The Condorcet Jury Theorem and Its Variations

Day 2: Extensions

Adrian Haret
a.haret@uva.nl

The ILLC,
University of Amsterdam

MoL Project
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1 Heterogeneous competences
2 Correlated voters
3 Strategic voting
4 More than Two Alternatives
5 Beyond
Recall the CJT

Theorem: The Condorcet Jury Theorem

If HOM, COM and IND are satisfied and $\frac{1}{2} < p < 1$, then, for any odd $n \geq 1$, it holds that:

(GBI) if $n > 1$, then $P(S_n \geq \frac{n+1}{2}) > p$;
(LIB) $P(S_{n+2} \geq \frac{n+3}{2}) > P(S_n \geq \frac{n+1}{2})$;
(ASY) $\lim_{n \to \infty} P(S_n \geq \frac{n+1}{2}) = 1$. 

Condorcet
A more general form of Claim GBI

**CONDORCET:** Claim GBI is a particular version of the idea that the performance of the group is better than the performance of each of its individual members:

\[ P(S_n \geq \frac{n + 1}{2}) > p_i, \text{ for all voters } i \in N. \]

**CONDORCET:** Which is what we’re aiming for, in general.
Recall, also, the assumptions

\((\text{COM})\) \(p_i > \frac{1}{2}\), for every \(i \in N\).

\((\text{HOM})\) \(p_i = p_j = p\), for any two agents \(i\) and \(j\) in \(N\).

\((\text{IND})\) \(P(X_i = u, X_j = v) = P(X_i = u)P(X_j = v)\), for any two agents \(i, j \in N\) and \(u, v \in \{0, 1\}\).
CONDORCET: What if we weaken some of these assumptions?
Heterogeneous competences
CONDORCET: What if voters have different competences?
Heterogeneous competences

**CONDORCET:** What if voters have different competences?

**CONDORCET:** Potentially above and below $\frac{1}{2}$. 
CONDORCET: We rely on more or less the same model as yesterday:

- $A = \{a_1, a_2\}$, the alternatives
- $a^* \in A$, the true alternative;
- set $N = \{1, \ldots, n\}$ of $n$ voters, where $n$ is odd;
- profile $v = (v_1, \ldots, v_n)$ of votes, where $v_i \in A$;
- agent $i$’s competence $P(v_i = a^*) = p_i$;
The model

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CONDORCET: We won’t be assuming that competences are all the same anymore.

- $p = (p_1, \ldots, p_n)$, vector of competences;
  - typically assuming that $p_1 \geq \cdots \geq p_n$;
- $\bar{p}$, the average competence, where:

$$\bar{p} = \frac{p_1 + \cdots + p_n}{n}.$$
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\[
\bar{\mathbf{p}} = \frac{p_1 + \cdots + p_n}{n}.
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CONDORCET: And let’s write $M(n, \mathbf{p})$ instead of $P(S_n \geq \frac{n+1}{2})$, for the probability that a majority out of the $n$ agents get it right, where $\mathbf{p} = (p_1 \ldots, p_n)$ is the vector of their competences.
The probability that the majority gets it right under these conditions

**CONDORCET:** If \( n = 3 \), then:

\[
M(3, p) = P(110) + P(101) + P(011) + P(111)
\]

\[
= p_1p_2(1 - p_2) + p_1(1 - p_2)p_3 + (1 - p_1)p_2p_3 + p_1p_2p_3
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= p_1p_2 + p_2p_3 + p_1p_3 - 2p_1p_2p_3.
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**OGF**: In general, we have:

\[
M(n, p) = \sum_{S \subseteq \mathbb{N}, |S| \geq \frac{n+1}{2}} \prod_{i \in S} p_i \prod_{i \notin S} (1 - p_i).
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Owen et al. [1989]
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**CONDORCET:** Do the conclusions of the CJT still hold?

Owen et al. [1989]
The conclusions of the CJT do not hold anymore

CONDORCET: If we take $p = (p_1, p_2, p_3) = (0.95, 0.8, 0.8)$, then:

$$M(3, p) = 0.944 < p_1.$$
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M(5, p') = 0.91 < M(3, p).
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CONDORCET: It’s also not true that larger groups are better (LIB) anymore.
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**J. PAROUSH:** Yes, if the voters are still reasonably competent.

**J. PAROUSH:** By that I mean that $p_i \geq \frac{1}{2} + \epsilon$, for some $\epsilon > 0$. 
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Parouch [1998]
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**J. Paroush:** And we know, from the CJT, that \( \lim_{n \to \infty} M(n, (\frac{1}{2} + \epsilon, \ldots, \frac{1}{2} + \epsilon)) = 1 \).

**J. Paroush:** So we’ve got ourselves a nice little result.

Paroush [1998]
Theorem: Paroush [1998]

If IND is satisfied, $p = (p_1, \ldots, p_n)$ is a vector of competences such that $p_i \geq \frac{1}{2} + \epsilon$, for some $\epsilon > 0$ then, for any odd $n$, it holds that:

$$\lim_{n \to \infty} M(n,p) = 1.$$
J. PAROUSH: Interestingly, it’s not enough to simply say that $p_i > \frac{1}{2}$.
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Paroush [1998]
An apparent CJT

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J. PAROUSH: See my paper for details.

Paroush [1998]
CONDORCET: This result still requires voters to be individually competent, though.
Still too strict?

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J. PAROUSH: True.
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J. PAROUTH: True.
OGF: We can address this.
Theorem: Owen et al. [1989]

If IND is satisfied and \( \bar{p} > \frac{1}{2} \) is fixed, then it holds that:

\[
\lim_{n \to \infty} M(n, p) = 1.
\]

OGF: If the average competence \( \bar{p} \) is greater than \( \frac{1}{2} \), then the asymptotic claim still holds.
The proof

OGF: To prove this result, we look at $M(n, p)$ as a function in the $p_i$’s, and use the derivatives with respect to each $p_i$ to understand how to maximize it.
But strange things still happen

\textbf{OGF:} If $\mathbf{p} = (0.72, 0.72, 0)$, then $\bar{\mathbf{p}} = 0.48$ and $M(3, \mathbf{p}) = 0.5184$.

\textbf{OGF:} If $\mathbf{p} = (1, 0.28, 0.28)$, then $\bar{\mathbf{p}} = 0.52$ and $M(3, 2\mathbf{p}) = 0.4816$. 

G. Owen

B. Grofman

S.L. Feld
More on this

There is, of course, more:
Berend and Paroush [1998], Ben-Yashar and Paroush [2000], Dietrich and List [2004],
Dietrich [2008]
Correlated voters
K.K. Ladha: The main weakness of CJT is that its assumption of independence is unreasonable.

K.K. Ladha: Votes will be correlated because the judges or experts:
- share common information,
- communicate with each other, and
- are influenced by various schools of thought or opinion leaders espousing the same or opposite positions.

Ladha [1992]
P. BOLAND: I propose:

- $X_1, \ldots, X_n$ random variables as before, with $X_i \in \{0, 1\}$;
- $P(X_i = 1) = p$, for every agent $i$, as in the classical CJT;
- voter 1 as the opinion leader, its decisions influencing the decision of the others;
- conditioned on $X_1$, $X_i$ and $X_j$ are conditionally independent, for $i, j \geq 2$;
- $0 \leq r \leq 1$, the correlation between $X_1$ and $X_i$, for $i \geq 2$;

Boland et al. [1989], Boland [1989]
A model with opinion leaders influencing the probabilities

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P. BOLAND: Think of $r$ as the probability that $X_i$ will follow $X_1$, for $i \geq 2$, with $0 \leq r \leq 1$.

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P. BOLAND: Let’s write $M(n, p, r)$ for the probability that a majority out of $n$ agents gets it right, with parameters $p$ and $r$.

Boland et al. [1989], Boland [1989]
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$$P(X_i = 1 \mid X_1 = 1) = r + p(1 - r)$$
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$$P(X_i = 1 \mid X_1 = 1) = r + p(1 - r)$$
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\begin{align*}
P(X_i = 1 \mid X_1 = 1) &= r + p(1 - r) \\
P(X_i = 0 \mid X_1 = 1) &= (1 - p)(1 - r) \\
P(X_i = 1 \mid X_1 = 0) &= (1 - r)p \\
P(X_i = 0 \mid X_1 = 0) &= r + (1 - r)(1 - p)
\end{align*}
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ME: The flow of influence is very much like in a Bayesian net.
When $n = 3$

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ME: So, in general:

$$P(X_1, X_2, X_3) = P(X_2 | X_1)P(X_3 | X_1)P(X_1).$$
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$$P(X_1, X_2, X_3) = P(X_2 | X_1)P(X_3 | X_1)P(X_1).$$

ME: Thus:

$$P(X_1 = 1, X_2 = 1, X_3 = 1) = P(X_2 = 1 | X_1 = 1)P(X_3 = 1 | X_1 = 1)P(X_1 = 1)$$

$$= (r + p(1 - r))^2 p;$$
When \( n = 3 \)

**ME:** The flow of influence is very much like in a Bayesian net.

**ME:** So, in general:

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P(X_1, X_2, X_3) = P(X_2 \mid X_1)P(X_3 \mid X_1)P(X_1).
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**ME:** Thus:

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= (r + p(1 - r))^2 p;
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\[
P(X_1 = 1, X_2 = 1, X_3 = 0) = P(X_2 = 1 \mid X_1 = 1)P(X_3 = 0 \mid X_1 = 1)P(X_1 = 1) \\
= (r + p(1 - r))(1 - p)(1 - r)p \\
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ME: So, in general:

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ME: Thus:

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$$P(X_1 = 1, X_2 = 1, X_3 = 0) = P(X_2 = 1 \mid X_1 = 1)P(X_3 = 0 \mid X_1 = 1)P(X_1 = 1) = (r + p(1 - r))(1 - p)(1 - r)p$$

$$= P(X_1 = 1, X_2 = 0, X_3 = 1);$$

$$P(X_1 = 0, X_2 = 1, X_3 = 1) = P(X_2 = 1 \mid X_1 = 0)P(X_3 = 1 \mid X_1 = 0)P(X_1 = 0) = (1 - r)^2 p^2 (1 - p).$$
The probability of a majority for the right answer, with $n = 3$

**ME:** We have here that:

$$M(3, p, r) = (r + p(1 − r))^2p + 2(r + p(1 − r))(1 − p)(1 − r)p + (1 − r)^2 p^2(1 − p)$$

$$= (-2p^3 + 3p^2 − p)r^2 + (4p^3 − 6p^2 + 2p)r − (2p^3 − 3p^2)$$

**ME:** We can see $M(3, p, r)$ as a function in $r$, and thus take the derivative with respect to $r$:

$$\frac{\partial M(3, p, r)}{\partial r} = (-4p^3 + 6p^2 − 2p)r + (4p^3 − 6p^2 + 2p)$$

$$= −2p(2p − 1)(p − 1)(r − 1)$$

$$< 0, \text{ if } 0 < r < 1 \text{ and } \frac{1}{2} < p < 1.$$ 

**ME:** Thus, if $\frac{1}{2} < p < 1$ and $0 < r \geq 1$, then $M(3, p, r)$ decreases as $r$ grows.
P. BOLAND: We can reproduce this computation in a smart way for any $n$. 
Group accuracy for competent voters diminished in the presence of a strong opinion leader

**Theorem: Boland et al. [1989], Boland [1989]**

If $0 \leq r_1 < r_2 \leq 1$, then, for any odd $n$, it holds that:

1. If $p > \frac{1}{2}$, then $M(n, p, r_1) > M(n, p, r_2)$;
2. If $p < \frac{1}{2}$, then $M(n, p, r_1) < M(n, p, r_2)$;
3. If $p = \frac{1}{2}$, then $M(n, p, r) = \frac{1}{2}$, for any $0 \geq r \geq 1$;

It also holds that, if $r < 1 - \frac{1}{2p}$, then:

$$\lim_{n \to \infty} M(n, p, r) = 1.$$
Theorem: Boland et al. [1989], Boland [1989]

If $0 \leq r_1 < r_2 \leq 1$, then, for any odd $n$, it holds that:

(1) if $p > \frac{1}{2}$, then $M(n, p, r_1) > M(n, p, r_2)$;
(2) if $p < \frac{1}{2}$, then $M(n, p, r_1) < M(n, p, r_2)$;
(3) if $p = \frac{1}{2}$, then $M(n, p, r) = \frac{1}{2}$, for any $0 \geq r \geq 1$;

It also holds that, if $r < 1 - \frac{1}{2p}$, then:

$$
\lim_{n \to \infty} M(n, p, r) = 1.
$$

P. BOLAND: If $p > \frac{1}{2}$, then the stronger the influence of the leader, the less likely the group is to make the correct decision.
K.K. LADHA: This is nice, but voters may be correlated for other reasons as well.
K.K. LADHA: This is nice, but voters may be correlated for other reasons as well.
K.K. LADHA: In most general terms, we can account for this using a correlation coefficient.
K.K. LADHA: I propose:

- \( X_1, \ldots, X_n \) random variables as before, with \( X_i \in \{0, 1\} \);
- \( P(X_i = 1) = p \), for every agent \( i \), as before;
- \( r_{ij} = P(X_i = 1, X_j = 1) \), the probability that \( i \) and \( j \) are both right;
- \( \rho_{ij} \), the coefficient of correlation between \( X_i \) and \( X_j \), where:

\[
\rho_{ij} = \frac{r_{ij} - p^2}{\sigma^2};
\]

- \( \bar{\rho} \), an aggregate correlation coefficient, defined as follows:

\[
\bar{\rho} = \frac{\sum_{i=1}^{n} \sum_{j \neq i}^{n} \rho_{ij}}{n(n - 1)}.
\]

Ladha [1992]
Let’s unpack this a bit

K.K. LADHA: $r_{ij}$ is a measure of how independent $i$ and $j$ vote:

$$\begin{cases} 
    r_{ij} = p^2, & \text{if } X_i \text{ and } X_j \text{ are independent,} \\
    r_{ij} < p^2, & \text{if } X_i \text{ and } X_j \text{ are negatively correlated,} \\
    r_{ij} > p^2, & \text{if } X_i \text{ and } X_j \text{ are positively correlated,}
\end{cases}$$
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K.K. LADHA: Recall from yesterday, \( \sigma^2 \) is the variance of \( X_i \), which in this case is \( p(1 - p) \).
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K.K. LADHA: Recall from yesterday, $\sigma^2$ is the variance of $X_i$, which in this case is $p(1 - p)$.

K.K. LADHA: If $n = 3$, then:

$$
\bar{\rho} = \frac{\rho_{12} + \rho_{13} + \rho_{23}}{3}.
$$
Applying this to the earlier model with an opinion leader

K.K. Ladha: For instance, for the model from Boland et al. [1989], Boland [1989], we get that:

\[ r_{12} = P(X_1 = 1, X_2 = 1) = P(X_2 = 1 \mid X_1 = 1)P(X_1 = 1) = pr + p^2 - p^2 r = r_{13} > p^2. \]
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K.K. LADHA: Similarly, we get:

\[ r_{23} = P(X_2 = 1, X_3 = 1 | X_1 = 1)P(X_1 = 1) + P(X_2 = 1, X_3 = 1 | X_1 = 0)P(X_1 = 0) \]
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K.K. LADHA: And \( \rho_{23} = r^2. \)

K.K. LADHA: And \( \bar{\rho} = \frac{r^2+2r}{3}. \)
K.K. LADHA: Correlation $\rho$ can occur because of an opinion leader, but it’s a more general notion.
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K.K. LADHA: It’s just a property of the distribution of values across the variables.
K.K. LADHA: It’s interesting because it sets a bound for the CJT to hold.
More on dependence between voters:
   Ladha [1993], Estlund [1994], Berg [1993a,b], Kaniovski [2010], Peleg and Zamir [2012], Pivato [2017]
As long as the correlation is not too high

**Theorem: Ladha [1992]**

For any odd $n$, it holds that if:

\[ \bar{\rho} < 1 - \frac{n}{n-1} \left( \frac{p - \frac{1}{4}}{p^2} \right), \]

then $M(n, p) > p$. 

K.K. Ladha
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then $M(n, p) > p.$

**K.K. Ladha:** If the votes are not highly correlated, then Claim GBI (groups are better than individuals) of the CJT still holds.
Strategic voting
Often left out, but

ASB: An important—but largely implicit—assumption in proofs of the CJT is that individuals vote ‘sincerely’.

Austen-Smith and Banks [1996]
Often left out, but

ASB: An important—but largely implicit—assumption in proofs of the CJT is that individuals vote ‘sincerely’.

CONDORCET: Why wouldn’t they?

Austen-Smith and Banks [1996]
We propose:

- as usual two alternatives, a and b, one of them the true alternative;
- voters $N = \{1, \ldots, n\}$, with prior probabilities $P(a) = \pi$ and $P(b) = 1 - \pi$;
  
  - $P(a)$ is the probability that $a$ is the true alternative;
- voter $i$ receives signal $s_i \in \{0, 1\}$, with:

  $$P(s_i = 0 \mid a) = q_a > \frac{1}{2};$$

  $$P(s_i = 1 \mid b) = q_b > \frac{1}{2};$$

- a profile $s = (s_1, \ldots, s_n)$ of signals;
A model with strategic voters

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- a profile $s = (s_1, \ldots, s_n)$ of signals;

ASB: So how do voters vote?
The most likely state, according to your own signal

ASB: Say voter $i$ receives a signal $s_i = 0$. 

Voter $i$ can use Bayes' rule to compute the odds for $a$ and $b$, given only its own signal $s_i = 0$:

$$P(a | s_1 = 0) = \frac{P(s_1 = 0 | a) P(a)}{P(s_1 = 0)} = q_a \pi P(s_1 = 0);$$

$$P(b | s_1 = 0) = \frac{P(s_1 = 0 | b) P(b)}{P(s_1 = 0)} = (1 - q_b) (1 - \pi) P(s_1 = 0).$$

Similar calculations can be made if $s_i = 1$.

Voter $i$ votes sincerely if it votes for the likeliest alternative, given its own signal $s_i$.

And we assume that, in general, for $s_i = 0$ then $a$ turns out to be more likely, and for $s_i = 1$ then $b$ turns out to be more likely.
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\]

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**ASB:** Similar calculations can be made if $s_i = 1$. 

D. Austen-Smith

J.S. Banks

*Personality, and the Conc* 

JEFFREY S. BANKS

*Note: This extract contains equations and mathematical notations. It describes a scenario where voters receive signals and use Bayes' rule to calculate the odds of their preferred option being the most likely state.*
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ASB: Say there are only three voters, and the signals are $s = (0, 1, 0)$. 
The most likely state, according to all signals

ASB: Say there are only three voters, and the signals are \( s = (0, 1, 0) \).

ASB: If voter 1 were to see \( s \), it could use Bayes' rule to update its probabilities:

\[
P(a \mid s_1 = 0, s_2 = 1, s_3 = 0) = \frac{P(s_1 = 0, s_2 = 1, s_3 = 0 \mid a) P(a)}{P(s_1 = 0, s_2 = 1, s_3 = 0)}
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\propto q_a^2 (1 - q_a) \pi.
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$$\propto q_a^2(1 - q_a)\pi.$$ 

ASB: Similarly for $b$:

$$P(b \mid s_1 = 0, s_2 = 1, s_3 = 0) \propto (1 - q_b)^2 q_b(1 - \pi).$$
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= \frac{P(s_1 = 0 \mid a) P(s_2 = 1 \mid a) P(s_3 = 0 \mid a) P(a)}{P(s_1 = 0, s_2 = 1, s_3 = 0)} \\
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\]

ASB: In this case, it can happen that \( b \) becomes more likely than \( a \).

ASB: So voter 1 has an incentive to vote against its own signal.
Thinking that you’re the decisive voter

**ASB:** If voter 1’s signal is $s_1 = 0$ and it’s thinking in game-theoretic terms, i.e., in which it wants to maximize its expected payoff (and the payoff depends on getting things right) then the only scenario voter 1 should care about is the one where it’s the decisive voter.

**J.S. Banks**
Thinking that you’re the decisive voter

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ASB: Because in all the others its vote doesn’t make a difference.
Thinking that you’re the decisive voter

**ASB:** If voter 1’s signal is $s_1 = 0$ and it’s thinking in game-theoretic terms, i.e., in which it wants to maximize its expected payoff (and the payoff depends on getting things right) then the *only* scenario voter 1 should care about is the one where it’s the decisive voter.

**ASB:** Because in all the others its vote doesn’t make a difference.

**ASB:** But in this case it can end up disregarding its own information!
The voters are fit for the CJT

ASB: Assuming that voters vote informatively, i.e., according to the signal they receive, the probability that voter $i$ gets it right is:

$$P(s_i = 0 \mid a)P(a) + P(s_i = 1 \mid b)P(b) = q_a\pi + q_b(1 - \pi)$$

$$> \frac{1}{2}.$$  

ASB: In other words, voters are competent.
The voters are fit for the CJT

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\[
P(s_i = 0 \mid a)P(a) + P(s_i = 1 \mid b)P(b) = q_a \pi + q_b (1 - \pi)
\]

\[
> \frac{1}{2}.
\]

ASB: In other words, voters are competent.

ASB: The probability that any two voters \( i \) and \( j \) both get it right is:

\[
P(s_i = 0, s_j = 0 \mid a)P(a) + P(s_i = 1, s_j = 1 \mid b)P(b) = q_a^2 \pi + q_b^2 (1 - \pi)
\]

\[
> (q_a \pi + q_b (1 - \pi))^2.
\]

ASB: Voters \( i \) and \( j \) are positively correlated.

ASB: But we can put bounds on this correlation.

ASB: In fact, we can show that the correlation falls within the bounds of Ladha [1992].
A perfect storm

ASB: Conditions are perfect for the CJT...
A perfect storm

**ASB:** Conditions are perfect for the CJT...

**ASB:** ...If only voters vote according to their own signal!
A perfect storm

ASB: Conditions are perfect for the CJT...
ASB: ...If only voters vote according to their own signal!
ASB: Herein lies the tragedy: parameters can be tweaked such that strategic agents end up voting against their signal.
ASB: Conditions are perfect for the CJT...
ASB: . . . If only voters vote according to their own signal!
ASB: Herein lies the tragedy: parameters can be tweaked such that strategic agents end up voting against their signal.
ASB: Messing up the CJT.
Theorem: Austen-Smith and Banks [1996]

The result shows exactly when sincere voting manages to be a Nash equilibrium in the associated Bayesian game.
FEDDERSEN & PESENDORFER: Sometimes the rational thing to do is to abstain.
Feddersen and Pesendorfer [1996]

MCLENNAN: Sometimes strategic voting can deliver the right result though.
McLennan [1998]
(possible assignment)
More than Two Alternatives
CONORCET: It would be nice to see what happens for more than two alternatives.
What if there are more than two alternatives

**CONDORCET:** It would be nice to see what happens for more than two alternatives.

**CONDORCET:** The problem with that, though, is that majority rule doesn’t make sense any more.
What if there are more than two alternatives

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CONDORCET: For instance, if the profile is $v = (a, a, b, b, c)$, there is no majority
What if there are more than two alternatives

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CONDORCET: For instance, if the profile is $\mathbf{v} = (a, a, b, b, c)$, there is no majority

LIST & GOODIN: Let’s look at the plurality rule.
LG: We propose:

- a set $A = \{a_1, \ldots, a_m\}$ of $m$ alternatives, with $a^* \in A$ the correct alternative;
- the set $N = \{1, \ldots, n\}$ of voters;
- each voter $i$ votes for an alternative $v_i \in A$;
- a profile $R = (v_1, \ldots, v_n)$ of votes;
- $P(v_i = a_j) = p_{ij}$, the voters' competence, i.e., the probability that voter $i$ votes for alternative $a_j$;
  - we write simply $P(a_j)$ when it doesn’t matter who is voting;
- the plurality rule $f_{plr}$ selects the alternative(s) that appear most often in $R$;
- $PL(n, a)$ is the probability that $a$ is the unique plurality winner out of a profile of size $n$. 

C. List
R.E. Goodin
Of course, we want to make some assumptions.

Voters have the same competences.

\((\text{gHOM})\) \( P(v_i = a_j) = P(v_k = a_j) \), i.e., \( p_{ij} = p_{kj} \), for any agents \( i, k \in N \).

Voters are competent, i.e., more likely to pick the correct option than not:

\((\text{gCOM})\) \( P(v_i = a^*) > P(v_i = a_i) \), for any voter \( i \in N \) and \( a_i \neq a^* \).

Then, as usual, voters vote independently.

\((\text{gIND})\) \( P(v_i = a_j, v_k = a_l) = P(v_i = a_j)P(v_k = a_l) \), for any \( a_i, a_l \in A \) and voters \( i, k \in N \).
An example

LG: For instance, take $A = \{a_1, a_2, a_3\}$, and assume that $a_1$ is the right alternative.
LG: Suppose the probabilities to select these alternatives are $(p_1, p_2, p_3)$, with $p_1 > p_2 > p_3$ and $p_1 + p_2 + p_3 = 1$
LG: And say we look at profiles of three voters.
LG: The probability that $a_1$ is the unique plurality winner is the probability that $a_1$ appears at least twice:

$$PL(3, a_1) = P(a_1, a_1, a_2) + P(a_1, a_1, a_3) + P(a_1, a_1, a_1)$$

$$= p_1^2 p_2 + p_1^2 p_3 + p_1^3$$

LG: Similarly, the probability that $a_2$ is the plurality winner is:

$$PL(3, a_2) = p_2^2 p_1 + p_2^2 p_3 + p_3^2$$

LG: It can be shown, then, that $PL(3, a_1) > PL(3, a_2)$.
LG: We can also do this all more generally.
Theorem: List and Goodin [2001]

If $g_{HOM}$, $g_{COM}$ and $g_{IND}$ are satisfied, then, for any integer $n$, it holds that:

1. $PL(n, a^*) > PL(n, a_i)$, for any alternative $a_i \neq a^*$;
2. $\lim_{n \to \infty} PL(n, a^*) = 1$. 

C. List  
R.E. Goodin
A more recent take on this problem

P. HUMMEL: See my paper.

Hummel [2010]
(possible assignment)
Beyond
Subjects we have not looked at...

But are great possible assignment topics:

- endogenous accuracy (Ben-Yashar and Nitzan [2001])
- state dependence (Ben-Yashar and Nitzan [1997, 2014])
- truth-tracking in other multi-agent scenarios:
  - judgment aggregation (Bovens and Rabinowicz [2006], De Clippel and Eliaz [2015], Hartmann and Sprenger [2012], Terzopoulou and Endriss [2019])
  - belief merging (Everaere et al. [2010])
  - liquid democracy (Kahng et al. [2018])
References


References III


