Jury Theorems, Cascades & All That

1. The Condorcet Jury Theorem

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1. The Wisdom of Crowds
2. The Condorcet Jury Theorem (CJT)
3. Beyond the Assumptions of the CJT
1. The Wisdom of Crowds
A mystery bridge

The bridge on the right connects Manhattan to which other New York borough?

- [ ] Brooklyn
- [ ] Queens
The many, who are not as individuals excellent men, nevertheless can, when they have come together, be better than the few best people, not individually but collectively, just as feasts to which many contribute are better than feasts provided at one person’s expense.

In other words: two (or more) heads are better than one.

Aristotle

Aristotle. Politics.
Aristotle

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No they’re not.

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Many examples of collective folly.
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The big idea

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Brexit, the *Everything Everywhere All at Once* movie…

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Aristotle. *Politics*.

And yet, and yet

If I may be allowed, I can personally attest to the accuracy of groups.

Francis Galton
And yet, and yet

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I’m talking, of course, about…
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The ox!
What happens in Plymouth...

Francis Galton

A weight-judging competition was carried on at the annual show of the West of England Fat Stock and Poultry Exhibition recently held at Plymouth.

A fat ox having been selected, competitors bought stamped and numbered cards, for 6d. each, on which to inscribe their respective names, addresses, and estimates of what the ox would weigh after it had been slaughtered and "dressed". Those who guessed most successfully received prizes.

About 800 tickets were issued, which were kindly lent me for examination after they had fulfilled their immediate purpose... [of which] there remained 787 for discussion.

Now the middlemost estimate is 1207 lb., and the weight of the dressed ox proved to be 1198 lb.; so the *vox populi* was in this case 9 lb., or 0.8 per cent. of the whole weight too high.

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By *middlemost* I mean what you might call today the *median*.
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...gets published in *Nature*

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**The ox**

The crowd was, on average, within 1 lb of the true weight!
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People have since pointed out that the *mean* was even more accurate: 1197 lbs.

The *ox*

The crowd was, on average, within 1 lb of the true weight!

This result is, I think, more creditable to the trust-worthiness of a democratic judgment than might have been expected.

---

Grandma

But can a crowd of bright students guess my age?
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In a previous iteration of this experiment there were 17 guesses, the average of which was 77.8.
But can a crowd of bright students guess my age?

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Whereas my real age was, and for about one more month still is, 81.
But can a crowd of bright students guess my age?

In a previous iteration of this experiment there were 17 guesses, the average of which was 77.8.

Whereas my real age was, and for about one more month still is, 81.

Not bad!
Beyond weighing oxes

James Surowiecki

There are many more examples of the wisdom of crowds at work.
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Like the market response to the *Challenger* disaster...

Or the finding of the *Scorpion* submarine.

James Surowiecki

Beyond weighing oxes

James Surowiecki

There are many more examples of the wisdom of crowds at work.

Like the market response to the Challenger disaster...

Or the finding of the Scorpion submarine.

See my book!

Another way of looking at democracy

When you give people democratic choice they end up doing something stupid.

Good statecraft is like flying a plane.

And you need a good pilot for that.

Plato. *The Republic*
Another way of looking at democracy

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Edwin Hutchins
Actually, flying a plane (or running a ship) requires a great deal of coordination and teamwork.

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Hélène Landemore

Similarly, modern societies need the input of as many and as diverse parties as possible to work well.

Plato. *The Republic*


Applications

**Justin Wolfers**

Prediction markets! Simple markets can be used to aggregate disperse information into efficient forecasts of uncertain future events.

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People buy and sell shares in future events (by a double auction). The price indicates the collective estimate of the probability of the event. See PredictIt.

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Prediction markets! Simple markets can be used to aggregate disperse information into efficient forecasts of uncertain future events.

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People buy and sell shares in future events (by a double auction). The price indicates the collective estimate of the probability of the event. See PredictIt.

**Justin Wolfers**

And other prediction platforms, like Metaculus and Good Judgment Open.

---

A plethora of books


Condorcet

The role of the government is to implement measures that are in the best interest of society.

Marie Jean Antoine Nicolas de Caritat, Marquis of Condorcet. *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix.* 1785
Where it all started

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But how to decide on what outcomes are good?

Democratic procedures can work well.

And I can show it using this newfangled theory of probabilities.

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2. The Condorcet Jury Theorem (CJT)
The model

agents, aka voters
\[ N = \{1, \ldots, n\} \]
\( n \) typically assumed odd

two alternatives
\[ A = \{a, b\} \]

the correct alternative
\[ a \]

voter \( i \)'s vote
\[ v_i \in A \]
\( i \)'s guess of the right answer

voter \( i \)'s competence
\[ p_i = \mathbb{P}[v_i = a], \text{ for } p_i \in [0, 1] \]
the probability that voter \( i \) gets the right answer

profile of votes
\[ \mathbf{v} = (v_1, \ldots, v_n) \]
we write profiles as words: \((a, a, b, a, \ldots) \rightsquigarrow aaba\ldots\)

voting rule
\[ F(\mathbf{v}) \mapsto x \in A \]

majority rule
\[ F_{\text{MAJ}}(\mathbf{v}) = x, \text{ such that } v_i = x \text{ for a strict majority of voters in } N \]
Assumptions of the CJT

Condorcet

I want to make some assumptions!

Assumption 1: Agents are competent, i.e., more likely than not to be correct.

Assumption 2: Competences are homogeneous.

Assumption 3: Agents vote independently of each other.

I claim that under these conditions, the majority tends to get it right.

**COM** \[ p_i > \frac{1}{2}, \text{ for every } i \in \mathbb{N} \]

**HOM** \[ p_i = p_j = p, \text{ for any two agents } i, j \in \mathbb{N} \]

**IND** \[ P[v_i = x, v_j = y] = P[v_i = x] \cdot P[v_j = y], \text{ for any two voters } i, j \in \mathbb{N} \text{ and } x, y \in A \]
Assumptions of the CJT

I want to make some assumptions!

One is that agents are competent, i.e., more likely than not to be correct.

$P_i > \frac{1}{2}, \text{ for every } i \in N.$
Assumptions of the CJT

One is that agents are *competent*, i.e., more likely than not to be correct. Then, that competences are *homogeneous*. I claim that under these conditions, the majority tends to get it right.

\[(\text{COM}) \quad p_i > \frac{1}{2}, \text{ for every } i \in N.\]

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I want to make some assumptions!

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And, finally, that agents vote *independently* of each other.

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\text{(COM)} \quad p_i > \frac{1}{2}, \text{ for every } i \in N.
\]

\[
\text{(HOM)} \quad p_i = p_j = p, \text{ for any two agents } i, j \in N.
\]

\[
\text{(IND)} \quad \mathbb{P}[v_i = x, v_j = y] = \mathbb{P}[v_i = x] \cdot \mathbb{P}[v_j = y], \text{ for any two voters } i, j \in N \text{ and } x, y \in A.
\]
Our goal for today

We want to understand:

\[ \mathbb{P}[F_{\text{MAJ}}(\mathbf{v}) = a], \]

i.e., the probability that the majority opinion is correct.
Profile contains only one voter: $\nu = (v_1)$

$$\mathbb{P}[F_{\text{MAJ}}(v_1) = a] = \mathbb{P}[v_1 = a]$$

$$= p$$

$$> \frac{1}{2}.$$  

(by HOM)  

(by COM)
Growing group accuracy for $\nu = (\nu_1)$

**Condorcet**

Note that as $p$ grows, so does group accuracy.
Growing group accuracy for $\nu = (\nu_1)$

**Condorcet**

Note that as $p$ grows, so does group accuracy.

Though when the group consists of only one agent, this is rather trivial.
Profile contains two voters: $v = (v_1, v_2)$
Profile contains two voters: $\nu = (\nu_1, \nu_2)$

Condorcet
Oh wait we’re not looking at this case.
Profile contains three voters: $\mathbf{\nu} = (v_1, v_2, v_3)$

\[ \mathbb{P}[F_{MAJ}(\mathbf{\nu}) = a] = \mathbb{P}[\mathbf{\nu} \in \{aab, aba, baa, aaa\}] \]
\[ = \mathbb{P}[\mathbf{\nu} = aab] + \mathbb{P}[\mathbf{\nu} = aba] + \mathbb{P}[\mathbf{\nu} = baa] + \mathbb{P}[\mathbf{\nu} = aaa] \]
\[ = \mathbb{P}[v_1 = a] \cdot \mathbb{P}[v_2 = a] \cdot \mathbb{P}[v_3 = b] + \]
\[ \mathbb{P}[v_1 = a] \cdot \mathbb{P}[v_2 = b] \cdot \mathbb{P}[v_3 = a] + \]
\[ \mathbb{P}[v_1 = b] \cdot \mathbb{P}[v_2 = a] \cdot \mathbb{P}[v_3 = a] + \]
\[ \mathbb{P}[v_1 = a] \cdot \mathbb{P}[v_2 = a] \cdot \mathbb{P}[v_3 = a] + \quad \text{(by IND)} \]
\[ = p \cdot p \cdot (1 - p) + p \cdot (1 - p) \cdot p + (1 - p) \cdot p \cdot p + p \cdot p \cdot p \quad \text{(by HOM)} \]
\[ = 3p^2(1 - p) + p^3 \]
\[ > p. \quad \text{(by COM)} \]
Profile contains three voters: $\mathbf{v} = (v_1, v_2, v_3)$

\[
\begin{align*}
P[F_{\text{MAJ}}(\mathbf{v}) = a] &= P[\mathbf{v} \in \{aab, aba, baa, aaa\}] \\
&= P[\mathbf{v} = aab] + P[\mathbf{v} = aba] + P[\mathbf{v} = baa] + P[\mathbf{v} = aaa] \\
&= P[v_1 = a] \cdot P[v_2 = a] \cdot P[v_3 = b] + \\
&\quad P[v_1 = a] \cdot P[v_2 = b] \cdot P[v_3 = a] + \\
&\quad P[v_1 = b] \cdot P[v_2 = a] \cdot P[v_3 = a] + \\
&\quad P[v_1 = a] \cdot P[v_2 = a] \cdot P[v_3 = a] \quad \text{(by IND)} \\
&= p \cdot p \cdot (1 - p) + p \cdot (1 - p) \cdot p + (1 - p) \cdot p \cdot p + p \cdot p \cdot p \quad \text{(by HOM)} \\
&= 3p^2(1 - p) + p^3 \\
&> p. \quad (?) \quad \text{(by COM)}
\end{align*}
\]
Growing group accuracy for $\nu = (\nu_1, \nu_2, \nu_3)$

**Condorcet**

By the croissants of my ancestors, I claim that as $p$ grows, so does group accuracy.
Growing group accuracy for $v = (v_1, v_2, v_3)$

Condorcet

By the croissants of my ancestors, I claim that as $p$ grows, so does group accuracy.

I also claim that a group of size 3 is more likely to be correct than a group of size 1.
Growing group accuracy for $\nu = (\nu_1, \nu_2, \nu_3)$

**Condorcet**

By the croissants of my ancestors, I claim that as $p$ grows, so does group accuracy.

I also claim that a group of size 3 is more likely to be correct than a group of size 1.

The accuracy of the group is greater than the accuracy of the individual voters!
Profile contains five voters: \( \nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \)

\[
\mathbb{P}[F_{\text{MAJ}}(\nu) = a] = \mathbb{P}[\nu \in \{aaabb, aabab, abaab, abba, aabba, ababa, baaba, abbaa, babaa, bbaaa\}] + \\
\mathbb{P}[\nu \in \{aaaab, aaaba, aabaa, abaaa, baaaa\}] + \\
\mathbb{P}[\nu \in \{aaaaa\}] \\
\cdots \\
= 10p^3(1 - p)^2 + 5p^4(1 - p) + p^5 \\
= \binom{5}{3}p^3(1 - p)^2 + \binom{5}{4}p^4(1 - p) + \binom{5}{5}p^5.
\]
Growing group accuracy for $\nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5)$

Condorcet

Again I claim that as $p$ grows, so does the group accuracy.
Growing group accuracy for $\mathbf{v} = (v_1, v_2, v_3, v_4, v_5)$

Condorcet

Again I claim that as $p$ grows, so does the group accuracy.

What is more, I claim that a group of size 5 is better than a group of size 3.
Growing group accuracy for \( \nu = (\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) \)

**Condorcet**

Again I claim that as \( p \) grows, so does the group accuracy.

What is more, I claim that a group of size 5 is better than a group of size 3.

The accuracy of the group gets better and better!
In general, for any profile $\nu = (\nu_1, \ldots, \nu_n)$ and odd $n$

$$
P[F_{\text{MAJ}}(\nu) = a] = P[\nu \text{ such that } > \frac{n}{2} \text{ agents vote for } a]
$$

$$= P[\nu \text{ s.t. } \lceil \frac{n}{2} \rceil + 1 \text{ agents vote for } a] + \cdots + P[\nu \text{ s.t. } n \text{ agents vote for } a]
$$

$$= \left( P\left[ \nu = a \underbrace{\ldots a}_{\lceil \frac{n}{2} \rceil + 1} b \ldots b \right] + \cdots + P\left[ \nu = b \underbrace{\ldots b}_{\lceil \frac{n}{2} \rceil + 1} a \ldots a \right] \right) + \cdots + P[\nu = a \ldots a]
$$

$$= \left( \binom{n}{\lceil \frac{n}{2} \rceil + 1} p^{\lceil \frac{n}{2} \rceil + 1} (1 - p)^n - (\lceil \frac{n}{2} \rceil + 1) \right) + \cdots + \left( \binom{n}{n - 1} p^{n-1} (1 - p)^1 + \binom{n}{n} p^n \right)
$$

$$= \sum_{i=\lceil \frac{n}{2} \rceil + 1}^{n} \binom{n}{i} p^i (1 - p)^{n-i}.\]
The Condorcet Jury Theorem (CJT)

**Theorem: The Condorcet Jury Theorem (CJT)**

If agents have the same accuracy $p > \frac{1}{2}$ and vote independently of each other, then, for odd $n$, it holds that:

**LIB** *Larger is better*, i.e., accuracy of the group improves as its size grows:

$$P[F_{\text{MAJ}}(v_1, \ldots, v_{n+2}) = a] > P[F_{\text{MAJ}}(v_1, \ldots, v_n) = a].$$

**GBI** *Groups are better than individuals*, i.e., accuracy of the group better than that of any of its members:

$$P[F_{\text{MAJ}}(v_1, \ldots, v_n) = a] > P[v_i = a], \text{ for } n \geq 3.$$

**ASY** The accuracy of the group approaches 1 *asymptotically*:

$$\lim_{n \to \infty} P[F_{\text{MAJ}}(v_1, \ldots, v_n) = a] = 1.$$
In other words

Condorcet

The larger the group, the more accurate.

In the limit, groups are infallible.

Provided there are no dumdums and people never talk to each other!
Our task for now

We want to prove this.
Keeping track of events with random variables

**indicator random variable**

\[ X_i = \begin{cases} 1, & \text{if agent } i \text{ gets it right, i.e., if } v_i = a \\ 0, & \text{otherwise} \end{cases} \]

per our assumption, \( \mathbb{P}[X_i = 1] = p \) and \( \Pr[X_i = 0] = 1 - p \)

**sum random variable**

\[ S_n = X_1 + \cdots + X_n \]
A note about the random variables

**Condorcet**

\[ S_n = k \] equivalent to \( k \) agents getting it right.

\[ S_n > k \] is equivalent to more than \( k \) agents getting it right.

Majority opinion correct if \( S_n > \lfloor n/2 \rfloor \).

\[ S_n = X_1 + \cdots + X_n \]
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Jacob Bernoulli

The \( X_i \)'s, if I may point out, are called Bernoulli variables.

\[
S_n = X_1 + \cdots + X_n
\]

\[
\mathbb{E}[X_i] = 1 \cdot p + 0 \cdot (1 - p) = p,
\]

\[
\mathbb{V}[X_i] = \mathbb{E}[(X_i - \mathbb{E}[X_i])^2] = (1 - p)^2 p + (0 - p)^2 (1 - p)
\]

\[
= p(1 - p).
\]
Warm-up: a recurrence relation for group accuracy

We want to think of the probability of a correct majority with \( n+2 \) voters in terms of the probability of a correct majority with \( n \) voters, i.e., as a recurrence relation.

More formally:

\[
P[S_5 > 2] = (1 - p)^2 \cdot P[S_3 > 2] + 2p(1 - p) \cdot P[S_3 > 1] + p^2 \cdot P[S_3 > 0].
\]
Warm-up: a recurrence relation for group accuracy

> We want to think of the probability of a correct majority with \( n+2 \) voters in terms of the probability of a correct majority with \( n \) voters, i.e., as a recurrence relation.
> Take \( n = 5 \).
Warm-up: a recurrence relation for group accuracy

We want to think of the probability of a correct majority with $n+2$ voters in terms of the probability of a correct majority with $n$ voters, i.e., as a recurrence relation.

Take $n = 5$.

If two of these voters are wrong then we need the remaining three to be correct if the group is to be correct:

$aaabb$

More formally:

$$P[S_5 > 2] = (1-p)^2 \cdot P[S_3 > 2] + 2p(1-p) \cdot P[S_3 > 1] + p^2 \cdot P[S_3 > 0].$$

We generalize this to $S_{n+2}$ and $S_n$, but first.
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If two of these voters are wrong then we need the remaining three to be correct if the group is to be correct:

$$aaabb$$

If one is right and the other is wrong (which can happen in two ways), we need at least two out of the other three to be right:

$$aaaba, aabba, ababa, baaba,$$

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$$aaaba, aabba, ababa, baaba,$$
$$aaaab, aabab, abaab, baaab.$$  

If both of them are right, we need at least one remaining voter to be correct:

$$abbaa, babaa, bbaaa, aabaa, abaaa, baaa, aaaaa.$$
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$$abbaa, babaa, bbaaa, aabaa, abaaa, baaaa, aaaaa.$$

More formally:

$$P[S_5 > 2] = (1 - p)^2 \cdot P[S_3 > 2] + 2p(1 - p) \cdot P[S_3 > 1] + p^2 \cdot P[S_3 > 0].$$
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We want to think of the probability of a correct majority with \( n+2 \) voters in terms of the probability of a correct majority with \( n \) voters, i.e., as a recurrence relation.

Take \( n = 5 \).

If two of these voters are wrong then we need the remaining three to be correct if the group is to be correct:

\[ aaabb \]

If one is right and the other is wrong (which can happen in two ways), we need at least two out of the other three to be right:

\[ aaaba, aabba, ababa, baaba, \]
\[ aaaab, aabab, abaab, baaab. \]

If both of them are right, we need at least one remaining voter to be correct:

\[ abbaa, babaa, bbaaa, aabaa, abaaa, baaaa, aaaaa. \]

More formally:

\[
\Pr[S_5 > 2] = (1 - p)^2 \cdot \Pr[S_3 > 2] + 2p(1 - p) \cdot \Pr[S_3 > 1] + p^2 \cdot \Pr[S_3 > 0].
\]

We generalize this to \( S_{n+2} \) and \( S_n \), but first...
Warm-up: Counting events

More-than-$k$-correct out of $n$ is the same as more-than-$(k+1)$ or exactly-$(k+1)$ correct:

$$\Pr[S_n > k] = \Pr[S_n > k+1] + \Pr[S_n = k+1].$$
Warm-up: Counting events

> More-than-$k$-correct out of $n$ is the same as more-than-($k+1$) or exactly-($k+1$) correct:

$$\Pr[S_n > k] = \Pr[S_n > k+1] + \Pr[S_n = k+1].$$

> In a slight variation, more-than-($k-1$) is the same as more-than-$k$ plus exactly-($k-1$), which, after rearranging the terms, yields:

$$\Pr[S_n > k] = \Pr[S_n > k-1] - \Pr[S_n = k-1].$$
Warm-up: Counting events

- More-than-$k$-correct out of $n$ is the same as more-than-$(k+1)$ or exactly-$(k+1)$ correct:

\[ P[S_n > k] = P[S_n > k+1] + P[S_n = k+1]. \]

- In a slight variation, more-than-$(k-1)$ is the same as more-than-$k$ plus exactly-$(k-1)$, which, after rearranging the terms, yields:

\[ P[S_n > k] = P[S_n > k-1] - P[S_n = k-1]. \]

- And the binomial terms exhibit symmetry:

\[ \binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}. \]
Proof that accuracy improves with group size (\textbf{LIB}): the equations

\textbf{Proof: Larger is Better (LIB)}

\begin{itemize}
\item Using the generalized versions of the previous identities, we have that:
\end{itemize}

\[ \Pr \left[ S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = (1-p)^2 \Pr \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor + 1 \right] + 2p(1-p) \Pr \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + p^2 \Pr \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right]. \tag{1} \]
Proof: Larger is Better \textbf{(LIB)}

> Using the generalized versions of the previous identities, we have that:

\[ \mathbb{P}\left[ S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = (1-p)^2 \mathbb{P}\left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor + 1 \right] + 2p(1-p) \mathbb{P}\left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + p^2 \mathbb{P}\left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right] . \tag{1} \]

> At the same time, the probability that more than \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) out of \( n \) voters are correct is the probability that more than \( \left\lfloor \frac{n}{2} \right\rfloor \) voters are correct, plus the probability that exactly \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) voters are correct:

\[ \mathbb{P}\left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right] = \mathbb{P}\left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + \left( \frac{n}{\left\lfloor \frac{n}{2} \right\rfloor} \right) p^{\left\lfloor n/2 \right\rfloor} (1-p)^{\left\lfloor n/2 \right\rfloor + 1} . \tag{2} \]
Proof: Larger is Better (LIB)

- Using the generalized versions of the previous identities, we have that:

\[
P \left[ S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = (1-p)^2 P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor +1 \right] + 2p(1-p) P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + p^2 P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right].
\]

- At the same time, the probability that more than \( \lfloor n/2 \rfloor - 1 \) out of \( n \) voters are correct is the probability that more than \( \lfloor n/2 \rfloor \) voters are correct, plus the probability that exactly \( \lfloor n/2 \rfloor \) voters are correct:

\[
P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right] = P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + \binom{n}{\lfloor n/2 \rfloor} p^{\lfloor n/2 \rfloor} (1-p)^{\lfloor n/2 \rfloor +1}.
\]

- The probability that more than \( \lfloor n/2 \rfloor \) out of \( n \) voters are correct is the probability of more than \( \lfloor n/2 \rfloor + 1 \) voters being correct plus the probability of exactly \( \lfloor n/2 \rfloor + 1 \) voters being correct, which gives us:

\[
P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor + 1 \right] = P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] - \binom{n}{\lfloor n/2 \rfloor +1} p^{\lfloor n/2 \rfloor +1} (1-p)^{\lfloor n/2 \rfloor}.
\]
Proof that accuracy improves with group size (LIB): the equations

**Proof: Larger is Better (LIB)**

- Using the generalized versions of the previous identities, we have that:

\[
P \left[ S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = (1-p)^2 P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor + 1 \right] + 2p(1-p) P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + p^2 P \left[ S_n > \left\lceil \frac{n}{2} \right\rceil - 1 \right]. \tag{1}
\]

- At the same time, the probability that more than \(\left\lfloor \frac{n}{2} \right\rfloor - 1\) out of \(n\) voters are correct is the probability that more than \(\left\lfloor \frac{n}{2} \right\rfloor\) voters are correct, plus the probability that exactly \(\left\lfloor \frac{n}{2} \right\rfloor\) voters are correct:

\[
P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor - 1 \right] = P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} p^{\left\lfloor \frac{n}{2} \right\rfloor} (1 - p)^{\left\lfloor \frac{n}{2} \right\rfloor + 1}. \tag{2}
\]

- The probability that more than \(\left\lfloor \frac{n}{2} \right\rfloor\) out of \(n\) voters are correct is the probability of more than \(\left\lfloor \frac{n}{2} \right\rfloor + 1\) voters being correct plus the probability of exactly \(\left\lfloor \frac{n}{2} \right\rfloor + 1\) voters being correct, which gives us:

\[
P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor + 1 \right] = P \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] - \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + 1} p^{\left\lfloor \frac{n}{2} \right\rfloor + 1} (1 - p)^{\left\lfloor \frac{n}{2} \right\rfloor}. \tag{3}
\]

- We plug (2) and (3) into (1), denoting:

\[\binom{n}{\left\lfloor \frac{n}{2} \right\rfloor} = \binom{n}{\left\lfloor \frac{n}{2} \right\rfloor + 1} = c.\]
Proof that accuracy improves with group size (LIB): the upshot

**Proof: Larger is Better (LIB)**

\( \Rightarrow \) We obtain:

\[
\mathbb{P} \left[ S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = \left( (1 - p)^2 + 2p(1 - p) + p^2 \right) \mathbb{P} \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] \\
= 1 + c \left( p^\lfloor n/2 \rfloor + 2 (1 - p)^\lfloor n/2 \rfloor + 1 - p^\lfloor n/2 \rfloor + 1 (1 - p)^\lfloor n/2 \rfloor + 2 \right) \\
= \mathbb{P} \left[ S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + c \cdot p^\lfloor n/2 \rfloor + 1 \cdot (1 - p)^\lfloor n/2 \rfloor + 1 \cdot (2p - 1).
\]
Proof: Larger is Better (LIB)

> We obtain:

\[
\begin{align*}
\mathbb{P}\left[S_{n+2} \geq \left\lfloor \frac{n+2}{2} \right\rfloor \right] &= \left( (1 - p)^2 + 2p(1 - p) + p^2 \right) \mathbb{P}\left[S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] \\
&\quad + c\left(p^{\lfloor n/2 \rfloor + 2} (1 - p)^{\lfloor n/2 \rfloor + 1} - p^{\lfloor n/2 \rfloor + 1} (1 - p)^{\lfloor n/2 \rfloor + 2} \right) \\
&= \mathbb{P}\left[S_n > \left\lfloor \frac{n}{2} \right\rfloor \right] + c \cdot p^{\lfloor n/2 \rfloor + 1} \cdot (1 - p)^{\lfloor n/2 \rfloor + 1} \cdot (2p - 1).
\end{align*}
\]

> Since \( \frac{1}{2} < p < 1 \), it holds that:

\[
c \cdot p^{\lfloor n/2 \rfloor + 1} \cdot (1 - p)^{\lfloor n/2 \rfloor + 1} \cdot (2p - 1) > 0,
\]
Proof that accuracy improves with group size (LIB): the upshot

**Proof: Larger is Better (LIB)**

> We obtain:

\[
P\left[S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] = \left( (1-p)^2 + 2p(1-p) + p^2 \right) \mathbb{P}\left[S_n > \frac{n}{2}\right] \]

\[
= 1 + c \left( p^{\lfloor n/2 \rfloor+2} (1-p)^{\lfloor n/2 \rfloor+1} - p^{\lfloor n/2 \rfloor+1} (1-p)^{\lfloor n/2 \rfloor+2} \right)
\]

\[
= \mathbb{P}\left[S_n > \frac{n}{2}\right] + c \cdot p^{\lfloor n/2 \rfloor+1} \cdot (1-p)^{\lfloor n/2 \rfloor+1} \cdot (2p - 1).
\]

> Since \( \frac{1}{2} < p < 1 \), it holds that:

\[
c \cdot p^{\lfloor n/2 \rfloor+1} \cdot (1-p)^{\lfloor n/2 \rfloor+1} \cdot (2p - 1) > 0,
\]

> and hence that:

\[
\mathbb{P}\left[S_{n+2} > \left\lfloor \frac{n+2}{2} \right\rfloor \right] > \mathbb{P}\left[S_n > \frac{n}{2}\right]. \quad \square
\]
Proof: Groups better than individuals (GBI)

Follows immediately from the fact that the larger a group gets the better it gets (LIB):

\[ \mathbb{P}[S_1 > 0] < \mathbb{P}[S_3 > 1] < \cdots < \mathbb{P}[S_n > \lfloor n/2 \rfloor] < \cdots \]
To prove the asymptotic claim, i.e., that in the limit accuracy is 1, we use the Law of Large Numbers.

The intuition for which is as follows.

Say we have random variables $X_i$ such that $X_i = \begin{cases} 
1, & \text{with probability } 0.02, \\
0, & \text{with probability } 0.98. 
\end{cases}$

The expected value of such a variable is:

$$E[X_i] = 1 \cdot 0.02 + 0 \cdot 0.98 = 0.02.$$ 

Now, if we sample $10^6$ such variables, then we'd expect around $0.02 \cdot 10^6 = 20000$ to have value 1. More to the point, we'd expect the average over many samples to be around 0.02.
The Law of Large Numbers

J. Bernoulli

**Theorem: The (Weak) Law of Large Numbers**

If $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) random variables such that $\mathbb{E}[X_i] = \mu$, then, for any $\varepsilon > 0$, it holds that:

$$\lim_{n \to \infty} \mathbb{P}\left[ \left| \frac{X_1 + \cdots + X_n}{n} - \mu \right| < \varepsilon \right] = 1.$$
In other words

The sample mean approaches the expected value, i.e., the ‘true’, theoretical mean.

Jacob Bernoulli

The ox
This probably explains what happened at the Plymouth county fair!
Proof: Group accuracy goes to 1 as $n$ goes to infinity (ASY)

- Recall that $\mu = \mathbb{E}[X_i] = p$, for $i \in N$.
- The Weak Law of Large Numbers gives us that, for any $\varepsilon > 0$:

$$\lim_{n \to \infty} \mathbb{P} \left[ \left| \frac{S_n}{n} - p \right| < \varepsilon \right] = 1,$$

which is not exactly what we want.
- But we can massage this into something closer to our goal:

$$\mathbb{P} \left[ \left| \frac{S_n}{n} - p \right| < \varepsilon \right] = \mathbb{P} \left[ -\varepsilon < \frac{S_n}{n} - p < \varepsilon \right] \leq \mathbb{P} \left[ -\varepsilon < \frac{S_n}{n} - p \right] = \mathbb{P} \left[ S_n > n(p - \varepsilon) \right].$$
Proof: Group accuracy goes to 1 as \( n \) goes to infinity (ASY)

> At this point we have that:

\[
\mathbb{P} \left( \left| \frac{S_n}{n} - p \right| < \varepsilon \right) \leq \mathbb{P} [S_n > n(p - \varepsilon)],
\]

\[
\rightarrow 1 \quad \text{for any } \varepsilon > 0.
\]

> Setting \( n(p - \varepsilon) = \lfloor n/2 \rfloor = (n-1)/2 \) we get:

\[
\varepsilon = p - \left( \frac{1}{2} + \frac{1}{2n} \right).
\]

> So now we plug \( \varepsilon = p - (1/2 + 1/2n) \) into (4) to infer that:

\[
\lim_{n \to \infty} \mathbb{P}[S_n > \lfloor n/2 \rfloor] = 1,
\]

as desired.
Summing up

Condorcet
Groups are better than their members.
**Summing up**

**Condorcet**

Groups are better than their members.

The larger the group, the better.

![Group accuracy vs individual competence](image)

- $p = \frac{1}{2}$
- For the same $n$, higher $p$ means higher accuracy.
Summing up

**Condorcet**

Groups are better than their members.

The larger the group, the better.

In the limit, performance is perfect.
Summing up

Condorcet
Groups are better than their members.
The larger the group, the better.
In the limit, performance is perfect.
And performance grows fast with the size of the group.

Provided $p > \frac{1}{2}$.
Also: for the same $n$, higher $p$ means higher accuracy.
Condorcet

Groups are better than their members.

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3. **Beyond the Assumptions of the CJT**
Beyond independence

It pains me to say it, but the CJT has a major blindspot: independent voter beliefs.
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Out there people interact and are exposed to common information sources, e.g., mass media.
Beyond independence

Condorcet

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Krishna K. Ladha

Introducing correlation between voters can make the optimistic results go away.


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Landemore, Hélène

At the same time, more and more evidence that deliberation is good for decision making.

---


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Hélène Landemore

At the same time, more and more evidence that deliberation is good for decision making.

Condorcet

We need better formal models!


And what is up with competence?

Condorcet

What does this $p$ even mean?

Can we rate people’s accuracies, especially if predicting rare or unique events?

Glenn Brier

Sure! Check out the Brier score.

Philip E. Tetlock

Some people seem to manage it: superforecasters.

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I still wonder though: is it realistic to assume that $p > \frac{1}{2}$?
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---

What would lead to competence being low?

What would be a reason for $p < \frac{1}{2}$?
What would lead to competence being low?

Condorcet

What would be a reason for $p < \frac{1}{2}$?

Daniel Kahneman

Biases!

You thought it was Brooklyn, didn’t you?

The bridge on the right connects Manhattan to which other New York borough?

- [ ] Brooklyn
- [x] Queens

Queensboro Bridge
What would lead to competence being low?

Daniel Kahneman
Biases!

What would lead to competence being low?

Daniel Kahneman
Biases!

Bryan Caplan
Most people cannot be relied upon to take good decisions.

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Jason Brennan
Especially when it comes to political issues.

What would lead to competence being low?

Daniel Kahneman

Biases!

Bryan Caplan

Most people cannot be relied upon to take good decisions.

Jason Brennan

Especially when it comes to political issues.

Hélène Landemore

Let's not exaggerate.


Beyond homogeneous competences

And what if voters do not all have the same $p$?
Beyond homogeneous competences

Condorcet

And what if voters do not all have the same $p$?

Bernard Grofman

It’s not so clear that the conclusions of the CJT still hold.
Beyond homogeneous competences

Condorcet
And what if voters do not all have the same $p$?

Bernard Grofman
It’s not so clear that the conclusions of the CJT still hold.

It gets kinda complicated.

Wrapping up

Condorcet
The CJT: a cornerstone of the idea that groups can be wise.
Wrapping up

Condorcet

The CJT: a cornerstone of the idea that groups can be wise.

But also feels like a fragile result, based on unrealistic assumptions.
Wrapping up

Condorcet

The CJT: a cornerstone of the idea that groups can be wise.

But also feels like a fragile result, based on unrealistic assumptions.

Can we find better results, for modern-day challenges?