61 Ξ-completeness

Suppose $\Xi \subseteq \mathcal{P}\omega$. A set A is Ξ-*complete* if $\forall X \in \Xi.\ X \leq_1 A$.

For example, since K is 1-complete (§48), and Σ_1 consists of the c.e. sets, K is Σ_1-complete. If $X \in \Pi_1$, then $\bar{X} \in \Sigma_1$ by (i) of §59, so $\bar{X} \leq_1 K$, hence $X \leq_1 \bar{K}$; so $\bar{K}$ is Π_1-complete.

62 Post's Theorem

We show that the arithmetical hierarchy is real by tying it to the jump hierarchy.

62.1 Theorem (Post). (i) A set is Σ_{n+1} iff it is c.e. in a Π_n-set iff it is c.e. in a Σ_n-set.
(ii) $\emptyset^{(n+1)}$ is Σ_{n+1}-complete.
(iii) A set is Σ_{n+1} iff it is c.e. in $\emptyset^{(n)}$.
(iv) A set B is Δ_{n+1} iff $B \leq_T \emptyset^{(n)}$.

Proof. (i) If $B \in \Sigma_{n+1}$, then by definition $B \in \Sigma_1^X$ for some $X \in \Pi_n$.

If B is c.e. in $X \in \Pi_n$, then, since $X \leq_T \bar{X}$, B is c.e. in $\bar{X} \in \Sigma_n$ (Jump Theorem (iv), §59 (i)).

If B is c.e. in $X \in \Sigma_n$, then $B = W_e^X$ for some index e, so

$$x \in B \Leftrightarrow \exists\sigma\exists s(\sigma \subset \chi_X \ \&\ \Phi_{e,s}^{\sigma}(x)\downarrow);$$

the second conjunct is computable, hence to show $B \in \Sigma_{n+1}$ it is enough to show $\sigma \subset \chi_X$ is Σ_{n+1}. It is in fact Δ_{n+1}:

$$\sigma \subset \chi_X \Leftrightarrow \forall y < |\sigma|\,((\sigma(y) = 1 \Rightarrow y \in X)\ \&\ (\sigma(y) = 0 \Rightarrow y \notin X));$$

the first conjunct is Σ_n, the second Π_n.
(ii) Induction on n. The case $n = 0$ is known (§48). Now assume $k > 0$ and $\emptyset^{(k)}$ is Σ_k-complete. Then $X \in \Sigma_{k+1}$ iff X is c.e. in a Σ_k-set iff, by induction hypothesis, X is c.e. in $\emptyset^{(k)}$, iff, by (iii) of the Jump Theorem, $X \leq_1 \emptyset^{(k+1)}$.
(iii) It is easy to see that being c.e. in $\emptyset$ is the same as being c.e. absolutely. The case $n > 0$ is immediate by (i) and (ii).
(iv) B is Δ_{n+1} iff B and $\bar{B}$ are both Σ_{n+1}, iff (by (iii)) B and $\bar{B}$ are both c.e. in $\emptyset^{(n)}$, iff (by the Relativized Complementation Theorem) $B \leq_T \emptyset^{(n)}$. ☒

62.2 Corollary (Hierarchy Theorem). For $n > 0$, $\Delta_n \subset \Sigma_n$ and $\Delta_n \subset \Pi_n$.

Proof. By clause (ii) of Post's Theorem, $\emptyset^{(n)} \in \Sigma_n$. By clause (ii) of the Jump Theorem, $\emptyset^{(n)} \not\leq_T \emptyset^{(n-1)}$. So by (iv) of Post's Theorem, $\emptyset^{(n)} \notin \Delta_n$. Dually, $\overline{\emptyset^{(n)}} \in \Pi_n - \Delta_n$. ☒