \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T})) \mathbin{{}^\circ\!\vee} (P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} \mathsf{F}) && \text{by IH}\\
&= (a \mathbin{{}^\circ\!\vee} (P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} \mathsf{T}) && \text{by La.A.1.2.2}\\
&= (a \mathbin{{}^\circ\!\vee} P^{\mathsf{U}}) \mathbin{{}_\circ\!\wedge} Q^{\mathsf{U}} && \text{by IH}
\end{aligned}
$$

□

Lemma A.1.4. *For all* $P \in P^{n*}$ *and* $Q \in P^{w*}$ $\text{EqFSCL}^{\mathsf{U}} \vdash P = P \mathbin{{}_\circ\!\wedge} x$ *and* $\text{EqFSCL}^{\mathsf{U}} \vdash Q = Q \mathbin{{}^\circ\!\vee} x$

Proof. We prove these claims by induction on the structure of P^{n*} and Q^{w*}.

In the base case for $n*$-terms, in the case of an $n\ell$-term of the form $(a \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}}$ we get:

$$
\begin{aligned}
((a \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}}) \mathbin{{}_\circ\!\wedge} x &= ((a \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} x && \text{by La.A.1.1}\\
&= ((a \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} (P^{\mathsf{U}} \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F}))) \mathbin{{}_\circ\!\wedge} x && \text{by (F10)}\\
&= ((a \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}}) \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}_\circ\!\wedge} x && \text{by La.A.1.1 and La.A.1.3}\\
&= (a \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}}) \mathbin{{}_\circ\!\wedge} (P^{\mathsf{U}} \mathbin{{}_\circ\!\wedge} x) && \text{by (F7)}\\
&= (a \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}}) \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}} && \text{by La.A.1.3}\\
&= (a \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) \mathbin{{}_\circ\!\wedge} (P^{\mathsf{U}} \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F})) && \text{by La.A.1.1 and La.A.1.3}\\
&= (a \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} (Q^{\mathsf{F}} \mathbin{{}_\circ\!\wedge} \mathsf{F}) && \text{by (F10)}\\
&= (a \mathbin{{}_\circ\!\wedge} P^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{F}} && \text{by La.A.1.1}
\end{aligned}
$$

In the case of $\neg a$, the proof proceeds the same. For the inductive step we assume the result holds for all $n*$-terms of lesser complexity than $P^* \mathbin{{}_\circ\!\wedge} P^{n*}$, $P^{b*} \mathbin{{}_\circ\!\wedge} P^{n*}$ and $P^{w*} \mathbin{{}_\circ\!\wedge} P^{n*}$. The proof for these cases follows directly from (F7)' and the induction hypothesis.

In the base case for $w*$-terms, in the case of a $w\ell$-term of the form $(a \mathbin{{}_\circ\!\wedge} P^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{U}}$ we get:

$$
\begin{aligned}
((a \mathbin{{}_\circ\!\wedge} P^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{U}}) \mathbin{{}^\circ\!\vee} x &= (a \mathbin{{}_\circ\!\wedge} P^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} (Q^{\mathsf{U}} \mathbin{{}^\circ\!\vee} x) && \text{by (F7)'}\\
&= (a \mathbin{{}_\circ\!\wedge} P^{\mathsf{T}}) \mathbin{{}^\circ\!\vee} Q^{\mathsf{U}} && \text{by La.A.1.3}
\end{aligned}
$$

In case of $\neg a$ the proof proceeds the same. For the inductive step we assume the result holds for all $w*$-terms of lesser complexity than $P^* \mathbin{{}^\circ\!\vee} P^{n*}$, $P^{b*} \mathbin{{}^\circ\!\vee} P^{n*}$ and $P^{w*} \mathbin{{}^\circ\!\vee} P^{n*}$. The proof for these cases follows directly from (F7) and the induction hypothesis. □

A.1.1 Correctness of the negation function

Lemma A.1.5. *For all* $P \in \mathit{SNF}$*, if* P *is a* T*-term then* $f^n(P)$ *is an* F*-term, if it is an* F*-term then* $f^n(P)$ *is a* T*-term, if it is an* U*-term then so is* $f^n(P)$*, and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^n(P) = \neg P.$$

Proof. For T-terms and F-terms see [7, La.A.2.3]. For U-terms this follows from (3.12), (3.13) and La.A.1.3.

□

Lemma A.1.6. *For all* $P \in \mathit{SNF}$*, if* P *is a* T*-$*$-term then so is* $f^n(P)$*, if it is a* T*-$n*$-term then* $f^n(P)$ *is a* T*-$w*$-term, if it is a* T*-$w*$-term then* $f^n(P)$ *is a* T*-$n*$-term, if it is a* T*-$b*$-term then so is* $f^n(P)$*, and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^n(P) = \neg P.$$

Proof. For the proof of the lemma for T-$*$-terms with (3.14)-(3.18) see [7, La.A.2.3]. We define a literal as an element of $P^{n\ell}, P^{w\ell}$ or $P^{b\ell}$. We will prove $f^n(P) = \neg P$ for P^{n*}, P^{w*} and P^{b*} by induction on the number of literals.

In the case of an $n\ell$-term of the form $(a \leftand P^{\mathsf{U}}) \leftor Q^{\mathsf{F}}$ we get:

$$
\begin{aligned}
f_2^n((a \leftand P^{\mathsf{U}}) \leftor Q^{\mathsf{F}}) &= (\neg a \leftand f^n(Q^{\mathsf{F}})) \leftor P^{\mathsf{U}} && \text{by (3.23) \textit{Note: This is a $w\ell$-term}}\\
&= (\neg a \leftand (f^n(Q^{\mathsf{F}}) \leftor \mathsf{T})) \leftor P^{\mathsf{U}} && \text{by La.A.1.1}\\
&= (\neg a \leftand (f^n(Q^{\mathsf{F}}) \leftor \mathsf{T})) \leftor (P^{\mathsf{U}} \leftand \mathsf{F}) && \text{by La.A.1.3}\\
&= (\neg a \leftor (P^{\mathsf{U}} \leftand \mathsf{F})) \leftand (f^n(Q^{\mathsf{F}}) \leftor \mathsf{T}) && \text{by La.A.1.2.2}\\
&= (\neg a \leftor P^{\mathsf{U}}) \leftand f^n(Q^{\mathsf{F}}) && \text{by La.A.1.3 and La.A.1.1}\\
&= (\neg a \leftor \neg P^{\mathsf{U}}) \leftand \neg Q^{\mathsf{F}} && \text{by La.A.1.3 and La.A.1.5}\\
&= \neg((a \leftand P^{\mathsf{U}}) \leftor Q^{\mathsf{F}}). && \text{by (F2) and its dual}
\end{aligned}
$$

$f_2^n((a \leftand P^{\mathsf{U}}) \leftor Q^{\mathsf{F}}) = (\neg a \leftand f^n(Q^{\mathsf{F}})) \leftor P^{\mathsf{U}}$ by (3.23), where $f^n(Q^{\mathsf{F}})$ is a T-term, so this is a $w\ell$-term. In the case of an $n\ell$-term of the form $(\neg a \leftand P^{\mathsf{U}}) \leftor Q^{\mathsf{F}}$ the proof proceeds the same, substituting $\neg a$ for a and applying (3.24) and (F3) where needed.

In the case of a $w\ell$-term of the form $(a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{U}}$ we have:

$$
\begin{aligned}
f_3^n((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{U}}) &= (\neg a \leftand Q^{\mathsf{U}}) \leftor f^n(P^{\mathsf{T}}) && \text{by (3.29) \textit{Note: This is an $n\ell$-term}}\\
&= (\neg a \leftand (Q^{\mathsf{U}} \leftor \mathsf{T})) \leftor f^n(P^{\mathsf{T}}) && \text{by La.A.1.3}\\
&= (\neg a \leftand (Q^{\mathsf{U}} \leftor \mathsf{T})) \leftor (f^n(P^{\mathsf{T}}) \leftand \mathsf{F}) && \text{by La.A.1.1}\\
&= (\neg a \leftor (f^n(P^{\mathsf{T}}) \leftand \mathsf{F})) \leftand (Q^{\mathsf{U}} \leftor \mathsf{T}) && \text{by La.A.1.2.2}\\
&= (\neg a \leftor f^n(P^{\mathsf{T}})) \leftand \neg Q^{\mathsf{U}} && \text{by La.A.1.3 and La.A.1.1}\\
&= (\neg a \leftor \neg P^{\mathsf{T}}) \leftand \neg Q^{\mathsf{U}} && \text{by La.A.1.5}\\
&= \neg((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{U}}). && \text{by (F2) and its dual}
\end{aligned}
$$

$f_3^n((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{U}}) = (\neg a \leftand Q^{\mathsf{U}}) \leftor f^n(P^{\mathsf{T}})$ by (3.29), where $f^n(P^{\mathsf{T}})$ is an F-term, so this is an $n\ell$-term. In the case of a $w\ell$-term of the form $(\neg a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{U}}$ the proof proceeds the same, substituting $\neg a$ for a and applying (3.30) and (F3) where needed.

In the case of a $b\ell$-term of the form $(P^{\ell} \leftand Q^{w\ell})$ we have:

$$
\begin{aligned}
f_4^n(P^{\ell} \leftand Q^{w\ell}) &= f_1^n(P^{\ell}) \leftor f_3^n(Q^{w\ell}) && \text{by (3.32) \textit{Note: This is a $b\ell$-term}}\\
&= \neg P^{\ell} \leftor \neg Q^{w\ell} && \text{see above and [7, La.A.2.3]}\\
&= \neg(P^{\ell} \leftand Q^{w\ell}) && \text{by (F2)}
\end{aligned}
$$

In the case of a $b\ell$-term of the form $(P^{\ell} \leftor Q^{n\ell})$:

$$
\begin{aligned}
f_4^n(P^{\ell} \leftor Q^{n\ell}) &= f_1^n(P^{\ell}) \leftand f_2^n(Q^{n\ell}) && \text{by (3.33) \textit{Note: This is a $b\ell$-term}}\\
&= \neg P^{\ell} \leftand \neg Q^{n\ell} && \text{see above and [7, La.A.2.3]}\\
&= \neg(P^{\ell} \leftor Q^{n\ell}) && \text{by (F2)}
\end{aligned}
$$

In the case of a $b\ell$-term of the form $(P^{w\ell} \leftand Q^{\ell})$ we first need to proof that $P^{w\ell} = (P^{w\ell} \leftor \mathsf{T})$:

$$\begin{aligned} f_4^n(P^{w\ell} \curlywedge Q^\ell) &= P^{w\ell} \curlywedge f_1^n(Q^\ell) && \text{by (3.34) } \textit{Note: This is a } b\ell\textit{-term} \\ &= P^{w\ell} \curlywedge \neg Q^\ell && \text{by [7, La.A.2.3]} \\ &= (P^{w\ell} \curlyvee \mathsf{T}) \curlywedge \neg Q^\ell && \text{by La.A.1.4} \\ &= \neg((P^{w\ell} \curlyvee \mathsf{T}) \curlywedge Q^\ell) && \text{by La.A.1.2.3} \\ &= \neg(P^{w\ell} \curlywedge Q^\ell) && \text{by La.A.1.4} \end{aligned}$$

For the inductive step we assume the result holds for all $n*$-terms with fewer literals than: $P^* \curlywedge Q^{n*}$, $P^{w*} \curlywedge Q^{n*}$ and $P^{b*} \curlywedge Q^{n*}$, for all $w*$-terms with fewer literals than: $P^* \curlyvee Q^{w*}$, $P^{n*} \curlyvee Q^{w*}$ and $P^{b*} \curlyvee Q^{w*}$ and for all $b*$-terms with fewer literals than: $P^* \curlywedge P^{w*}, P^{w*} \curlywedge P^*, P^* \curlyvee P^{n*}, P^* \curlywedge P^{b*}, P^{b*} \curlywedge P^*, P^{b*} \curlywedge P^{b*}, P^{w*} \curlywedge P^{b*}, P^{b*} \curlywedge P^{w*}, P^{b*} \curlyvee P^*, P^* \curlyvee P^{b*}, P^{b*} \curlyvee P^{n*}$ and $P^{b*} \curlyvee P^{b*}$.

In the case of an $n*$-term of the form $P^* \curlywedge Q^{n*}$ we get:

$$\begin{aligned} f_2^n(P^* \curlywedge Q^{n*}) &= f_1^n(P^*) \curlyvee f_2^n(Q^{n*}) && \text{by (3.20) } \textit{Note: This is a } w*\textit{-term} \\ &= \neg P^* \curlyvee \neg Q^{n*} && \text{by IH} \\ &= \neg(P^* \curlywedge Q^{n*}) && \text{by dual of (F2)} \end{aligned}$$

The proof for $P^{w*} \curlywedge Q^{n*}$ and $P^{b*} \curlywedge Q^{n*}$ follows the same pattern using (3.21) and (3.22) instead of (3.20).

In the case of a $w*$-term of the form $P^* \curlyvee Q^{w*}$

$$\begin{aligned} f_3^n(P^* \curlyvee Q^{w*}) &= f_1^n(P^*) \curlywedge f_3^n(Q^{w*}) && \text{by (3.26) } \textit{Note: This is an } n*\textit{-term} \\ &= \neg P^* \curlywedge \neg Q^{w*} && \text{by IH} \\ &= \neg(P^* \curlyvee Q^{w*}) && \text{by (F2)} \end{aligned}$$

The proof for $P^{n*} \curlyvee Q^{w*}$ and $P^{b*} \curlyvee Q^{w*}$ follows the same pattern using (3.27) and (3.28) instead of (3.26).

In the case of a $b*$-term of the form $(P^{b*} \curlyvee Q^{n*})$:

$$\begin{aligned} f_4^n(P^{b*} \curlyvee Q^{n*}) &= f_4^n(P^{b*}) \curlywedge f_2^n(Q^{n*}) && \text{by (3.37) } \textit{Note: This is a } b*\textit{-term} \\ &= \neg P^{b*} \curlywedge \neg Q^{n*} && \text{by IH} \\ &= \neg(P^{b*} \curlyvee Q^{n*}) && \text{by (F2)} \end{aligned}$$

The proof for $P^* \curlyvee Q^{n*}$, $P^{b*} \curlyvee Q^*, P^* \curlyvee Q^{b*}$ and $P^{b*} \curlyvee Q^{b*}$ follows the same pattern using(3.33), (3.35), (3.41) and (3.39) instead of (3.37).

In the case of a $b*$-term of the form $(P^{b*} \curlywedge Q^{w*})$ we get:

$$\begin{aligned} f_4^n(P^{b*} \curlywedge Q^{w*}) &= f_4^n(P^{b*}) \curlyvee f_3^n(Q^{w*}) && \text{by (3.43) } \textit{Note: This is a } b*\textit{-term} \\ &= \neg P^{b*} \curlyvee \neg Q^{w*} && \text{By IH} \\ &= \neg(P^{b*} \curlywedge Q^{w*}) && \text{by dual of (F2)} \end{aligned}$$

The proof for $P^* \curlywedge Q^{w*}$, $P^{b*} \curlywedge Q^*, P^* \curlywedge P^{b*}$ and $P^{b*} \curlywedge P^{b*}$ follows the same pattern (3.32), (3.36), (3.42) and (3.38) instead of (3.43).

In the case of a $b*$-term of the form $(P^{w*} \curlywedge Q^{b*})$ we get:

$$\begin{aligned} f_4^n(P^{w*} \curlywedge Q^{b*}) &= P^{w*} \curlywedge f_4^n(Q^{b*}) && \text{by (3.40) } \textit{Note: This is a } b*\textit{-term} \\ &= P^{w*} \curlywedge \neg Q^{b*} && \text{by IH} \\ &= (P^{w*} \curlyvee \mathsf{T}) \curlywedge \neg Q^{b*} && \text{by La.A.1.4} \\ &= \neg((P^{w*} \curlyvee \mathsf{T}) \curlywedge Q^{b*}) && \text{by La.A.1.2.3} \\ &= \neg(P^{w*} \curlywedge Q^{b*}) && \text{by La.A.1.4} \end{aligned}$$

The proof for $P^{w*} \leftand Q^{*}$ follows the same pattern using (3.34) instead of (3.40).

For T-$n*$-terms:

$$
\begin{aligned}
f^n(P^{\mathsf{T}} \leftand Q^{n*}) &= P^{\mathsf{T}} \leftand f_2^n(Q^{n*}) && \text{by (3.19)}\\
&= P^{\mathsf{T}} \leftand \neg Q^{n*} && \text{by IH}\\
&= (P^{\mathsf{T}} \leftor \mathsf{T}) \leftand \neg Q^{n*} && \text{by La.A.1.1}\\
&= \neg((P^{\mathsf{T}} \leftor \mathsf{T}) \leftand Q^{n*}) && \text{by La.A.1.2.3}\\
&= \neg(P^{\mathsf{T}} \leftand \neg Q^{n*}) && \text{by La.A.1.1}
\end{aligned}
$$

The proof of T-$w*$-terms and T-$b*$-terms follows the same pattern using (3.25) and (3.31) instead of (3.19). So f^n on each T-a-term will return a T-a-term. □

Hence, for all $P \in \mathit{SNF}$, $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^n(P) = \neg P$.

A.1.2 Correctness of the conjunction function

Lemma A.1.7. *For all $P \in P^{n*}$ and $Q \in P^{w*}$ $\mathrm{EqFSCL}^{\mathsf{U}} \vdash P \leftor x = \neg P \leftand x$ and $\mathrm{EqFSCL}^{\mathsf{U}} \vdash Q \leftand x = \neg Q \leftor x$*

Proof. In the case of $P^{n*} \leftor x$ the proof is as follows:

$$
\begin{aligned}
P^{n*} \leftor x &= \neg(\neg P^{n*} \leftand \neg x) && \text{by (F2)'}\\
&= \neg((\neg P^{n*} \leftor \mathsf{T}) \leftand \neg x) && \text{by La.A.1.6 and La.A.1.4}\\
&= (\neg P^{n*} \leftor \mathsf{T}) \leftand x && \text{by La.A.1.2.3 and (F3)}\\
&= \neg P^{n*} \leftand x && \text{by La.A.1.6 and La.A.1.4}
\end{aligned}
$$

The proof for $P^{w*} \leftand x$ follows by duality. □

Lemma A.1.8. *For any T-term P and $Q \in \mathit{SNF}$, $f^c(P, Q)$ has the same grammatical category as Q and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \leftand Q.$$

Proof. We will prove this lemma by induction on the complexity of the T-term. For the base case we see that $f^c(\mathsf{T}, P) = P$ by (3.44), which is clearly of the same grammatical category as P. Derivable equality can be done by using (F4). For the inductive step, we assume that the result holds for all T-terms of lesser complexity than $(a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}$. If the second argument is a T-term or an F-term see [7, La.A.2.4]. If the second argument is an U-term, we prove derivable equality as follows:

$$
\begin{aligned}
&f^c((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}, R^{\mathsf{U}})\\
&\quad= (a \leftor f^c(Q^{\mathsf{T}}, R^{\mathsf{U}})) \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{U}}) && \text{by (3.47) \textit{Note this is an} }\mathsf{U}\text{-\textit{term}}\\
&\quad= (a \leftor (Q^{\mathsf{T}} \leftand R^{\mathsf{U}})) \leftand (P^{\mathsf{T}} \leftand R^{\mathsf{U}}) && \text{by IH}\\
&\quad= (a \leftor ((Q^{\mathsf{T}} \leftor \mathsf{T}) \leftand R^{\mathsf{U}})) \leftand\\
&\qquad ((P^{\mathsf{T}} \leftor \mathsf{T}) \leftand R^{\mathsf{U}}) && \text{by La.A.1.1}\\
&\quad= (a \leftor ((Q^{\mathsf{T}} \leftor \mathsf{T}) \leftand (R^{\mathsf{U}} \leftand \mathsf{F}))) \leftand\\
&\qquad ((P^{\mathsf{T}} \leftor \mathsf{T}) \leftand (R^{\mathsf{U}} \leftand \mathsf{F})) && \text{by La.A.1.3}\\
&\quad= ((a \leftand (P^{\mathsf{T}} \leftor \mathsf{T})) \leftor (Q^{\mathsf{T}} \leftor \mathsf{T})) \leftand (R^{\mathsf{U}} \leftand \mathsf{F}) && \text{by La.A.1.2.5}\\
&\quad= ((a \leftand (P^{\mathsf{T}} \leftor \mathsf{T})) \leftor (Q^{\mathsf{T}} \leftor \mathsf{T})) \leftand R^{\mathsf{U}} && \text{by La.A.1.3}\\
&\quad= ((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}) \leftand R^{\mathsf{U}}. && \text{by La.A.1.1}
\end{aligned}
$$

If the second argument is a T-a-term, we prove derivable equality as follows:

$$
\begin{aligned}
f^c((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}, R^{\mathsf{T}} \leftand S^a) &= f^c((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}, R^{\mathsf{T}}) \leftand S^a. && \text{by (3.48)}\\
&= ((a \leftand f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \leftor f^c(Q^{\mathsf{T}}, R^{\mathsf{T}})) \leftand S^a && \text{by (3.45)}\\
&= ((a \leftand (P^{\mathsf{T}} \leftand R^{\mathsf{T}})) \leftor (Q^{\mathsf{T}} \leftand R^{\mathsf{T}})) \leftand S^a && \text{by IH}\\
&= ((a \leftand (P^{\mathsf{T}} \leftand (R^{\mathsf{T}} \leftor \mathsf{T})))\leftor\\
&\quad (Q^{\mathsf{T}} \leftand (R^{\mathsf{T}} \leftor \mathsf{T}))) \leftand S^a && \text{by La.A.1.1}\\
&= (((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}) \leftand (R^{\mathsf{T}} \leftor \mathsf{T})) \leftand S^a && \text{by La.A.1.2.4}\\
&= (((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}) \leftand R^{\mathsf{T}}) \leftand S^a. && \text{by La.A.1.1}\\
&= ((a \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{T}}) \leftand (R^{\mathsf{T}} \leftand S^a). && \text{by (F7)}
\end{aligned}
$$

□

Lemma A.1.9. *For any* F*-term* P *and* $Q \in SNF$*,* $f^c(P,Q)$ *is an* F*-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. The grammatical result is immediate by (3.49) and the claim about derivable equality follows from Lemma A.1.1, (F7) and (F6) □

Lemma A.1.10. *For any* U*-term* P *and* $Q \in SNF$*,* $f^c(P,Q)$ *is an* U*-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. The grammatical result is immediate by (3.50) and the claim about derivable equality follows from Lemma A.1.3, (F7) and (FU2) □

Lemma A.1.11. *For any* T*-$*$-term* P *and* T*-term* Q*,* $f^c(P,Q)$ *is a* T*-$*$-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. See [7, La.A.2.6]: f^c_{1a} as defined in (3.52)-(3.55) is the same as f^c_1 defined in that paper. □

Lemma A.1.12. *For any* T*-$*$-term* P *and* F*-term* Q*,* $f^c(P,Q)$ *is an* F*-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. See [7, La.A.2.7]: f^c_{1b} as defined in (3.57)-(3.60) is the same as f^c_2 defined in that paper. □

Lemma A.1.13. *For any* T*-$*$-term* P *and* T*-$*$-term* Q*,* $f^c(P,Q)$ *is a* T*-$*$-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. See [7, La.A.2.8]: f^c_{1d} as defined in (3.67)-(3.69) is the same as f^c_3 defined in that paper. □

Lemma A.1.14. *For any* T*-$*$-term* P *and* U*-term* Q*,* $f^c(P,Q)$ *is a* T*-$n*$-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. By (3.61), Lemma A.1.8 and (F7) it suffices to prove that $f^c_{1c}(P^*, Q^{\mathsf{U}})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{1c}(P^*, Q^{\mathsf{U}}) = (P^* \leftand Q^{\mathsf{U}})$. This will be proven by induction on the number of ℓ-terms in P^*.

In the case of an ℓ-term of the form $(\hat{a} \leftand P^{\mathsf{T}}) \leftor Q^{\mathsf{F}}$ with $\hat{a} \in \{a, \neg a\}$ we get:

$$
\begin{aligned}
& f^c_{1c}((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{F}}, R^{\mathsf{U}}) \\
&= (\hat{a} \circ\!\wedge f^c(P^{\mathsf{T}}, R^{\mathsf{U}})) \circ\!\vee Q^{\mathsf{F}} && \text{by (3.62), (3.63) } \textit{Note: This is an } n\ell\textit{-term} \\
&= (\hat{a} \circ\!\wedge (P^{\mathsf{T}} \circ\!\wedge R^{\mathsf{U}})) \circ\!\vee Q^{\mathsf{F}} && \text{by La.A.1.8} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\wedge R^{\mathsf{U}}) \circ\!\vee Q^{\mathsf{F}} && \text{by (F7)} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\wedge (R^{\mathsf{U}} \circ\!\vee \mathsf{T})) \circ\!\vee (Q^{\mathsf{F}} \circ\!\wedge \mathsf{F}) && \text{by La.A.1.3 and La.A.1.1} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee (Q^{\mathsf{F}} \circ\!\wedge \mathsf{F})) \circ\!\wedge (R^{\mathsf{U}} \circ\!\vee \mathsf{T}) && \text{by La.A.1.2.2} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{F}}) \circ\!\wedge R^{\mathsf{U}}. && \text{by La.A.1.3 and La.A.1.1}
\end{aligned}
$$

For the inductive step we assume the result holds for all $*$-terms with less ℓ terms than $P^* \circ\!\wedge P^d$ and $P^* \circ\!\vee P^c$.

$$
\begin{aligned}
f^c_{1c}(P^* \circ\!\vee Q^c, R^{\mathsf{U}}) &= f^n_2(f^c_{1c}(P^*, R^{\mathsf{U}})) \circ\!\wedge f^c_{1c}(Q^c, R^{\mathsf{U}}) && \text{by (3.65) } \textit{Note: This is an } n*\textit{-term} \\
&= f^c_{1c}(P^*, R^{\mathsf{U}}) \circ\!\vee f^c_{1c}(Q^c, R^{\mathsf{U}}) && \text{by La.A.1.6 and La.A.1.7} \\
&= (P^* \circ\!\wedge R^{\mathsf{U}}) \circ\!\vee (Q^c \circ\!\wedge R^{\mathsf{U}}) && \text{by IH} \\
&= (P^* \circ\!\wedge (R^{\mathsf{U}} \circ\!\vee \mathsf{T})) \circ\!\vee (Q^c \circ\!\wedge (R^{\mathsf{U}} \circ\!\vee \mathsf{T})) && \text{by La.A.1.3} \\
&= (P^* \circ\!\vee Q^c) \circ\!\wedge (R^{\mathsf{U}} \circ\!\vee \mathsf{T}) && \text{by (F10)'} \\
&= (P^* \circ\!\vee Q^c) \circ\!\wedge R^{\mathsf{U}} && \text{by La.A.1.3}
\end{aligned}
$$

For $P^* \circ\!\wedge P^d$ the proof follows directly from (3.64), the induction hypothesis and (F7). □

Lemma A.1.15. *For any* T-$n*$*-term* P *and* $Q \in$ *SNF,* $f^c(P, Q)$ *is a* T-$n*$*-term and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \circ\!\wedge Q.$$

Proof. The gramatical result follows from (3.90) and the claim about derivable equality follows from Lemma A.1.4 and (F7). □

Lemma A.1.16. *For any* T-$b*$*-term and* T-$w*$*-term* P *and* T*-term* Q*,* $f^c(P, Q)$ *has the same grammatical category as* P *and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \circ\!\wedge Q.$$

Proof. For T-$w*$-terms: By (3.91) and (F7) it suffices to prove that $f^c_{3a}(P^{w*}, Q^{\mathsf{T}})$ is a $w*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) = (P^{w*} \circ\!\wedge Q^{\mathsf{T}})$. For T-$b*$-terms: By (3.133), Lemma A.1.8 and (F7) it suffices to prove that $f^c_{4a}(P^{b*}, Q^{\mathsf{T}})$ is a $b*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) = (P^{b*} \circ\!\wedge Q^{\mathsf{T}})$. This will be proven by induction on the number of *literals* in P^{w*} and P^{b*}.

In the case of a $w\ell$-term of the form $(\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{U}}$ with $\hat{a} \in \{a, \neg a\}$ we get:

$$
\begin{aligned}
& f^c_{3a}((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{U}}, R^{\mathsf{T}}) \\
&= (\hat{a} \circ\!\wedge f^c(P^{\mathsf{T}}, R^{\mathsf{T}})) \circ\!\vee Q^{\mathsf{U}} && \text{by (3.92), (3.93) } \textit{Note: This is a } w\ell\textit{-term} \\
&= (\hat{a} \circ\!\wedge (P^{\mathsf{T}} \circ\!\wedge R^{\mathsf{T}})) \circ\!\vee Q^{\mathsf{U}} && \text{by La.A.1.8} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\wedge R^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{U}} && \text{by (F7)} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\wedge (R^{\mathsf{T}} \circ\!\vee \mathsf{T})) \circ\!\vee (Q^{\mathsf{U}} \circ\!\wedge \mathsf{F}) && \text{by La.A.1.3 and La.A.1.1} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee (Q^{\mathsf{U}} \circ\!\wedge \mathsf{F})) \circ\!\wedge (R^{\mathsf{T}} \circ\!\vee \mathsf{T}) && \text{by La.A.1.2.2} \\
&= ((\hat{a} \circ\!\wedge P^{\mathsf{T}}) \circ\!\vee Q^{\mathsf{U}}) \circ\!\wedge R^{\mathsf{T}}. && \text{by La.A.1.3 and La.A.1.1}
\end{aligned}
$$

In the case of a $b\ell$-term of the form $(P^{\ell} \circ\!\wedge Q^{w\ell})$ we get:

$$
\begin{aligned}
f_{4a}^c(P^\ell \leftand Q^{w\ell}, V^{\mathsf{T}}) &= P^\ell \leftand f_{3a}^c(Q^{w\ell}, V^{\mathsf{T}}) && \text{by (3.134) } \textit{Note: This is a } b\ell\textit{-term}\\
&= P^\ell \leftand (Q^{w\ell} \leftand V^{\mathsf{T}}) && \text{see } w\ell\text{-term above}\\
&= (P^\ell \leftand Q^{w\ell}) \leftand V^{\mathsf{T}} && \text{by (F7)}
\end{aligned}
$$

In the case of a $b\ell$-term of the form $(P^{w\ell} \leftand Q^\ell)$ the proof follows the same pattern using (3.136) and Lemma A.1.11.

In the case of a $b\ell$-term of the form $(P^\ell \leftor Q^{n\ell})$ we get:

$$
\begin{aligned}
& f_{4a}^c(P^\ell \leftor Q^{n\ell}, V^{\mathsf{T}}) && \\
&= f_{1a}^c(P^\ell, V^{\mathsf{T}}) \leftor Q^{n\ell} && \text{by (3.135) } \textit{Note: This is a } b\ell\textit{-term}\\
&= (P^\ell \leftand V^{\mathsf{T}}) \leftor Q^{n\ell} && \text{by La.A.1.11}\\
&= (P^\ell \leftand V^{\mathsf{T}}) \leftor (Q^{n\ell} \leftand \mathsf{F}) && \text{by La.A.1.4}\\
&= (P^\ell \leftor (Q^{n\ell} \leftand \mathsf{F})) \leftand (V^{\mathsf{T}} \leftor (Q^{n\ell} \leftand \mathsf{F})) && \text{by (F10)}\\
&= (P^\ell \leftor Q^{n\ell}) \leftand V^{\mathsf{T}} && \text{by La.A.1.1 and La.A.1.4}
\end{aligned}
$$

For the inductive step we assume the result holds for all $w*$-terms of lesser complexity than: $P^* \leftor Q^{w*}$, $P^{n*} \leftor Q^{w*}$ and $P^{b*} \leftor Q^{w*}$ and for all $b*$-terms of lesser complexity than: $P^* \leftand P^{w*}, P^{w*} \leftand P^*, P^* \leftor P^{n*}, P^* \leftand P^{b*}, P^{b*} \leftand P^*, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}, P^{b*} \leftand P^{w*}$, $P^{b*} \leftor P^*, P^* \leftor P^{b*}, P^{b*} \leftor P^{n*}$ and $P^{b*} \leftor P^{b*}$.

In the case of a $w*$-term of the form $P^* \leftor Q^{w*}$ we get:

$$
\begin{aligned}
& f_{3a}^c(P^* \leftor Q^{w*}, R^{\mathsf{T}}) && \\
&= f_{1a}^c(P^*, R^{\mathsf{T}}) \leftor f_{3a}^c(Q^{w*}, R^{\mathsf{T}}) && \text{by (3.94) } \textit{Note: This is a } w*\textit{-term}\\
&= (P^* \leftand R^{\mathsf{T}}) \leftor (Q^{w*} \leftand R^{\mathsf{T}}) && \text{by IH and La.A.1.11}\\
&= (P^* \leftand (R^{\mathsf{T}} \leftor \mathsf{T})) \leftor (Q^{w*} \leftand (R^{\mathsf{T}} \leftor \mathsf{T})) && \text{by La.A.1.1}\\
&= (P^* \leftor Q^{w*}) \leftand (R^{\mathsf{T}} \leftor \mathsf{T}) && \text{by (F10)'}\\
&= (P^* \leftor Q^{w*}) \leftand R^{\mathsf{T}} && \text{by La.A.1.1}
\end{aligned}
$$

The proof for a $w*$-term of the form $P^{b*} \leftor Q^{w*}$ has a similar patter using (3.96) instead of (3.94). The proof for the $b*$-terms $P^{b*} \leftor Q^{b*}$, $P^{b*} \leftor Q^*$, $P^* \leftor Q^{b*}$ also has a similar pattern using (3.141), (3.137) and (3.143) instead of (3.94).

In the case of a $w*$-term of the form $P^{n*} \leftor Q^{w*}$

$$
\begin{aligned}
& f_{3a}^c(P^{n*} \leftor Q^{w*}, R^{\mathsf{T}}) && \\
&= P^{n*} \leftor f_{3a}^c(Q^{w*}, R^{\mathsf{T}}) && \text{by (3.95) } \textit{Note: This is a } w*\textit{-term}\\
&= P^{n*} \leftor (Q^{w*} \leftand R^{\mathsf{T}}) && \text{by IH}\\
&= (P^{n*} \leftand (R^{\mathsf{T}} \leftor \mathsf{T})) \leftor (Q^{w*} \leftand (R^{\mathsf{T}} \leftor \mathsf{T})) && \text{By La.A.1.4 and La.A.1.1}\\
&= (P^{n*} \leftor Q^{w*}) \leftand (R^{\mathsf{T}} \leftor \mathsf{T}) && \text{By (F10)'}\\
&= (P^{n*} \leftor Q^{w*}) \leftand R^{\mathsf{T}} && \text{By La.A.1.1}
\end{aligned}
$$

In the case of a $b*$-term of the form $P^* \leftor Q^{n*}$ or $P^{b*} \leftor Q^{n*}$ the proof has the same pattern using (3.135) and (3.139) instead of (3.95).The proof for $P^* \leftand P^{w*}, P^{w*} \leftand P^*, P^{b*} \leftand P^*, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}$, $P^{b*} \leftand P^{w*}$ and $P^* \leftand P^{b*}$ follows directly from (3.134), (3.136), (3.138), (3.140), (3.142), (3.144), (3.145) the induction hypothesis, A.1.11 and F7.

□

Lemma A.1.17. *For any* T-$b*$*-term or* T-$w*$*-term* P *and* F*-term* Q, $f^c(P, Q)$ *is a* T-$n*$*-term and*

$$
\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \leftand Q.
$$

Proof. For T-$w*$-terms: By (3.97) and (F7) it suffices to prove that $f^c_{3b}(P^{w*}, Q^{\mathsf{F}})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{3b}(P^{w*}, Q^{\mathsf{F}}) = (P^{w*} \mathbin{\circ\!\!\!\wedge} Q^{\mathsf{F}})$. For T-$b*$-terms: By (3.146) and (F7) it suffices to prove that $f^c_{4b}(P^{b*}, Q^{\mathsf{F}})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{4b}(P^{b*}, Q^{\mathsf{F}}) = (P^{b*} \mathbin{\circ\!\!\!\wedge} Q^{\mathsf{F}})$. This will be proven by induction on the number of *literals* in P^{w*} and P^{b*}.

In the case of a $w\ell$-term of the form $(\hat{a} \mathbin{\circ\!\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\!\vee} Q^{\mathsf{U}}$ with $\hat{a} \in \{a, \neg a\}$ we get:

$$\begin{aligned}
f^c_{3b}((\hat{a} \mathbin{\circ\!\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\!\vee} Q^{\mathsf{U}}, R^{\mathsf{F}}) &= (\hat{a} \mathbin{\circ\!\!\!\vee} Q^{\mathsf{U}}) \mathbin{\circ\!\!\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{F}}) && \text{by (3.98), (3.99) \textit{Note: This is an $n\ell$-term}}\\
&= (\hat{a} \mathbin{\circ\!\!\!\vee} Q^{\mathsf{U}}) \mathbin{\circ\!\!\!\wedge} (P^{\mathsf{T}} \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}}) && \text{by La.A.1.8}\\
&= ((\hat{a} \mathbin{\circ\!\!\!\vee} Q^{\mathsf{U}}) \mathbin{\circ\!\!\!\wedge} P^{\mathsf{T}}) \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}} && \text{by (F7)}
\end{aligned}$$

In the case of a $b\ell$-term of the form $(P^{\ell} \mathbin{\circ\!\!\!\wedge} Q^{w\ell})$ the equality follows directly from the $w\ell$ term above and (3.147). In the case of a $b\ell$-term of the form $(P^{w\ell} \mathbin{\circ\!\!\!\wedge} Q^{\ell})$ proof follows directly from A.1.12, the $w\ell$ term above and (3.149). In the case of a $b\ell$-term of the form $(P^{\ell} \mathbin{\circ\!\!\!\vee} Q^{n\ell})$ we get:

$$\begin{aligned}
f^c_{4b}((P^{\ell} \mathbin{\circ\!\!\!\vee} Q^{n\ell}), R^{\mathsf{F}}) &= f^n_1(f^c_{1a}(P^{\ell}, f^n(R^{\mathsf{F}}))) \mathbin{\circ\!\!\!\wedge} Q^{n\ell} && \text{by (3.148) \textit{Note: This is an $n\ell$-term}}\\
&= f^n_1(P^{\ell} \mathbin{\circ\!\!\!\wedge} f^n(R^{\mathsf{F}})) \mathbin{\circ\!\!\!\wedge} Q^{n\ell} && \text{by La.A.1.11}\\
&= (\neg P^{\ell} \mathbin{\circ\!\!\!\vee} R^{\mathsf{F}}) \mathbin{\circ\!\!\!\wedge} Q^{n\ell} && \text{by La.A.1.5, La.A.1.6 and (F2)}\\
&= (\neg P^{\ell} \mathbin{\circ\!\!\!\vee} (R^{\mathsf{F}} \mathbin{\circ\!\!\!\wedge} \mathsf{F})) \mathbin{\circ\!\!\!\wedge} (Q^{n\ell} \mathbin{\circ\!\!\!\wedge} \mathsf{F}) && \text{by La.A.1.1 and La.A.1.4}\\
&= (P^{\ell} \mathbin{\circ\!\!\!\vee} (Q^{n\ell} \mathbin{\circ\!\!\!\wedge} \mathsf{F})) \mathbin{\circ\!\!\!\wedge} (R^{\mathsf{F}} \mathbin{\circ\!\!\!\wedge} \mathsf{F}) && \text{by La.A.1.2.1}\\
&= (P^{\ell} \mathbin{\circ\!\!\!\vee} Q^{n\ell}) \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}} && \text{by La.A.1.1 and La.A.1.4}
\end{aligned}$$

For the inductive step we assume the result holds for all $w*$-terms of lesser complexity than: $P^* \mathbin{\circ\!\!\!\vee} Q^{w*}$, $P^{n*} \mathbin{\circ\!\!\!\vee} Q^{w*}$ and $P^{b*} \mathbin{\circ\!\!\!\vee} Q^{w*}$ and for all $b*$-terms of lesser complexity than: $P^* \mathbin{\circ\!\!\!\wedge} P^{w*}, P^{w*} \mathbin{\circ\!\!\!\wedge} P^*, P^* \mathbin{\circ\!\!\!\vee} P^{n*}, P^* \mathbin{\circ\!\!\!\wedge} P^{b*}, P^{b*} \mathbin{\circ\!\!\!\wedge} P^*, P^{b*} \mathbin{\circ\!\!\!\wedge} P^{b*}, P^{w*} \mathbin{\circ\!\!\!\wedge} P^{b*}, P^{b*} \mathbin{\circ\!\!\!\wedge} P^{w*}, P^{b*} \mathbin{\circ\!\!\!\vee} P^*, P^* \mathbin{\circ\!\!\!\vee} P^{b*}, P^{b*} \mathbin{\circ\!\!\!\vee} P^{n*}$ and $P^{b*} \mathbin{\circ\!\!\!\vee} P^{b*}$.

In the case of a $w*$-term of the form $P^* \mathbin{\circ\!\!\!\vee} Q^{w*}$ we get:

$$\begin{aligned}
&f^c_{3b}(P^* \mathbin{\circ\!\!\!\vee} Q^{w*}, R^{\mathsf{F}})\\
&= f^n_1(f^c_{1a}(P^*, f^n(R^{\mathsf{F}}))) \mathbin{\circ\!\!\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) && \text{by (3.100) \textit{Note: This is an n*-term}}\\
&= f^n_1(P^* \mathbin{\circ\!\!\!\wedge} f^n(R^{\mathsf{F}})) \mathbin{\circ\!\!\!\wedge} (Q^{w*} \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}}) && \text{by IH and La.A.1.11}\\
&= (\neg P^* \mathbin{\circ\!\!\!\vee} R^{\mathsf{F}}) \mathbin{\circ\!\!\!\wedge} (Q^{w*} \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}}) && \text{by La.A.1.5, La.A.1.6 and (F2)}\\
&= (\neg P^* \mathbin{\circ\!\!\!\vee} (R^{\mathsf{F}} \mathbin{\circ\!\!\!\wedge} \mathsf{F})) \mathbin{\circ\!\!\!\wedge} (Q^{w*} \mathbin{\circ\!\!\!\wedge} (R^{\mathsf{F}} \mathbin{\circ\!\!\!\wedge} \mathsf{F})) && \text{by La.A.1.1}\\
&= (P^* \mathbin{\circ\!\!\!\vee} Q^{w*}) \mathbin{\circ\!\!\!\wedge} (R^{\mathsf{F}} \mathbin{\circ\!\!\!\wedge} \mathsf{F}) && \text{by [7, La.2.1.6]}\\
&= (P^* \mathbin{\circ\!\!\!\vee} Q^{w*}) \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}} && \text{by La.A.1.1}
\end{aligned}$$

In the case of a $w*$-term of the form $P^{b*} \mathbin{\circ\!\!\!\vee} Q^{w*}$ the proof follows the same pattern, using (3.102) instead of (3.100) and using Lemma A.1.16 instead of A.1.11.

In the case of a $w*$-term of the form $P^{n*} \mathbin{\circ\!\!\!\vee} Q^{w*}$ we get:

$$\begin{aligned}
f^c_{3b}(P^{n*} \mathbin{\circ\!\!\!\vee} Q^{w*}, R^{\mathsf{F}}) &= f^n_2(P^{n*}) \mathbin{\circ\!\!\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) && \text{by (3.101) \textit{Note: This is an $n*$-term}}\\
&= \neg P^{n*} \mathbin{\circ\!\!\!\wedge} f^c_{3b}(Q^{w*}, R^{\mathsf{F}}) && \text{by La.A.1.6}\\
&= \neg P^{n*} \mathbin{\circ\!\!\!\wedge} (Q^{w*} \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}}) && \text{by IH}\\
&= (\neg P^{n*} \mathbin{\circ\!\!\!\wedge} Q^{w*}) \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}} && \text{by (F7)}\\
&= (P^{n*} \mathbin{\circ\!\!\!\vee} Q^{w*}) \mathbin{\circ\!\!\!\wedge} R^{\mathsf{F}} && \text{by La.A.1.7}
\end{aligned}$$

In the case of a $b*$-term of the form $P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{*}$ we get:

$$\begin{aligned}
&f^c_{4b}(P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{*}, R^{\mathsf{F}}) && \\
&= f^c_{4b}(f^n_4(f^c_{4a}(P^{b*}, f^n(R^{\mathsf{F}}))), f^c_{1b}(Q^{*}, R^{\mathsf{F}})) && \text{by (3.150) } \textit{Note: This is an } n*\textit{-term} \\
&= f^c_{4b}(f^n_4(P^{b*} \mathbin{{}^{\circ}\!\wedge} f^n(R^{\mathsf{F}})), (Q^{*} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{F}})) && \text{by La.A.1.16 and La.A.1.12} \\
&= f^n_4(P^{b*} \mathbin{{}^{\circ}\!\wedge} f^n(R^{\mathsf{F}})) \mathbin{{}^{\circ}\!\wedge} (Q^{*} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{F}}) && \text{by IH} \\
&= (\neg P^{b*} \mathbin{{}^{\circ}\!\vee} R^{\mathsf{F}}) \mathbin{{}^{\circ}\!\wedge} (Q^{*} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{F}}) && \text{by La.A.1.5, La.A.1.6 and (F2)} \\
&= (\neg P^{b*} \mathbin{{}^{\circ}\!\vee} (R^{\mathsf{F}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} (Q^{*} \mathbin{{}^{\circ}\!\wedge} (R^{\mathsf{F}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) && \text{by La. A.1.1} \\
&= (P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{*}) \mathbin{{}^{\circ}\!\wedge} (R^{\mathsf{F}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) && \text{by [7, La.2.1.6]} \\
&= (P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{*}) \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{F}} && \text{by La. A.1.1}
\end{aligned}$$

The proof for $P^{b*} \mathbin{{}^{\circ}\!\vee} P^{b*}$ and $P^{*} \mathbin{{}^{\circ}\!\vee} P^{b*}$ follows the same pattern using (3.154) and (3.156) instead of (3.150). The proof for $P^{b*} \mathbin{{}^{\circ}\!\vee} P^{n*}$ and $P^{*} \mathbin{{}^{\circ}\!\vee} P^{n*}$ also follows the same pattern using (3.152) and (3.148) instead of (3.150) and using Lemma A.1.7. The proof for $P^{*} \mathbin{{}^{\circ}\!\wedge} P^{w*}, P^{w*} \mathbin{{}^{\circ}\!\wedge} P^{*}, P^{b*} \mathbin{{}^{\circ}\!\wedge} P^{*}, P^{b*} \mathbin{{}^{\circ}\!\wedge} P^{b*}, P^{w*} \mathbin{{}^{\circ}\!\wedge} P^{b*}$, $P^{b*} \mathbin{{}^{\circ}\!\wedge} P^{w*}$ and $P^{*} \mathbin{{}^{\circ}\!\wedge} P^{b*}$ follows directly from (3.147), (3.149), (3.151), (3.153), (3.155), (3.157), (3.158) the induction hypothesis, A.1.12 and F7. □

Lemma A.1.18. *For any* T*-*$b*$*-term* P *and* U*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$n*$*-term and for any* T*-*$w*$*-term* P *and* U*-term* Q*,* $f^c(P, Q)$ *is an* U*-term and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \mathbin{{}^{\circ}\!\wedge} Q.$$

Proof. For T-$w*$-terms: By (3.103), Lemma A.1.8 and (F7) it suffices to prove that $f^c_{3c}(P^{w*}, Q^{\mathsf{U}})$ is an U-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{3c}(P^{w*}, Q^{\mathsf{U}}) = (P^{w*} \mathbin{{}^{\circ}\!\wedge} Q^{\mathsf{F}})$. For T-$b*$-terms: By (3.159) and (F7) it suffices to prove that $f^c_{4c}(P^{b*}, Q^{\mathsf{U}})$ is an $n*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{4c}(P^{b*}, Q^{\mathsf{U}}) = (P^{b*} \mathbin{{}^{\circ}\!\wedge} Q^{\mathsf{U}})$. This will be proven by induction on the number of *literals* in P^{w*} and P^{b*}.

In the case of a $w\ell$-term of the form $(\hat{a} \mathbin{{}^{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}$ with $\hat{a} \in \{a, \neg a\}$ we get:

$$\begin{aligned}
&f^c_{3c}((\hat{a} \mathbin{{}^{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}, R^{\mathsf{U}}) && \\
&= (\hat{a} \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} f^c(P^{\mathsf{T}}, R^{\mathsf{U}}) && \text{by (3.104), (3.105) } \textit{Note: This is an } \mathsf{U}\textit{-term} \\
&= (\hat{a} \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} (P^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}}) && \text{by Lemma A.1.8} \\
&= ((\hat{a} \mathbin{{}^{\circ}\!\vee} Q^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} P^{\mathsf{T}}) \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}} && \text{by (F7)}
\end{aligned}$$

In the case of a $b\ell$-term of the form $(P^{\ell} \mathbin{{}^{\circ}\!\wedge} Q^{w\ell})$ the equality follows directly from the proof for the $w\ell$-term above, (F7) and (3.160). In the case of a $b\ell$-term of the form $(P^{w\ell} \mathbin{{}^{\circ}\!\wedge} Q^{\ell})$ proof follows directly from A.1.14, (F7) and (3.162).

In the case of a $b\ell$-term of the form $(P^{\ell} \mathbin{{}^{\circ}\!\vee} Q^{n\ell})$ we get:

$$\begin{aligned}
&f^c_{4c}((P^{\ell} \mathbin{{}^{\circ}\!\vee} Q^{n\ell}), R^{\mathsf{U}}) && \\
&= f^n_2(f^c_{1c}(P^{\ell}, R^{\mathsf{U}})) \mathbin{{}^{\circ}\!\wedge} Q^{n\ell} && \text{by (3.161) } \textit{Note: This is an } n*\textit{-term} \\
&= f^n_2(P^{\ell} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} Q^{n\ell} && \text{by La.A.1.11} \\
&= (\neg P^{\ell} \mathbin{{}^{\circ}\!\vee} R^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} Q^{n\ell} && \text{by La.A.1.5, La.A.1.6, (F2) and La.A.1.3} \\
&= (\neg P^{\ell} \mathbin{{}^{\circ}\!\vee} (R^{\mathsf{U}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} (Q^{n\ell} \mathbin{{}^{\circ}\!\wedge} (R^{\mathsf{U}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) && \text{by La.A.1.3 and La.A.1.4} \\
&= (\neg P^{\ell} \mathbin{{}^{\circ}\!\vee} (R^{\mathsf{U}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} ((Q^{n\ell} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) && \text{by (F6)} \\
&= (P^{\ell} \mathbin{{}^{\circ}\!\vee} ((Q^{n\ell} \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}}) \mathbin{{}^{\circ}\!\wedge} \mathsf{F})) \mathbin{{}^{\circ}\!\wedge} (R^{\mathsf{U}} \mathbin{{}^{\circ}\!\wedge} \mathsf{F}) && \text{by La. A.1.2.1} \\
&= (P^{\ell} \mathbin{{}^{\circ}\!\vee} Q^{n\ell}) \mathbin{{}^{\circ}\!\wedge} R^{\mathsf{U}} && \text{by La.A.1.3 and La.A.1.4}
\end{aligned}$$

For the inductive step we assume the result holds for all $w*$-terms of lesser complexity than: $P^{*} \mathbin{{}^{\circ}\!\vee} Q^{w*}$, $P^{n*} \mathbin{{}^{\circ}\!\vee} Q^{w*}$ and $P^{b*} \mathbin{{}^{\circ}\!\vee} Q^{w*}$ and for all $b*$-terms of lesser complexity than: $P^{*} \mathbin{{}^{\circ}\!\wedge}$

$P^{w*}, P^{w*} \leftand P^*, P^* \leftor P^{n*}, P^* \leftand P^{b*}, P^{b*} \leftand P^*, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}, P^{b*} \leftand P^{w*}, P^{b*} \leftor P^*, P^* \leftor P^{b*}, P^{b*} \leftor P^{n*}$ and $P^{b*} \leftor P^{b*}$.

In the case of a $w*$-term of the form $P^* \leftor Q^{w*}$:

$f^c_{3c}(P^* \leftor Q^{w*}, R^{\mathsf{U}})$	
$= f^c_{3c}(f^n_2(f^c_{1c}(P^*, R^{\mathsf{U}})), f^c_{3c}(Q^{w*}, R^{\mathsf{U}}))$	by (3.106) *Note: This is an* U*-term*
$= f^n_2(f^c_{1c}(P^*, R^{\mathsf{U}})) \leftand (Q^{w*} \leftand R^{\mathsf{U}})$	By IH
$= f^n_2(P^* \leftand R^{\mathsf{U}}) \leftand (Q^{w*} \leftand R^{\mathsf{U}})$	by La.A.1.14
$= (\neg P^* \leftor R^{\mathsf{U}}) \leftand (Q^{w*} \leftand R^{\mathsf{U}})$	by La.A.1.6, F2 and La.A.1.3
$= (\neg P^* \leftor (R^{\mathsf{U}} \leftand \mathsf{F})) \leftand (Q^{w*} \leftand (R^{\mathsf{U}} \leftand \mathsf{F}))$	by La.A.1.3
$= (P^* \leftor Q^{w*}) \leftand (R^{\mathsf{U}} \leftand \mathsf{F})$	by [7, La.2.1.6]
$= (P^* \leftor Q^{w*}) \leftand R^{\mathsf{U}}$	by La.A.1.3

In the case of a $w*$-term of the form $P^{b*} \leftor Q^{w*}$ the proof proceeds in a similar manner, using (3.108) instead of (3.106) and using the induction hypothesis instead of Lemma A.1.14.

In the case of a $w*$-term of the form $P^{n*} \leftor Q^{w*}$ we get:

$f^c_{3c}(P^{n*} \leftor Q^{w*}, R^{\mathsf{U}}) = f^c_{3c}(f^n_2(P^{n*}), f^c_{3c}(Q^{w*}, R^{\mathsf{U}}))$	by (3.107) *Note: This is an* U*-term*
$= f^n_2(P^{n*}) \leftand (Q^{w*} \leftand R^{\mathsf{U}})$	By IH
$= \neg P^{n*} \leftand (Q^{w*} \leftand R^{\mathsf{U}})$	by La.A.1.6
$= (\neg P^{n*} \leftand Q^{w*}) \leftand R^{\mathsf{U}}$	by (F7)
$= (P^{n*} \leftor Q^{w*}) \leftand R^{\mathsf{U}}$	by La.A.1.7

In the case of a $b*$-term of the form $P^{b*} \leftor Q^*$ we get:

$f^c_{4c}(P^{b*} \leftor Q^*, R^{\mathsf{U}})$	
$= f^n_4(f^c_{4c}(P^{b*}, R^{\mathsf{U}})) \leftand f^c_{1c}(Q^*, R^{\mathsf{U}})$	by (3.163) *Note: This is an* $n*$*-term*
$= f^n_4(P^{b*} \leftand R^{\mathsf{U}}) \leftand (Q^* \leftand R^{\mathsf{U}})$	by IH and La.A.1.14
$= (\neg P^{b*} \leftor R^{\mathsf{U}}) \leftand (Q^* \leftand R^{\mathsf{U}})$	by La.A.1.5, La.A.1.6, La.A.1.3 and (F2)
$= (\neg P^{b*} \leftor (R^{\mathsf{U}} \leftand \mathsf{F})) \leftand (Q^* \leftand (R^{\mathsf{U}} \leftand \mathsf{F}))$	by La. A.1.3
$= (P^{b*} \leftor Q^*) \leftand (R^{\mathsf{U}} \leftand \mathsf{F})$	by [7, La.2.1.6]
$= (P^{b*} \leftor Q^*) \leftand R^{\mathsf{U}}$	by La. A.1.3

The proof for $P^{b*} \leftor P^{b*}$ and $P^* \leftor P^{b*}$ follows the same pattern using (3.167) and (3.169) instead of (3.163). The proof for $P^{b*} \leftor P^{n*}$ and $P^* \leftor P^{n*}$ also follows the same pattern using (3.165) and (3.161) instead of (3.163) and using (A.1.7). The proof for $P^* \leftand P^{w*}, P^{w*} \leftand P^*, P^{b*} \leftand P^*, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}$, $P^{b*} \leftand P^{w*}$ and $P^* \leftand P^{b*}$ follows directly from (3.160), (3.162), (3.164), (3.166), (3.168), (3.170),(3.171) the induction hypothesis, A.1.14 and F7.

□

Lemma A.1.19. *For any* T*-*$*$*-term* P *and* T*-*$w*$*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$b*$*-term and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \leftand Q.$$

Proof. By (3.75) and (F7) it suffices to prove that $f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand R^{w*})$ is a $b*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{1f}(P^*, Q^{\mathsf{T}} \leftand R^{w*}) = P^* \leftand Q^{\mathsf{T}} \leftand R^{w*})$. The proof follows directly from (3.76), La.A.1.11 and (F7).

□

Lemma A.1.20. *For any* T*-*$*$*-term* P *and* T*-*$b*$*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$b*$*-term and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \leftand Q.$$

Proof. By (3.77) and (F7) it suffices to prove that $f^c_{1g}(P^*, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{b*})$ is a $b*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{1g}(P^*, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{b*}) = (P^* \mathbin{{}^{\circ}\!\wedge} Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{b*})$. In the case of a $b*$-term of the form $R^* \mathbin{{}^{\circ}\!\vee} S^{n*}$, $R^{b*} \mathbin{{}^{\circ}\!\vee} S^*$, $R^{b*} \mathbin{{}^{\circ}\!\vee} S^{n*}$, $R^{b*} \mathbin{{}^{\circ}\!\vee} S^{b*}$ or $R^* \mathbin{{}^{\circ}\!\vee} S^{b*}$ the proof follows directly from (3.79), (3.81), (3.83), (3.85), (3.87), Lemma A.1.11 and (F7). In the case of a $b*$-term of the form $R^* \mathbin{{}^{\circ}\!\wedge} S^{w*}$ or $R^* \mathbin{{}^{\circ}\!\wedge} S^{b*}$ the proof follows directly from (3.78), (3.88), Lemma A.1.13 and (F7). In the case of a $b*$-term of the form $R^{w*} \mathbin{{}^{\circ}\!\wedge} S^*$ and $R^{w*} \mathbin{{}^{\circ}\!\wedge} S^{b*}$ the proof follows directly from (3.80), (3.86), Lemma A.1.19 and (F7).

In the case of a $b*$-term of the form $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^{w*}$, $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^{b*}$ or $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^*$ we need to use induction on the number of $b\ell$-terms in $b*$. For the base case we have $R^\ell \mathbin{{}^{\circ}\!\vee} S^{n\ell}$, $R^{w\ell} \mathbin{{}^{\circ}\!\wedge} S^\ell$, $R^\ell \mathbin{{}^{\circ}\!\wedge} S^{w\ell}$. We can prove $R^\ell \mathbin{{}^{\circ}\!\vee} S^{n\ell}$ with (3.79), Lemma A.1.11 and (F7). We can prove $R^\ell \mathbin{{}^{\circ}\!\wedge} S^{w\ell}$ with (3.78), Lemma A.1.13 and (F7). We can prove $R^{w\ell} \mathbin{{}^{\circ}\!\wedge} S^\ell$ with (3.80), Lemma A.1.19 and (F7). For the inductive step we need to assume the result holds for all $b*$-terms of lesser complexity than: $P^* \mathbin{{}^{\circ}\!\wedge} P^{w*}, P^{w*} \mathbin{{}^{\circ}\!\wedge} P^*, P^* \mathbin{{}^{\circ}\!\vee} P^{n*}, P^* \mathbin{{}^{\circ}\!\wedge} P^{b*}, P^{b*} \mathbin{{}^{\circ}\!\wedge} P^*, P^{b*} \mathbin{{}^{\circ}\!\wedge} P^{b*}, P^{w*} \mathbin{{}^{\circ}\!\wedge} P^{b*}, P^{b*} \mathbin{{}^{\circ}\!\wedge} P^{w*}, P^{b*} \mathbin{{}^{\circ}\!\vee} P^*, P^* \mathbin{{}^{\circ}\!\vee} P^{b*}, P^{b*} \mathbin{{}^{\circ}\!\vee} P^{n*}$ and $P^{b*} \mathbin{{}^{\circ}\!\vee} P^{b*}$. The proof of $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^{w*}$, $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^{b*}$ and $R^{b*} \mathbin{{}^{\circ}\!\wedge} S^*$ follows directly from (3.82), (3.84), (3.89), the induction hypothesis and (F7). □

Lemma A.1.21. *For any* T*-*$w*$*-term* P *and* T*-*$*$*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$b*$*-term and:*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \mathbin{{}^{\circ}\!\wedge} Q.$$

Proof. By (3.109) and (F7) it suffices to prove that $f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*)$ is a $b*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*) = P^{w*} \mathbin{{}^{\circ}\!\wedge} (Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*)$. We will prove this by induction on the number of ℓ-terms in R^*. For the base case we have $f^c_{3d}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^\ell)$ the proof follows from (3.110), (F7) and Lemma A.1.16. For the inductive step we assume the result holds for all $*$-terms of lesser complexity than $R^* \mathbin{{}^{\circ}\!\wedge} S^d$ and $R^* \mathbin{{}^{\circ}\!\vee} S^c$. In the case of $R^* \mathbin{{}^{\circ}\!\vee} S^c$ the proof follows directly from (3.112), (F7) and Lemma A.1.16. In the case of $R^* \mathbin{{}^{\circ}\!\wedge} S^d$ the proof follows directly from (3.111), (F7) and the induction hypothesis.

□

Lemma A.1.22. *For any* T*-*$b*$*-term* P *and* T*-*$*$*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$b*$*-term and:*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \mathbin{{}^{\circ}\!\wedge} Q.$$

Proof. By (3.172) and (F7) it suffices to prove that $f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*)$ is a $b*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*) = P^{b*} \mathbin{{}^{\circ}\!\wedge} (Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^*)$. We will prove this by induction on the number of ℓ-terms in R^*. For the base case we have $f^c_{4d}(P^{b*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^\ell)$ the proof follows from (3.173), (F7) and Lemma A.1.16. For the inductive step we assume the result holds for all $*$-terms of lesser complexity than $R^* \mathbin{{}^{\circ}\!\wedge} S^d$ and $R^* \mathbin{{}^{\circ}\!\vee} S^c$. In the case of $R^* \mathbin{{}^{\circ}\!\wedge} S^d$ the proof follows directly from (3.174), (F7) and the induction hypothesis. In the case of $R^* \mathbin{{}^{\circ}\!\vee} S^c$ the proof follows directly from (3.175), (F7) and Lemma A.1.16.

□

Lemma A.1.23. *For any* T*-*$w*$*-term* P *and* T*-*$w*$*-term* Q*,* $f^c(P, Q)$ *is a* T*-*$w*$*-term and*

$$\text{EqFSCL}^{\mathsf{U}} \vdash f^c(P, Q) = P \mathbin{{}^{\circ}\!\wedge} Q.$$

Proof. By (3.118) and (F7) it suffices to prove that $f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{w*})$ is a $w*$-term and that $\text{EqFSCL}^{\mathsf{U}} \vdash f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{w*}) = P^{w*} \mathbin{{}^{\circ}\!\wedge} (Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{w*})$.

$$\begin{aligned}
f^c_{3f}(P^{w*}, Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{w*}) &= f^n_3(f^c_{3a}(P^{w*}, Q^{\mathsf{T}})) \mathbin{{}^{\circ}\!\vee} R^{w*} && \text{by (3.119) } \textit{Note: This is a } w*\textit{-term} \\
&= \neg f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \mathbin{{}^{\circ}\!\vee} R^{w*} && \text{by La.A.1.6} \\
&= f^c_{3a}(P^{w*}, Q^{\mathsf{T}}) \mathbin{{}^{\circ}\!\wedge} R^{w*} && \text{by La.A.1.7} \\
&= (P^{w*} \mathbin{{}^{\circ}\!\wedge} Q^{\mathsf{T}}) \mathbin{{}^{\circ}\!\wedge} R^{w*} && \text{by La.A.1.16} \\
&= P^{w*} \mathbin{{}^{\circ}\!\wedge} (Q^{\mathsf{T}} \mathbin{{}^{\circ}\!\wedge} R^{w*}) && \text{by (F7)}
\end{aligned}$$

□

Lemma A.1.24. *For any* T*-b∗-term* P *and* T*-w∗-term* Q, $f^c(P,Q)$ *is a* T*-b∗-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. By (3.181) and (F7) it suffices to prove that $f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*})$ is a $w*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*}) = P^{b*} \leftand (Q^{\mathsf{T}} \leftand R^{w*})$.

The proof for f^c_{4f} follows directly from:

$$\begin{aligned} f^c_{4f}(P^{b*}, Q^{\mathsf{T}} \leftand R^{w*}) &= f^c_{4a}(P^{b*}, Q^{\mathsf{T}}) \leftand R^{w*} && \text{by (3.182) \textit{Note: This is a b∗-term}} \\ &= (P^{b*} \leftand Q^{\mathsf{T}}) \leftand R^{w*} && \text{by La.A.1.16} \\ &= P^{b*} \leftand (Q^{\mathsf{T}} \leftand R^{w*}) && \text{by (F7)} \end{aligned}$$

□

Lemma A.1.25. *For any* T*-∗-term* P *and* T*-n∗-term* Q, $f^c(P,Q)$ *is a* T*-n∗-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. By (3.70) and (F7) it suffices to prove that $f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand R^{n*})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{1e}(P^*, Q^{\mathsf{T}} \leftand R^{n*}) = P^* \leftand (Q^{\mathsf{T}} \leftand R^{n*})$. In the case of an $n\ell$-term the proof follows from (F7), Lemma A.1.13 and (3.71). In the case of an $n*$-term of the form $R^* \leftand S^{n*}$ the proof follows from (3.72), Lemma A.1.13 and (F7). In the case of an $n*$-term of the form $R^{w*} \leftand S^{n*}$ the proof follows from (3.73), Lemma A.1.19 and (F7). Furthermore, in the case of an $n*$-term of the form $R^{b*} \leftand S^{n*}$ the proof follows from (3.74), Lemma A.1.20 and (F7). □

Lemma A.1.26. *For any* T*-w∗-term* P *and* T*-b∗-term* Q, $f^c(P,Q)$ *is a* T*-b∗-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. By (3.120) and (F7) it suffices to prove that $f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*})$ is a $b*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{3g}(P^{w*}, Q^{\mathsf{T}} \leftand R^{b*}) = P^{w*} \leftand (Q^{\mathsf{T}} \leftand R^{b*})$. In the case of a $b*$-term of the form $R^* \leftor S^{n*}$, $R^{b*} \leftor S^*$, $R^{b*} \leftor S^{n*}$, $R^{b*} \leftor S^{b*}$ or $R^* \leftor S^{b*}$ the proof follows directly from (3.122), (3.124), (3.126), (3.128), (3.130), Lemma A.1.16 and (F7). In the case of a $b*$-term of the form $R^* \leftand S^{w*}$ or $R^* \leftand S^{b*}$ the proof follows directly from (3.121), (3.131), Lemma A.1.21 and (F7). In the case of a $b*$-term of the form $R^{w*} \leftand S^*$ and $R^{w*} \leftand S^{b*}$ the proof follows directly from (3.123), (3.129), Lemma A.1.23 and (F7).

In the case of a $b*$-term of the form $R^{b*} \leftand S^{w*}$, $R^{b*} \leftand S^{b*}$ or $R^{b*} \leftand S^*$ we need to use induction on the number of $b\ell$-terms in $b*$. For the base case we have $R^\ell \leftor S^{n\ell}$, $R^{w\ell} \leftand S^\ell$, $R^\ell \leftand S^{w\ell}$. We can prove $R^\ell \leftor S^{n\ell}$ with (3.122), Lemma A.1.16 and (F7). We can prove $R^\ell \leftand S^{w\ell}$ with (3.121), Lemma A.1.21 and (F7). We can prove $R^{w\ell} \leftand S^\ell$ with (3.123), Lemma A.1.23 and (F7). For the inductive step we need to assume the result holds for all $b*$-terms of lesser complexity than: $P^* \leftand P^{w*}, P^{w*} \leftand P^*, P^* \leftor P^{n*}, P^* \leftand P^{b*}, P^{b*} \leftand P^*, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}, P^{b*} \leftand P^{w*}, P^{b*} \leftor P^*, P^* \leftor P^{b*}, P^{b*} \leftor P^{n*}$ and $P^{b*} \leftor P^{b*}$. The proof of $R^{b*} \leftand S^{w*}$, $R^{b*} \leftand S^{b*}$ and $R^{b*} \leftand S^*$ follows directly from (3.125), (3.127), (3.132), the induction hypothesis and (F7). □

Lemma A.1.27. *For any* T*-b∗-term* P *and* T*-b∗-term* Q, $f^c(P,Q)$ *is a* T*-b∗-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c(P,Q) = P \leftand Q.$$

Proof. By (3.184) and (F7) it suffices to prove that $f^c_{4g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*})$ is a $b*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^c_{3g}(P^{b*}, Q^{\mathsf{T}} \leftand R^{b*}) = P^{b*} \leftand (Q^{\mathsf{T}} \leftand R^{b*})$. In the case of a $b*$-term of the form $R^* \leftor S^{n*}$, $R^{b*} \leftor S^*$, $R^{b*} \leftor S^{n*}$, $R^{b*} \leftor S^{b*}$ or $R^* \leftor S^{b*}$ the proof follows directly from (3.186), (3.188), (3.190), (3.192), (3.194), Lemma A.1.16 and (F7). In the case of a $b*$-term of the form $R^* \leftand S^{w*}$ or $R^* \leftand S^{b*}$ the proof follows directly from (3.185), (3.195), Lemma A.1.22 and (F7). In the case of a $b*$-term of the form $R^{w*} \leftand S^*$ and $R^{w*} \leftand S^{b*}$ the proof follows directly from (3.187), (3.193), Lemma A.1.24 and (F7).

In the case of a $b*$-term of the form $R^{b*} \leftand S^{w*}$, $R^{b*} \leftand S^{b*}$ or $R^{b*} \leftand S^{*}$ we need to use induction on the number of $b\ell$-terms in $b*$. For the base case we have $R^{\ell} \leftor S^{n\ell}$, $R^{w\ell} \leftand S^{\ell}$, $R^{\ell} \leftand S^{w\ell}$. We can prove $R^{\ell} \leftor S^{n\ell}$ with (3.186), Lemma A.1.16 and (F7). We can prove $R^{\ell} \leftand S^{w\ell}$ with (3.185), Lemma A.1.22 and (F7). We can prove $R^{w\ell} \leftand S^{\ell}$ with (3.187), Lemma A.1.24 and (F7). For the inductive step we need to assume the result holds for all $b*$-terms of lesser complexity than: $P^{*} \leftand P^{w*}, P^{w*} \leftand P^{*}, P^{*} \leftor P^{n*}, P^{*} \leftand P^{b*}, P^{b*} \leftand P^{*}, P^{b*} \leftand P^{b*}, P^{w*} \leftand P^{b*}, P^{b*} \leftand P^{w*}, P^{b*} \leftor P^{*}, P^{*} \leftor P^{b*}, P^{b*} \leftor P^{n*}$ and $P^{b*} \leftor P^{b*}$. The proof of $R^{b*} \leftand S^{w*}$, $R^{b*} \leftand S^{b*}$ and $R^{b*} \leftand S^{*}$ follows directly from (3.189), (3.191), (3.196), the induction hypothesis and (F7). □

Lemma A.1.28. *For any* T-*$w*$-term* P *and* T-*$n*$-term* Q*,* $f^{c}(P,Q)$ *is a* T-*$n*$-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^{c}(P,Q) = P \leftand Q.$$

Proof. By (3.113) and (F7) it suffices to prove that $f^{c}_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand R^{n*})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^{c}_{3e}(P^{w*}, Q^{\mathsf{T}} \leftand R^{n*}) = P^{w*} \leftand (Q^{\mathsf{T}} \leftand R^{n*})$. If R is an $n\ell$-term the proof follows from (3.114), Lemma (A.1.16) and (F7). In the case of an $n*$-term of the form $R^{*} \leftand S^{n*}$ the proof follows from (3.115), Lemma A.1.21 and (F7). If R is an $n*$-term of the form $R^{w*} \leftand S^{n*}$ the proof follows from (3.116), Lemma A.1.23 and (F7). Furthermore, if R is an $n*$-term of the form $R^{b*} \leftand S^{n*}$ the proof follows from (3.117), Lemma A.1.26 and (F7). □

Lemma A.1.29. *For any* T-*$b*$-term* P *and* T-*$n*$-term* Q*,* $f^{c}(P,Q)$ *is a* T-*$n*$-term and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^{c}(P,Q) = P \leftand Q.$$

Proof. By (3.176) and (F7) it suffices to prove that $f^{c}_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand R^{n*})$ is an $n*$-term and that $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^{c}_{4e}(P^{b*}, Q^{\mathsf{T}} \leftand R^{n*}) = P^{b*} \leftand (Q^{\mathsf{T}} \leftand R^{n*})$. In the case of an $n\ell$-term the proof follows from (3.177), Lemma A.1.16 and (F7). In the case of an $n*$-term of the form $R^{*} \leftand S^{n*}$ the proof follows from (3.178), Lemma A.1.22 and (F7). In the case of an $n*$-term of the form $R^{w*} \leftand S^{n*}$ the proof follows from (3.179), Lemma A.1.24 and (F7). Furthermore, in the case of an $n*$-term of the form $R^{b*} \leftand S^{n*}$ the proof follows from (3.180), Lemma A.1.27 and (F7). □

Lemma A.1.30. *For any* $P \in \mathit{SNF}$ *and* $Q \in \mathit{SNF}$*,* $f_{c}(P,Q)$ *is in SNF and* $\mathrm{EqFSCL}^{\mathsf{U}} \vdash f^{c}(P,Q) = P \leftand Q$.

Proof. Every term in *SNF* is either a T-term, a F-term, an U-term or an T-a-term. The proof for f_{c} in case P is a T-term, a F-term, an U-term and $Q \in \mathit{SNF}$ follows from A.1.8, A.1.9, A.1.10. In the case that P is a T-a-term and Q is a T-term the proof follows from A.1.11, A.1.16 and A.1.15. In the case that Q is an F-term the proof follows from A.1.12, A.1.17 and A.1.15. In the case that Q is an U-term the proof follows from A.1.14, A.1.18 and A.1.15. With this the claim is proven for all terms except the combinations of T-a-terms.

In the case where P is a T-a-term and Q is a T-a-term the proof follows from A.1.13, A.1.15, A.1.19, A.1.20, A.1.21, A.1.23, A.1.22, A.1.24, A.1.25, A.1.26, A.1.27, A.1.28 and A.1.29 □

Theorem 3.2.1 *(from Chapter 3) For any* $P \in \mathcal{S}_{A}{}^{U}$*,* $f(P)$ *terminates,* $f(P) \in \mathit{SNF}$ *and*

$$\mathrm{EqFSCL}^{\mathsf{U}} \vdash f(P) = P.$$

Proof. If P is an atom, the result follows from (3.1), (F4), (F5) and its dual. If P is T, F or U the result follows from (3.2), (3.3) or (3.4). For the other terms we need to use induction on the structure of P. For the inductive case we get the result from (3.5), (3.6), (3.7), (F2), Lemma A.1.5, Lemma A.1.6 and Lemma A.1.30. □