

Intuitionistic Subframe Formulas, *NNIL*-Formulas and
n-universal Models

MSc Thesis (*Afstudeerscriptie*)

written by

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Chapter 1

Introduction

In this thesis, we investigate intuitionistic subframe formulas and *NNIL*-formulas by using the technique of n -universal models. Intuitionistic subframe formulas axiomatize subframe logics which are intermediate logics characterized by a class of frames closed under subframes. Zakharyashev introduced in [27], [29] the subframe formulas by using $[\wedge, \rightarrow]$ -formulas, which contain only \wedge and \rightarrow as connectives. It then follows that subframe logics are axiomatized by $[\wedge, \rightarrow]$ -formulas.

NNIL-formulas are the formulas that have **no nesting of implications to the left**. Visser, de Jongh, van Benthem and Renardel de Lavalette proved in [25] that *NNIL*-formulas are exactly the formulas preserved under taking submodels. The topic of this thesis was inspired by N. Bezhanishvili [3] who used the insight that *NNIL*-formulas are then preserved under subframes as well to introduce subframe formulas in the *NNIL*-form. It was proved in [3] that *NNIL*-formulas are sufficient to axiomatize subframe logics.

This thesis is set up in a way to be able to connect the results on subframe formulas defined by $[\wedge, \rightarrow]$ -formulas and *NNIL*-formulas by using n -universal models as a uniform method. Our original intention to throw new light on subframe logics by the use of *NNIL*-formulas was barely realized, but we do provide new insights on the *NNIL*-formulas themselves.

Chapter 2 gives a background on intuitionistic propositional logic and its Kripke, algebraic and topological semantics. In Chapter 3, we discuss n -universal models $\mathcal{U}(n)$ of **IPC** by giving proofs of known theorems in a uniform manner including a direct and very perspicuous proof of the fact that the n -universal model of **IPC** is isomorphic to the upper part of the n -Henkin model. This then also gives a method for a new proof (Theorem

3.4.9) of Jankov's theorem in [19] on **KC**. In Chapter 4, we summarize classic and recent results on subframe logics and subframe formulas. In Chapter 5, we investigate properties of the $[\wedge, \rightarrow]$ -fragment of **IPC** consisting of $[\wedge, \rightarrow]$ -formulas only. This chapter is based on the results in Diego [12], de Bruijn [7] and Hendriks [16]. We redefined the *exact model* defined in [16] by using the n -universal models of **IPC** and give a uniform treatment of known results.

In Chapter 6, we give an algorithm to translate every *NNIL*-formula to a $[\wedge, \rightarrow]$ -formula in such a way that they are equivalent on frames. We study subsimulations between models and construct representative models for equivalence classes of rooted generated submodels of $\mathcal{U}(n)$ induced by two-way subsimulations. We construct finite n -universal models $\mathcal{U}(n)^{NNIL}$ for *NNIL*-formulas with n variables by the representative models and prove the related properties. As a consequence, the theorem that formulas preserved under subsimulations are equivalent to *NNIL*-formulas proved in [25] becomes a natural corollary of the properties of $\mathcal{U}(n)^{NNIL}$. Finally, we obtain the subframe logics axiomatized by two-variable *NNIL*-formulas by observing the structure of $\mathcal{U}(2)^{NNIL}$. Although it is not yet clear how to generalize the result for the model $\mathcal{U}(2)^{NNIL}$ and the subframe logics axiomatized by *NNIL*(p, q)-formulas to the models $\mathcal{U}(n)^{NNIL}$ for any $n \in \omega$, this result clearly suggests that the $\mathcal{U}(n)^{NNIL}$ models are a good tool for future work on subframe logics.

Chapter 2

Preliminaries

2.1 Intuitionistic logic

The language of the intuitionistic propositional calculus (**IPC**) consists of propositional variables p_0, p_1, \dots , logical connectives $\wedge, \vee, \rightarrow$ and a constant \perp . Formulas of **IPC** are constructed in the standard way. We write $\neg\varphi$, $\varphi \leftrightarrow \psi$ and \top as abbreviations of $\varphi \rightarrow \perp$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ and $\neg\perp$. Denote by **FORM** the set of all well-formed formulas. Symbols $\varphi, \psi, \chi, \dots$ represent formulas and $\Gamma, \Delta, \Theta, \dots$ represent sets of formulas. We write $\varphi(\vec{p})$ to indicate that the propositional variables of φ are among \vec{p} , where $\vec{p} = (p_0, \dots, p_n)$. We use $PV(\varphi)$ to denote the set of propositional variables that occur in φ . The set of all propositional variables of **IPC** is denoted by **PROP**.

The *Intuitionistic propositional calculus* **IPC** is the smallest set of formulas that contains the axioms:

1. $p \rightarrow (q \rightarrow p)$,
2. $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$,
3. $p \wedge q \rightarrow p, p \wedge q \rightarrow q$
4. $p \rightarrow p \vee q, q \rightarrow p \vee q$,
5. $p \rightarrow (q \rightarrow (p \wedge q))$,
6. $(p \rightarrow r) \rightarrow ((q \rightarrow r) \rightarrow (p \vee q \rightarrow r))$,

7. $\perp \rightarrow p$.

and is closed under the inference rules Modus Ponens (**MP**):

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi},$$

and Substitution (**Sub**):

$$\frac{\varphi(p_0, \dots, p_n)}{\varphi(\psi_0, \dots, \psi_n)}.$$

If $\varphi \in \mathbf{IPC}$, then we write $\vdash_{\mathbf{IPC}} \varphi$ (or simply $\vdash \varphi$ if it causes no confusion) and say that φ is a *theorem* of **IPC**. A *theory* T is a set of sentences with the property $T \vdash \varphi \Rightarrow \varphi \in T$. Let $Th(\Gamma) = \{\varphi : \Gamma \vdash \varphi\}$ and call it a *theory* of Γ . If $\Gamma = \{\varphi\}$ a singleton, then we will write $Th(\varphi)$ instead of $Th(\Gamma)$.

Let **CPC** denote the *classical propositional calculus*. A set of formulas L closed under **MP** and **Sub** is called an intermediate logic if $\mathbf{IPC} \subseteq L \subseteq \mathbf{CPC}$.

2.2 Kripke semantics

Definition 2.2.1. An *intuitionistic Kripke frame* is a pair $\mathfrak{F} = \langle W, R \rangle$ consisting of a nonempty set W and a partial order R on W .

In a Kripke frame \mathfrak{F} , if wRv , then v is called a *successor* of w . If in addition there is no other point $u \in W$ such that wRu and uRv , then v is called an *immediate successor* of w . Denote the set of all immediate successors of w by S_w . The *depth* $d(w) \in [1, +\infty)$ of a point w is defined as usual. The *depth of a frame* $\mathfrak{F} = \langle W, R \rangle$ $d(\mathfrak{F})$ is defined as $d(\mathfrak{F}) = \max\{d(w) : w \in W\}$. For any point w in \mathfrak{F} , we define

$$R(w) = \{v \in W : wRv\},$$

$$R^{-1}(w) = \{v \in W : vRw\}.$$

We sometimes also write $w \uparrow$ and $w \downarrow$ instead of $R(w)$ and $R^{-1}(w)$, respectively. An *endpoint* w is a point with $R(w) = \{w\}$ and has depth 1.

Definition 2.2.2. An *intuitionistic Kripke model* is a triple $\mathfrak{M} = \langle W, R, V \rangle$ such that $\langle W, R \rangle$ is an intuitionistic Kripke frame, and V is an intuitionistic

valuation, which is a partial map $V : \text{PROP} \rightarrow \wp(W)$ satisfying the *persistence* condition: if $w \in V(p)$ and wRv , then $v \in V(p)$. Usually V is used as a total map. We call a model \mathfrak{M} an n -model if $\text{dom}(V) = \{p_1, \dots, p_n\}$.

\mathfrak{M} is also called a model on the frame $\mathfrak{F} = \langle W, R \rangle$.

Let w be a point in a Kripke model $\mathfrak{M} = \langle W, R, V \rangle$. We inductively define a relation $\mathfrak{M}, w \models \varphi$ as follows:

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$;
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$;
- $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$;
- $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff for all $v \in W$ such that wRv , $\mathfrak{M}, v \models \varphi$ implies $\mathfrak{M}, v \models \psi$;
- $\mathfrak{M}, w \not\models \perp$.

If $\mathfrak{M}, w \models \varphi$, we say that “ φ is true at w in \mathfrak{M} ” or “ φ is satisfied at w in \mathfrak{M} ”. If φ is true at every point w in \mathfrak{M} , we write $\mathfrak{M} \models \varphi$ and say that “ φ is true in \mathfrak{M} ” or “ \mathfrak{M} is a model of φ ”. If φ is true in every model on \mathfrak{F} , we write $\mathfrak{F} \models \varphi$ and say that “ φ is valid in \mathfrak{F} ” or “ \mathfrak{F} is a frame of φ ”. For a point w in \mathfrak{M} , we denote the set $\{\varphi : \mathfrak{M}, w \models \varphi\}$ by $Th_{\mathfrak{M}}(w)$ or simply $Th(w)$.

The n -canonical model (or n -Henkin model) of **IPC** is the model $\mathcal{H}(n) = \langle W^c, \subseteq, V^c \rangle$, where W^c is the set of all consistent theories with the disjunction property, and V^c is defined by $\Gamma \in V^c(p)$ iff $p \in \Gamma$.

Many of the Kripke frames considered in the thesis will be *rooted* frames, i.e. frames that have least nodes, *roots*. For proofs of the following two theorems, see e.g. [6], [15].

Theorem 2.2.3. *IPC is sound and complete with respect to the class of all rooted Kripke frames.*

Theorem 2.2.4. *IPC has the finite model property.*

2.3 Operations on Kripke frames and Kripke models

There are three truth-preserving operators on Kripke models and they all have their corresponding operations on Kripke frames.

Definition 2.3.1. Let $\mathfrak{F} = \langle W, R \rangle$ be a frame. A set $V \subseteq W$ is called an *upward closed subset* or an *upset*, if for every $w \in V$ and $v \in W$, wRv implies $v \in V$. Denote the set of all upsets in \mathfrak{F} by $Up(\mathfrak{F})$.

Definition 2.3.2. A Kripke frame $\mathfrak{G} = \langle V, S \rangle$ is called a *subframe* of a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ if $V \subseteq W$ and S is the restriction of R to V ($S = R \upharpoonright V$, in symbols), i.e., $S = R \cap V^2$. The subframe \mathfrak{G} is a *generated subframe* of \mathfrak{F} if V is an upward closed subset of W .

Definition 2.3.3. A Kripke model $\mathfrak{N} = \langle W', R', V' \rangle$ is called a *submodel* of a Kripke model $\mathfrak{M} = \langle W, R, V \rangle$ if $\langle W', R' \rangle$ is a subframe of $\langle W, R \rangle$ and $V' = V \upharpoonright W'$, i.e. $V'(p) = V(p) \cap W'$ for every propositional variable p . \mathfrak{N} is called a *generated submodel* of \mathfrak{M} if $\langle W', R' \rangle$ is a generated subframe of $\langle W, R \rangle$ and $V' = V \upharpoonright W'$.

Let $\mathfrak{M} = \langle W, R, V \rangle$ be a Kripke model, X be a subset of W . We denote the submodel of \mathfrak{M} generated by X by \mathfrak{M}_X . In the case that $X = \{w\}$, we will only write \mathfrak{M}_w for the rooted model generated by $w \in W$.

Definition 2.3.4. Let $\mathfrak{F} = \langle W, R \rangle$ and $\mathfrak{G} = \langle V, S \rangle$ be two Kripke frames. A map f from W to V is called a *p-morphism* of \mathfrak{F} to \mathfrak{G} if it satisfies the following conditions:

(R1) For any $w, u \in W$, wRu implies $f(w)Sf(u)$;

(R2) $f(w)Sv'$ implies $\exists v \in W (wRv \wedge f(v) = v')$.

A surjective p-morphism on frames is also called a *reduction*. If there is a surjective p-morphism from \mathfrak{F} to \mathfrak{G} , then we say that \mathfrak{F} is *reducible* to \mathfrak{G} , and call \mathfrak{G} a *p-morphic image* or a *reduct* of \mathfrak{F} .

Definition 2.3.5. A p-morphism f of \mathfrak{F} to \mathfrak{G} is called a *p-morphism* of a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ to a model $\mathfrak{N} = \langle \mathfrak{G}, V' \rangle$ if for every $p \in \text{PROP}$, $w \in W$,

(R0) $w \in V(p) \iff f(w) \in V'(p)$.

Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame.

- Assume w, v are points in W such that $R(w) = R(v) \cup \{w\}$. Define a Kripke frame $\mathfrak{G} = \langle W', R' \rangle$ by taking $W' = W \setminus \{w\}$, $R' = R \upharpoonright W'$. Define a map $\alpha : W \rightarrow W'$ by taking

$$\alpha(x) = \begin{cases} x, & x \neq w \\ v, & x = w. \end{cases}$$

Then α is a p-morphism. We call a function like α an α -reduction.

- Assume w, v are points in W such that $R(w) \setminus \{w\} = R(v) \setminus \{v\}$. Define a Kripke frame $\mathfrak{G} = \langle W', R' \rangle$ by taking $W' = W \setminus \{w\}$, $R' = R \upharpoonright W' \cup \{(x, v) : (x, w) \in R\}$. Define a map $\beta : W \rightarrow W'$ by taking

$$\beta(x) = \begin{cases} x, & x \neq w \\ v, & x = w. \end{cases}$$

Then β is a p-morphism. We call a function like β a β -reduction.

The next theorem was first proved in [8].

Theorem 2.3.6. *If f is a proper p-morphism of \mathfrak{F} onto \mathfrak{G} , then there exists a sequence f_1, \dots, f_n of α - and β -reductions such that $f = f_1 \circ \dots \circ f_n$.*

Definition 2.3.7. Let $\{\mathfrak{F}_i = \langle W_i, R_i \rangle : i \in I\}$ be a family of Kripke frames such that $W_i \cap W_j = \emptyset$ for all $i \neq j$. The *disjoint union* of the family $\{\mathfrak{F}_i : i \in I\}$ is the frame $\biguplus_{i \in I} \mathfrak{F}_i = \langle \bigcup_{i \in I} W_i, \bigcup_{i \in I} R_i \rangle$.

Definition 2.3.8. Let $\{\mathfrak{M}_i = \langle \mathfrak{F}_i, V_i \rangle : i \in I\}$ be a family of disjoint Kripke models. The *disjoint union* of the family $\{\mathfrak{M}_i : i \in I\}$ is the model $\biguplus_{i \in I} \mathfrak{M}_i = \langle \biguplus_{i \in I} \mathfrak{F}_i, \bigcup_{i \in I} V_i \rangle$.

The following theorem shows that the three operations on Kripke models are truth preserving. For a proof, see e.g. [6].

Theorem 2.3.9.

- If \mathfrak{N} is a generated submodel of \mathfrak{M} , then for every point w in \mathfrak{N} and every formula φ ,

$$\mathfrak{N}, w \models \varphi \iff \mathfrak{M}, w \models \varphi.$$

- If f is a p-morphism of a model \mathfrak{M} to a model \mathfrak{N} , then for every point w in \mathfrak{M} and every formula φ ,

$$\mathfrak{M}, w \models \varphi \iff \mathfrak{N}, f(w) \models \varphi.$$

- If w_i is a point in \mathfrak{M}_i for $i \in I$ and every formula φ ,

$$\mathfrak{M}_i, w_i \models \varphi \iff \biguplus_{i \in I} \mathfrak{M}_i, w_i \models \varphi.$$

2.4 Algebraic semantics

Definition 2.4.1. A *Heyting algebra* $\mathfrak{A} = \langle A, \vee, \wedge, \rightarrow, 0 \rangle$ is a distributive lattice $\langle A, \vee, \wedge, 0 \rangle$ with a binary operator \rightarrow , a *Heyting implication*, defined as:

$$c \leq a \rightarrow b \text{ iff } a \wedge c \leq b.$$

Let \mathfrak{A} be a Heyting algebra. A function $v : \text{PROP} \rightarrow A$ is called a *Heyting valuation*. It can be extended in the standard way from PROP to FORM. We say a formula φ is *true* in \mathfrak{A} under v , if $v(\varphi) = 1$; φ is *valid* in \mathfrak{A} , if φ is true under every valuation in \mathfrak{A} .

For a proof of the following theorem, see e.g. [6].

Theorem 2.4.2. $\vdash_{\text{IPC}} \varphi$ iff φ is valid in every Heyting algebra.

2.5 Topological semantics

Definition 2.5.1. A *topological space* is a pair $\mathcal{X} = \langle X, \tau \rangle$, where $X \neq \emptyset$ and τ is a collection of subsets of X satisfying:

- $\emptyset, X \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$;
- if $\{U_i\}_{i \in I} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$.

Elements of τ are called *open sets*. U is called a *closed set* if $X \setminus U$ is open. A subset which is both closed and open is called a *clopen set*. Denote the set of all clopens of X by $\mathcal{CO}(X)$.

Let Int denote the *interior* operator of \mathcal{X} . A function $v : \text{PROP} \rightarrow \tau$ is called a *valuation*. It can be extended from PROP to FORM as follows:

- $v(\varphi \wedge \psi) = v(\varphi) \cap v(\psi)$;
- $v(\varphi \vee \psi) = v(\varphi) \cup v(\psi)$;
- $v(\perp) = \emptyset$;
- $v(\varphi \rightarrow \psi) = \text{Int}((X \setminus v(\varphi)) \cup v(\psi))$.

Note that $v(\varphi) \in \tau$ for every φ . We say a formula φ is *true* in \mathcal{X} under ν , if $v(\varphi) = X$; φ is *valid* in \mathcal{X} , if φ is true under every valuation in \mathcal{X} .

For a proof of the following theorem, see e.g. [21].

Theorem 2.5.2. $\vdash_{\text{IPC}} \varphi$ iff φ is valid in every topological space.

2.6 The connection of descriptive frames with Heyting algebra and Heyting spaces

2.6.1 Descriptive frames and Heyting algebras

Definition 2.6.1. An *intuitionistic general frame* is a triple $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, where $\langle W, R \rangle$ is a Kripke frame and \mathcal{P} is a family of upsets containing \emptyset and closed under \cap , \cup and the following operation \supset : for every $X, Y \subseteq W$,

$$X \supset Y = \{x \in W : \forall y \in W (xRy \wedge y \in X \rightarrow y \in Y)\}$$

Elements of the set \mathcal{P} are called *admissible sets*.

General frames and Heyting algebras have a close connection. For proofs of the following two theorems, see e.g. [6].

Theorem 2.6.2. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a general frame. The algebra $\langle \mathcal{P}, \cap, \cup, \supset, \emptyset \rangle$ is a Heyting algebra and is called the dual of \mathfrak{F} , denoted by \mathfrak{F}^+ .

Theorem 2.6.3. Let \mathfrak{A} be a Heyting algebra, define $\mathfrak{A}_+ = \langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, \mathcal{P}_{\mathfrak{A}} \rangle$ as follows:

- (i) $W_{\mathfrak{A}} = \{\nabla \subseteq A : \nabla \text{ is a prime filter of } \mathfrak{A}\}$,
- (ii) $\nabla_1 R_{\mathfrak{A}} \nabla_2$ iff $\nabla_1 \subseteq \nabla_2$,
- (iii) $\mathcal{P}_{\mathfrak{A}} = \{\hat{a} : a \in A\}$, where $\hat{a} = \{\nabla \in W_{\mathfrak{A}} : a \in \nabla\}$.

Then \mathfrak{A}_+ is a general frame called the dual of \mathfrak{A} . Furthermore, $\mathfrak{A} \cong (\mathfrak{A}_+)^+ = \langle \mathcal{P}_{\mathfrak{A}}, \cap, \cup, \supset, \emptyset \rangle$.

The preceding theorem means that by applying the two operators $(\)_+$ and $(\)^+$ consecutively, one can go from a Heyting algebra through a general frame and then back to itself. However, in general, for any general frame \mathfrak{F} , it is not necessary that $(\mathfrak{F}^+)_+ \cong \mathfrak{F}$. So we have the following definition.

Definition 2.6.4. A *descriptive frame* \mathfrak{F} is a general frame satisfying $(\mathfrak{F}^+)_+ \cong \mathfrak{F}$.

Descriptive frames can also be defined frame-theoretically.

Definition 2.6.5. An intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is called *refined* if for any $x, y \in W$,

$$\forall X \in \mathcal{P} (x \in X \rightarrow y \in X) \Rightarrow xRy,$$

or equivalently,

$$\neg xRy \Rightarrow \exists X \in \mathcal{P} (x \in X \wedge y \notin X).$$

Definition 2.6.6. A family \mathcal{X} of sets has the *finite intersection property* if every finite subfamily $\mathcal{X}' \subseteq \mathcal{X}$ has a nonempty intersection, i.e., $\bigcap \mathcal{X}' \neq \emptyset$.

Definition 2.6.7. An intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ is called *compact*, if for any families $\mathcal{X} \subseteq \mathcal{P}$ and $\mathcal{Y} \subseteq \overline{\mathcal{P}} = \{W - X : X \in \mathcal{P}\}$ for which $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property, $\bigcap (\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$.

For a proof of the next theorem, see e.g. [6].

Theorem 2.6.8. *An intuitionistic general frame \mathfrak{F} is a descriptive frame iff it is refined and compact.*

The three truth-preserving operators on Kripke frames can be generalized to descriptive frames case.

Definition 2.6.9. A descriptive frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is called a *generated subframe* of a descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ if it satisfies the following conditions:

(S1) $\langle V, S \rangle$ is a generated subframe of $\langle W, R \rangle$,

(S2) $\mathcal{Q} = \{U \cap V : U \in \mathcal{P}\}$.

Definition 2.6.10. Given descriptive frames $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$, we call a surjective p-morphism f of $\langle W, R \rangle$ onto $\langle V, S \rangle$ a *surjective p-morphism* (or a *reduction*) of \mathfrak{F} onto \mathfrak{G} , if it also satisfies the following condition:

(R3) $\forall X \in \mathcal{Q}, f^{-1}(X) \in \mathcal{P}$.

Definition 2.6.11. Let $\{\mathfrak{F}_i = \langle W_i, R_i, \mathcal{P}_i \rangle\}_{i=1}^n$ be a finite family of disjoint descriptive frames. The *disjoint union* of the family $\{\mathfrak{F}_i\}_{i=1}^n$ is the frame $\biguplus_{i=1}^n \mathfrak{F}_i = \langle W, R, \mathcal{P} \rangle$, where $\langle W, R \rangle = \biguplus_{i=1}^n \langle W_i, R_i \rangle$ and $\mathcal{P} = \bigcup_{i=1}^n \mathcal{P}_i$.

The following theorem is well-known. For a proof, see e.g. [6].

Theorem 2.6.12. *Let \mathfrak{A} and \mathfrak{B} be Heyting algebras, and \mathfrak{F} and \mathfrak{G} descriptive frames. Then*

1. (a) \mathfrak{A} is a homomorphic image of \mathfrak{B} iff \mathfrak{A}_+ is isomorphic to a generated subframe of \mathfrak{B}_+ ;
- (b) \mathfrak{A} is isomorphic to a subalgebra of \mathfrak{B} iff \mathfrak{A}_+ is a p -morphic image of \mathfrak{B}_+ ;
- (c) $(\prod_{i=1}^n \mathfrak{A}_i)_+$ is isomorphic to the disjoint union $\biguplus_{i=1}^n (\mathfrak{A}_i)_+$;
2. (a) \mathfrak{F} is isomorphic to a generated subframe of \mathfrak{G} iff \mathfrak{F}^+ is a homomorphic image of \mathfrak{G}^+ ;
- (b) \mathfrak{F} is a p -morphic image of \mathfrak{G} iff \mathfrak{F}^+ is isomorphic to a subalgebra of \mathfrak{G}^+ ;
- (c) $(\biguplus_{i=1}^n \mathfrak{F}_i)^+$ is isomorphic to $\prod_{i=1}^n \mathfrak{F}_i^+$.

Given any descriptive frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, it is not necessary that the general frame generated by any subset of W is descriptive (for more details, see [26]). However, admissible subsets do generate descriptive frames.

Lemma 2.6.13. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a descriptive frame. A general frame $\mathfrak{G} = \langle W', R', \mathcal{P}' \rangle$ generated by an admissible subset of \mathfrak{F} is descriptive.*

Proof. It suffices to show that \mathfrak{G} is refined and compact. For refinedness, suppose $\neg w R' v$ for some $w, v \in W'$. Since \mathcal{P} is refined, there exists $U \in \mathcal{P}$ such that $w \in U$ and $v \notin U$. Let $U' = U \cap W'$. Then by (S2), $U' \in \mathcal{P}'$. And we have $w \in U'$ and $v \notin U'$.

For compactness, for any families $\mathcal{X} \subseteq \mathcal{P}'$ and $\mathcal{Y} \subseteq \overline{\mathcal{P}'} = \{W' \setminus U' : U' \in \mathcal{P}'\}$, suppose $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property.

Note that since $W' \in \mathcal{P}$, we have that $\mathcal{X} \subseteq \mathcal{P}' \subseteq \mathcal{P}$. Define

$$\mathcal{Y}^* = \{W \setminus U' : U' \in \mathcal{P}', W' \setminus U' \in \mathcal{Y}\} \subseteq \overline{\mathcal{P}}.$$

Take any $X_1, \dots, X_n \in \mathcal{X}$, and $Y_1^*, \dots, Y_k^* \in \mathcal{Y}^*$. We know that $X_i = X_i \cap W'$ for all $1 \leq i \leq n$, and there exist $Y_1, \dots, Y_k \in \mathcal{Y}$ such that

$$Y_j = Y_j^* \cap W' (1 \leq j \leq k).$$

Observe that

$$\bigcap_{i=1}^n X_i \cap \bigcap_{j=1}^k Y_j = \bigcap_{i=1}^n X_i \cap \bigcap_{j=1}^k Y_j^* \cap W' = \bigcap_{i=1}^n (X_i \cap W') \cap \bigcap_{j=1}^k Y_j^* = \bigcap_{i=1}^n X_i \cap \bigcap_{j=1}^k Y_j^*.$$

So, the fact that $\mathcal{X} \cup \mathcal{Y}$ has the finite intersection property implies that $\mathcal{X} \cup \mathcal{Y}^*$ has the finite intersection property.

Similarly, we also have that

$$\bigcap (\mathcal{X} \cup \mathcal{Y}) = \bigcap \mathcal{X} \cap \bigcap \mathcal{Y} = \bigcap \mathcal{X} \cap \bigcap \mathcal{Y}^* = \bigcap (\mathcal{X} \cup \mathcal{Y}^*).$$

Thus, by the compactness of \mathfrak{F} , it holds that $\bigcap (\mathcal{X} \cup \mathcal{Y}^*) \neq \emptyset$, from which it follows that $\bigcap (\mathcal{X} \cup \mathcal{Y}) \neq \emptyset$. \square

2.6.2 Descriptive frames and Heyting spaces

Definition 2.6.14. A Heyting space is a triple $\mathcal{X} = \langle X, \tau, R \rangle$, where $\langle X, \tau \rangle$ is a Stone space, and R is a partial order on X such that

- for each $x \in X$, $R(x)$ is closed;
- for each $U \in \mathcal{CO}(X)$, $R^{-1}(U) \in \mathcal{CO}(X)$.

The following two theorems follow from [13].

Theorem 2.6.15. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a descriptive frame. Let $\tau_{\mathcal{P}}$ be the topology generated by the sub-basis $\mathcal{P} \cup \overline{\mathcal{P}}$, where

$$\overline{\mathcal{P}} = \{W \setminus U : U \in \mathcal{P}\}.$$

Then $\mathcal{X} = \langle W, \tau_{\mathcal{P}}, R \rangle$ is a Heyting spaces.

Theorem 2.6.16. Let $\mathcal{X} = \langle X, \tau, R \rangle$ be a Heyting space. Define

$$\mathcal{P}_{\mathcal{X}} = \{U \subseteq X : U \text{ is a clopen upset}\}.$$

Then $\mathfrak{F} = \langle X, R, \mathcal{P}_{\mathcal{X}} \rangle$ is a descriptive frame.

In this connection, we can define the topological counterparts of the three truth-preserving operators of descriptive frames. Let $\mathcal{X} = \langle X, \tau, R \rangle$ and $\mathcal{Y} = \langle Y, \nu, S \rangle$ be Heyting spaces.

Definition 2.6.17. \mathcal{Y} is called a *generated subframe* of \mathcal{X} if $\langle Y, \nu, S \rangle$ is a generated subframe of $\langle X, R \rangle$ and $\langle Y, \nu \rangle$ is a closed subspace of $\langle X, \tau \rangle$.

Definition 2.6.18. A map $f : X \rightarrow Y$ is called a *continuous p-morphism* of \mathcal{X} to \mathcal{Y} if f is continuous and a p-morphism of $\langle X, R \rangle$ to $\langle Y, S \rangle$.

Definition 2.6.19. Let $\{\mathcal{X}_i = \langle X_i, \tau_i, R_i \rangle\}_{i=1}^n$ be a family of disjoint Heyting spaces. The *disjoint union* of the family $\{\mathcal{X}_i\}_{i=1}^n$ is the frame $\biguplus_{i \in I} \mathcal{X}_i = \langle X, \tau, R \rangle$, where $\langle X, R \rangle = \biguplus_{i=1}^n \langle X_i, R_i \rangle$ and $\langle X, \tau \rangle = \bigoplus_{i=1}^n \langle X_i, \tau_i \rangle$ (the topological sum).

2.6.3 Heyting algebras and Heyting spaces

For proofs of the following two theorems, see [13].

Theorem 2.6.20. Let $\mathfrak{A} = \langle A, \vee, \wedge, \rightarrow, 0 \rangle$ be a Heyting algebra. Let $X_{\mathfrak{A}}$ be the set of all prime filters of \mathfrak{A} , and $\tau_{\mathfrak{A}}$ be a topology generated by the basis $\{\hat{a}, A \setminus \hat{a}\}_{a \in A}$, where

$$\hat{a} = \{\nabla \in X_{\mathfrak{A}} : a \in \nabla\}.$$

Then $\mathcal{X} = \langle X_{\mathfrak{A}}, \tau_{\mathfrak{A}}, \subseteq \rangle$ is a Heyting space.

Theorem 2.6.21. Let $\mathcal{X} = \langle X, \tau, R \rangle$ be a Heyting space. The algebra $\langle \mathcal{CO}(X), \cup, \cap, \rightarrow, \emptyset \rangle$ is a Heyting algebra, where \rightarrow is defined as

$$U \rightarrow V = X \setminus R^{-1}(U \setminus V).$$

2.6.4 Duality of categories DF, HA and HS

Let

- DF be the category of descriptive frames and descriptive p-morphisms;
- HA be the category of Heyting algebras and Heyting homomorphisms;
- HS be the category of Heyting spaces and continuous p-morphisms.

Theorem 2.6.22. DF is dually equivalent to HA, and HS is dually equivalent to HA.

2.7 The variety of Heyting algebras

2.7.1 HS = SH

Definition 2.7.1. A nonempty class K of algebras is called a *variety* if it is closed under subalgebras, homomorphic images and direct products.

For any class K of algebras, let $\mathbf{V}(K)$ denote the smallest variety containing K . And let $\mathbf{S}(K)$, $\mathbf{H}(K)$ and $\mathbf{P}(K)$ denote the smallest class containing K which is closed under subalgebras, homomorphic images and direct products respectively.

Theorem 2.7.2. $\mathbf{V} = \mathbf{HSP}$.

Theorem 2.7.3. *The class of all Heyting algebras forms a variety \mathbf{HA} .*

\mathbf{HA} has the property that $\mathbf{HS} = \mathbf{SH}$, which follows from the congruence extension property of \mathbf{HA} .

Definition 2.7.4. We say an algebra \mathfrak{A} has *the congruence extension property (CEP)* if for every subalgebra \mathfrak{B} of \mathfrak{A} , and every $\theta \in \text{Con } \mathfrak{B}$ ($\text{Con } \mathfrak{B}$: the set of all congruences of \mathfrak{B}), there is a $\varphi \in \text{Con } \mathfrak{A}$ such that $\theta = \varphi \cap \mathfrak{B}^2$. A variety V has the **CEP** if every algebra in V has the **CEP**.

Definition 2.7.5. Let $f : \mathfrak{A} \rightarrow \mathfrak{B}$ be a homomorphism. Then the *kernel* of f , $\ker(f)$, is defined by

$$\ker(f) = \{(a, b) \in A^2 : f(a) = f(b)\}.$$

For a proof of the following theorem see e.g. [1].

Theorem 2.7.6. *The variety of Heyting algebras \mathbf{HA} has the congruence extension property.*

Theorem 2.7.7. *If a variety V has the **CEP**, then for any $K \subseteq V$, $\mathbf{HS}(K) = \mathbf{SH}(K)$.*

Proof. It suffices to show that $\mathbf{HS}(K) \subseteq \mathbf{SH}(K)$. For any $\mathfrak{A} \in \mathbf{HS}(K)$, there exist $\mathfrak{B}, \mathfrak{C} \in K$ such that \mathfrak{C} is a subalgebra of \mathfrak{B} and there exists surjective homomorphism $f : \mathfrak{C} \rightarrow \mathfrak{A}$. Note that $\ker(f) \in \text{Con } \mathfrak{C}$. By the congruence extension property, there exists $\theta \in \text{Con } \mathfrak{B}$ such that

$$\ker(f) = \theta \cap \mathfrak{C}^2. \tag{2.1}$$

It follows from $\mathfrak{C} \in \mathbf{S}(\mathfrak{B})$ and (2.1) that

$$\mathfrak{C}/\ker f = \mathfrak{C}/\theta = \{[c]_\theta : c \in C\}.$$

By the Homomorphism Theorem we have that $\mathfrak{A} \cong \mathfrak{C}/\ker(f)$. Consider the set $\mathfrak{B}/\theta = \{[b]_\theta : b \in B\}$. We have that $\mathfrak{C}/\ker(f) \in \mathbf{S}(\mathfrak{B}/\theta)$. Together with the fact that $\mathfrak{B}/\theta \in \mathbf{H}(\mathfrak{B})$, we obtain $\mathfrak{A} \in \mathbf{SH}(K)$. \square

Corollary 2.7.8. *For any Heyting algebra $\mathfrak{A} \in \mathbf{HA}$, $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$.*

By the duality result, we have the following corollary.

Corollary 2.7.9. *A descriptive frame \mathfrak{F} is a p -morphic image of a generated subframe of a descriptive frame \mathfrak{G} iff \mathfrak{F} is a generated subframe of a p -morphic image of \mathfrak{G} .*

2.7.2 Heyting algebras and intermediate logics

For every intermediate logic L , since the three operators \mathbf{H} , \mathbf{S} and \mathbf{P} are truth-preserving, the class of all Heyting algebras that validates L forms a variety. We have the following theorem in universal algebra. For a proof, see e.g. [5].

Theorem 2.7.10. *Every variety of algebras is generated by its finitely generated subdirectly irreducible algebras.*

Note that by the duality theorems in the preceding sections, subdirectly irreducible Heyting algebras correspond to rooted descriptive frames. Thus, the above theorem gives the following corollary in the descriptive frames context.

Corollary 2.7.11. *Every intermediate logic is complete with respect to finitely generated rooted descriptive frames.*

2.7.3 Free Heyting algebras

Definition 2.7.12. Given a variety V of algebras and a set X , the free algebra in V generated by $X \subseteq F(X)$ is the algebra $\mathbf{F}_V(X)$ satisfying for any $\mathfrak{A} \in V$, any map $f : X \rightarrow \mathfrak{A}$ can be extended uniquely to a homomorphism $\bar{f} : \mathbf{F}(X) \rightarrow \mathfrak{A}$.

If $|X| = |Y|$, then $\mathbf{F}_V(X) \cong \mathbf{F}_V(Y)$ (see e.g. [5]). So we will only speak of *the* free algebra in V on a certain number of generators and denote such a free algebra by $\mathbf{F}_V(n)$ if the cardinality of the set of generators is n . The n -generated free Heyting algebra is then denoted by $\mathbf{F}_{\mathbf{HA}}(n)$ or $\mathbf{F}(n)$ for short.

Chapter 3

n -universal models of IPC

In this chapter, we discuss n -universal models $\mathcal{U}(n)$ of **IPC**. The descriptive frame $\mathbf{F}(n)_+$ dual to the n -generated free Heyting algebra $\mathbf{F}(n)$ is isomorphic to the n -Henkin frame $\mathcal{H}(n)$.

In Section 3.1, we recall the definition of n -universal models by giving a more mathematical definition than the one in [3]. N. Bezhanishvili gave in [3] an algebraic proof of the fact that the upper part of $\mathcal{H}(n)$ is isomorphic to $\mathcal{U}(n)$. In Section 3.2, we prove it directly from a model-theoretic point of view. In Section 3.3, we prove that $\mathcal{H}(n)$ and $\mathcal{U}(n)$ are “connected” in the sense that every infinite upset of $\mathcal{H}(n)$ has an infinite intersection in $\mathcal{U}(n)$. As a preparation of this result, we give a frame-based new proof of Jankov’s theorem in [18]. The proof uses the de Jongh formulas and the idea of the proof is inspired by the algebraic proof in de Jongh [8]. It then turns out that the idea of this proof can be generalized to prove Jankov’s theorem on **KC** proved in [19]. In Section 3.4, we give this alternative proof by making slight modifications to the de Jongh formulas in such a way that the new formulas are negation-free and satisfy a theorem similar to Jankov’s theorem in [18].

3.1 n -universal models of IPC

In this section we recall the definition of an n -universal model by giving a mathematical definition. Throughout this section, we will talk about the valuation of point w in a n -model \mathfrak{M} by using the terminology *color*. In general, an n -color is a 0-1-sequence $c_1 \cdots c_n$ of length n . If the length is

understandable from the context, we will only talk about a *color* instead of an n -color. The set of all n -colors is denoted by \mathbf{C}^n . The *projections* $\pi_i : \mathbf{C}^n \rightarrow \{0, 1\}$ ($1 \leq i \leq n$) are defined by $\pi_i(c_1 \cdots c_n) = c_i$. We define an ordering on the colors as follows:

$$c_1 \cdots c_n \leq c'_1 \cdots c'_n \text{ iff } c_i \leq c'_i \text{ for each } 1 \leq i \leq n.$$

We write $c_1 \cdots c_n < c'_1 \cdots c'_n$ if $c_1 \cdots c_n \leq c'_1 \cdots c'_n$ but $c_1 \cdots c_n \neq c'_1 \cdots c'_n$.

A *coloring* on a nonempty set W is a function $col : W \rightarrow \mathbf{C}^n$. Colorings and valuations on frames are in one-one correspondence. Given a $\mathfrak{M} = \langle W, R, V \rangle$, we can describe the valuation of a point by the coloring $col_V : W \rightarrow \mathbf{C}^n$, defined by $col_V(w) = c_1 \cdots c_n$, where for each $1 \leq i \leq n$,

$$c_i = \begin{cases} 1, & w \in V(p_i); \\ 0, & w \notin V(p_i). \end{cases}$$

We call $col_V(w)$ the *color of w under V* . On the other hand, given a coloring col on a frame $\mathfrak{F} = \langle W, R \rangle$, we can define a valuation $V_{col} : \text{PROP} \rightarrow \wp(W)$ on \mathfrak{F} by taking

$$V_{col}(p_i) = col^{-1}(\{c_1 \cdots c_n \in \mathbf{C}^n : c_i = 1\}).$$

In any frame $\mathfrak{F} = \langle W, R \rangle$, we say that a subset $X \subseteq W$ *totally covers* a point $w \in W$, denoted by $w \prec X$, if X is the set of all immediate successors of w . We will just write $w \prec v$ in the case that $w \prec \{v\}$. A subset $X \subseteq W$ is called an *anti-chain* if $|X| > 1$ and for every $w, v \in X$, $w \neq v$ implies that $\neg wRv$ and $\neg vRw$. Let R^+ denote the transitive and reflexive closure of a relation R .

Now, we define every layer of an n -universal model $\mathcal{U}(n)$ inductively. We first give a less mathematical definition as follows:

Definition 3.1.1.

- The first layer, $\mathcal{U}(n)^0 = \langle U(n)^0, R^0, V^0 \rangle$ consists of 2^n points with 2^n different n -colors, and the relation $R^0 = \emptyset$.
- The model $\mathcal{U}(n)^{k+1} = \langle U(n)^{k+1}, R^{k+1}, V^{k+1} \rangle$ of the first $k + 1$ layers of $\mathcal{U}(n)$ is obtained from $\mathcal{U}(n)^k$ by adding the following elements:

- for each point $w \in U(n)^k \setminus U(n)^{k-1}$ and each color $s < col(w)$, we introduce a unique new point $v_{w,s}$ into $U(n)^{k+1}$. Extend the coloring col_{V^k} such that $col_{V^{k+1}}(v_{w,s}) = s$. Extend R^k to the smallest partial order satisfying $v_{w,s} \prec w$.
- for each finite anti-chain X in $U(n)^k$ such that $X \not\subseteq U(n)^{k-1}$ and each color s with $s \leq col(w)$ for all $w \in X$, we introduce a unique new point $v_{X,s}$ into $U(n)^{k+1}$. Extend the coloring col_{V^k} such that $col_{V^{k+1}}(v_{X,s}) = s$. Extend R^k to the smallest partial order satisfying $v_{X,s} \prec X$.
- Let $U(n)^{k+1}$, col^{k+1} and R^{k+1} be extensions of $U(n)^k$, col^k and R^k according to the above regulations respectively.

Define the n -universal model $\mathcal{U}(n) = \langle U(n), R, V \rangle$ by taking

$$U(n) = \bigcup_{i \in \omega} U(n)^i, \quad R = \left(\bigcup_{i \in \omega} R^i \right)^+, \quad col = \bigcup_{i \in \omega} col^i \quad \text{and} \quad V = V_{col}.$$

Having the intuition of what an n -universal model $\mathcal{U}(n)$ looks like in mind, we now define every layer of $\mathcal{U}(n)$ precisely.

Definition 3.1.2.

- The first layer $\mathcal{U}(n)^0 = \langle U(n)^0, R^0, V^0 \rangle$ of $\mathcal{U}(n)$ is defined by taking

$$U(n)^0 = \{w_1, \dots, w_{2^n}\}, \quad R^0 = \emptyset,$$

and letting $V^0 = V_{col^0}$, where the coloring $col^0 : U(n)^0 \rightarrow \mathbf{C}^n$ is a bijection.

- The model $\mathcal{U}(n)^{k+1} = \langle U(n)^{k+1}, R^{k+1}, V^{k+1} \rangle$ of the first $k+1$ layers of $\mathcal{U}(n)$ is defined as follows:

- Let

$$E^{k+1} = \{v_{w,s} : w \in U(n)^k \setminus U(n)^{k-1}, \quad s \in \mathbf{C}^n \text{ and } s < col(w)\},$$

$$S^{k+1} = \{(v_{w,s}, w) : v_{w,s} \in E^{k+1}\}.$$

Define a coloring $col_0^{k+1} : E^{k+1} \rightarrow \mathbf{C}^n$ by taking

$$col_0^{k+1}(v_{w,s}) = s.$$

– Let

$$F^{k+1} = \{v_{X,s} : X \text{ is a finite anti-chain in } U(n)^k \text{ s.t. } X \not\subseteq U(n)^{k-1}, \\ s \in \mathbf{C}^n \text{ and } s \leq \text{col}(w) \text{ for all } w \in X\},$$

$$T^{k+1} = \{(v_{X,s}, w) : v_{X,s} \in F^{k+1} \text{ and } w \in X\}.$$

Define a coloring $\text{col}_1^{k+1} : F^{k+1} \rightarrow \mathbf{C}^n$ by taking

$$\text{col}_1^{k+1}(v_{X,s}) = s.$$

– Finally, let

$$U(n)^{k+1} = U^k \cup E^{k+1} \cup F^{k+1},$$

$$R^{k+1} = (R^k \cup S^{k+1} \cup T^{k+1})^+,$$

$$\text{col}^{k+1} = \text{col}^k \cup \text{col}_0^{k+1} \cup \text{col}_1^{k+1},$$

$$V^{k+1} = V_{\text{col}^{k+1}}.$$

It is easy to see from the construction that every $\mathcal{U}(n)^k$ is finite. As a consequence, for any finite subset X of $U(n)$, the generated submodel $\mathcal{U}(n)_X$ is finite. In particular, $\mathcal{U}(n)_w$ is finite for any point w in $U(n)$.

The 1-universal model is also called *Rieger-Nishimura ladder*, which is depicted in Figure 3.1.

3.2 n -universal models and n -Henkin models

Let $\text{Upper}(\mathfrak{M})$ denote the submodel $\mathfrak{M}_{\{w \in W : d(w) < \omega\}}$ generated by all the points with finite depth. It is known that the n -universal model is isomorphic to the finite part of the n -Henkin model $\text{Upper}(\mathcal{H}(n))$. N. Bezhanishvili gave in [3] an algebraic proof of this fact. In this section, we prove it directly. Two important lemmas are that every finite model can be mapped p -morphically to a generated submodel of $\mathcal{U}(n)$, and that $\mathcal{U}(n)_w$ is isomorphic to the submodel of $\mathcal{H}(n)$ generated by the theory of the de Jongh formula of w .

Lemma 3.2.1. *Let f be a p -morphism of $\mathfrak{M} = \langle W, R, V \rangle$ onto $\mathfrak{N} = \langle W', R', V' \rangle$. If $w \in W$, then $f \upharpoonright R(w)$ is a p -morphism of \mathfrak{M}_w onto $\mathfrak{N}_{f(w)}$.*

Proof. Trivial. □

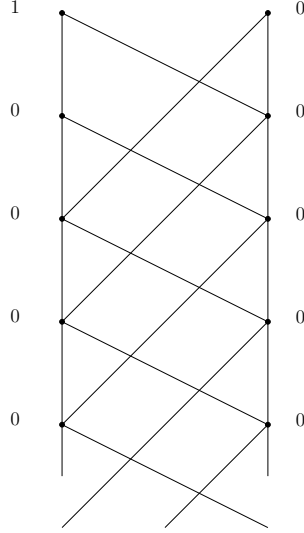


Figure 3.1: Rieger-Nishimura ladder

Lemma 3.2.2. *Given Kripke models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{N} = \langle W', R', V' \rangle$, and two points $w, v \in W$. Let f be a p -morphism of \mathfrak{M}_w to \mathfrak{N} , and g be a p -morphism of \mathfrak{M}_v to \mathfrak{N} . If for each $u \in \text{dom}(f) \cap \text{dom}(g)$, $f(u) = g(u)$, then $f \cup g$ is a p -morphism of $\mathfrak{M}_{\{w,v\}}$ to \mathfrak{N} .*

Proof. Trivial. □

The next theorem shows that every finite model can be mapped p -morphically onto a generated submodel of $\mathcal{U}(n)$. In the sequel, we will often refer to this useful theorem.

Theorem 3.2.3. *For any finite rooted Kripke n -model \mathfrak{M} , there exists a unique $w \in \mathcal{U}(n)$ and a p -morphism of \mathfrak{M} onto $\mathcal{U}(n)_w$.*

Proof. Let $\mathfrak{M} = \langle W, R \rangle$ and $\mathcal{U}(n) = \langle \mathcal{U}(n), \preceq \rangle$. We prove the lemma by induction on the depth of \mathfrak{M} . If $d(\mathfrak{M}) = 1$, then \mathfrak{M} is a single node x with $\text{col}(x) = i_1 \dots i_n$. By the definition of $\mathcal{U}(n)$, there exists a unique $w \in \mathcal{U}(n)^0$ such that $\text{col}(w) = \text{col}(x)$. Clearly, the map $f : \{w\} \rightarrow \{x\}$, defined as $f(w) = x$, is a unique p -morphism of \mathfrak{M} onto $\mathcal{U}(n)_w$.

If $d(\mathfrak{M}) = k + 1$, for any $x \in W$ with $d(x) = k + 1$, let $\{x_1, \dots, x_m\}$ be the set of all x 's immediate successors. By the induction hypothesis, we have that for each $1 \leq i \leq m$, there exist a unique w_i and a unique p -morphism f_i of \mathfrak{M}_{x_i} onto $\mathcal{U}(n)_{w_i}$.

For any $1 \leq i, j \leq m$, we have

$$y \in \text{dom}(f_i) \cap \text{dom}(f_j) \implies f_i(y) = f_j(y). \quad (3.1)$$

Indeed, since $d(y) \leq k$, by the induction hypothesis, there exist a unique w_y and a unique p-morphism f_y of \mathfrak{M}_y onto $\mathcal{U}(n)_{w_y}$. Now by Lemma 3.2.1, $f_i \upharpoonright R(y)$ and $f_j \upharpoonright R(y)$ are p-morphisms of \mathfrak{M}_y onto $\mathcal{U}(n)_{f_i(y)}$ and $\mathcal{U}(n)_{f_j(y)}$ respectively. Then, by the uniqueness, we must have that $f_i \upharpoonright R(y) = f_j \upharpoonright R(y)$, i.e. (3.1) holds.

Put

$$f = f_1 \cup \dots \cup f_m.$$

By (3.1), f is a well-defined function. Moreover, by Lemma 3.2.2, f is a p-morphism of $\mathfrak{M}_{\{x_1, \dots, x_m\}}$ onto $\mathcal{U}(n)_{\{w_1, \dots, w_m\}}$. And clearly, f is unique.

Let $X = \{w_{i_1}, \dots, w_{i_l}\}$ be a subset of $\{w_1, \dots, w_m\}$ such that for any $w_i \in \{w_{i_1}, \dots, w_{i_l}\}$ and any $w_j \in \{w_1, \dots, w_m\}$,

$$w_i \neq w_j \implies \neg w_j \preceq w_i.$$

Observe that X is either a singleton or an anti-chain. If X is a singleton $\{w_{i_1}\}$, then since

$$\text{col}(x) \leq \text{col}(x_i) = \text{col}(w_i) \quad (3.2)$$

for any $1 \leq i \leq m$, there are two cases. Case 1: $\text{col}(w_{i_1}) = \text{col}(x)$. Then define $g = f \cup \{(x, w_{i_1})\}$. It follows from (3.2) that g is a unique surjective p-morphism.

Case 2: $\text{col}(x) < \text{col}(w_{i_1})$. Then by (3.2) and the definition of $\mathcal{U}(n)$, there exists a unique point $w \in U(n)$ such that $\text{col}(w) = \text{col}(x)$ and $w \prec w_{i_1}$, which implies that

$$w \prec w_i \text{ for any } w_i \in \{w_1, \dots, w_m\}. \quad (3.3)$$

Define $g = f \cup \{(x, w)\}$. It follows from (3.2) and (3.3) that g is a unique surjective p-morphism.

If X is an anti-chain, then by (3.2) and the definition of $\mathcal{U}(n)$, there exists a unique point $w \in U(n)$ such that

$$\text{col}(w) = \text{col}(x) \text{ and } w \prec X,$$

so (3.3) holds. Define $g = f \cup \{(x, w)\}$. It follows from (3.2) and (3.3) that g is a unique surjective p-morphism. \square

The next theorem implies that $\mathcal{U}(n)$ is a countermodel of every non-theorem of **IPC**. This fact explains the meaning of $\mathcal{U}(n)$ being an “universal model”.

Theorem 3.2.4. *For any formula $\varphi(\vec{p})$, $\mathcal{U}(n) \models \varphi$ iff $\vdash_{\mathbf{IPC}} \varphi$.*

Proof. “ \Leftarrow ” is trivial. Suppose $\not\vdash_{\mathbf{IPC}} \varphi(\vec{p})$. Then there exists a finite model \mathfrak{M} and a point $w \in \mathfrak{M}$ such that $\mathfrak{M}, w \not\models \varphi(\vec{p})$. By Theorem 3.2.3, there exists a p-morphism f of \mathfrak{M} to $\mathcal{U}(n)$. Hence, $\mathcal{U}(n), f(w) \not\models \varphi(\vec{p})$. \square

For any point w in an n -model \mathfrak{M} , if $\{w_1, \dots, w_m\}$ is the set of all immediate successors of w , then we let

$$\text{prop}(w) := \{p_i : w \models p_i, 1 \leq i \leq n\},$$

$$\text{notprop}(w) := \{q_i : w \not\models q_i, 1 \leq i \leq n\},$$

$$\text{newprop}(w) := \{r_j : w \not\models r_j \text{ and } w_i \models r_j \text{ for each } 1 \leq i \leq m, \text{ for } 1 \leq j \leq n\}^1.$$

Next, we define the de Jongh formulas, which were first introduced in [8].

Definition 3.2.5. Let w be a point in $\mathcal{U}(n)$. we inductively define *de Jongh formulas* φ_w and ψ_w .

If $d(w) = 1$, then let

$$\varphi_w := \bigwedge \text{prop}(w) \wedge \bigwedge \{\neg p_k : w \not\models p_k, 1 \leq k \leq n\}$$

and

$$\psi_w := \neg \varphi_w.$$

If $d(w) > 1$, and $\{w_1, \dots, w_m\}$ is the set of all immediate successors of w , then define

$$\varphi_w := \bigwedge \text{prop}(w) \wedge \left(\bigvee \text{newprop}(w) \vee \bigvee_{i=1}^m \psi_{w_i} \rightarrow \bigvee_{i=1}^m \varphi_{w_i} \right)$$

and

$$\psi_w := \varphi_w \rightarrow \bigvee_{i=1}^m \varphi_{w_i}.$$

¹Note that if w is an endpoint, $\text{newprop}(w) = \text{notprop}(w)$.

The most important properties of the de Jongh formulas are revealed in the next theorem. It was first proved in [8].

Theorem 3.2.6. *For every $w \in \mathcal{U}(n) = \langle U(n), R, V \rangle$, we have that*

- $V(\varphi_w) = R(w)$;
- $V(\psi_w) = U(n) \setminus R^{-1}(w)$.

Now, we recall the notion of an isomorphism between two Kripke models.

Definition 3.2.7. Let $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}' = \langle W', R', V' \rangle$ be two Kripke models. A bijection $f : W \rightarrow V$ is called an *isomorphism* from \mathfrak{M} onto \mathfrak{M}' if the following conditions are satisfied.

- For any $p \in \text{PROP}$ and any $x \in W$, $x \in V(p)$ iff $f(x) \in V'(p)$.
- For any $x, y \in W$, xRy iff $f(x)R'f(y)$.

We say that \mathfrak{M} is isomorphic to \mathfrak{M}' , in symbols $\mathfrak{M} \cong \mathfrak{M}'$, if there is an isomorphism from \mathfrak{M} to \mathfrak{M}' .

Remark 3.2.8. *An injective and surjective p -morphism is an isomorphism.*

Lemma 3.2.9. *Let f be a isomorphism of $\mathfrak{M} = \langle W, R, V \rangle$ onto $\mathfrak{N} = \langle W', R', V' \rangle$. If $w \in W$, then $f \upharpoonright R(w)$ is an isomorphism of \mathfrak{M}_w onto $\mathfrak{N}_{f(w)}$.*

Proof. Trivial. □

The next lemma is important in the proof of the main theorem of this section.

Lemma 3.2.10. *For any $w \in \mathcal{U}(n)$, let φ_w be a de Jongh formula. Then we have that $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$.*

Proof. Let $\mathcal{U}(n) = \langle U(n), R, V \rangle$ and $\mathcal{H}(n) = \langle H(n), R', V' \rangle$. Define a map $f : \mathcal{U}(n)_w \rightarrow \mathcal{H}(n)_{Cn(\varphi_w)}$ by taking

$$f(v) = Cn(\varphi_v).$$

We show that f is an isomorphism.

First for any $v \in U(n)$, any formula θ , we have that

$$\begin{aligned}
\theta \in Cn(\varphi_v) &\iff \vdash_{\mathbf{IPC}} \varphi_v \rightarrow \theta \\
&\iff \mathcal{U}(n) \models \varphi_v \rightarrow \theta \text{ (by Theorem 3.2.4)} \\
&\iff \forall u \in V(\varphi_v), \mathcal{U}(n), u \models \theta \\
&\iff \forall u \in R(v), \mathcal{U}(n), u \models \theta \text{ (by Theorem 3.2.6)} \\
&\iff \mathcal{U}(n), v \models \theta.
\end{aligned} \tag{3.4}$$

It follows that $v \in V(p) \iff Cn(\varphi_v) \in V'(p)$ and that

$$\begin{aligned}
uRv &\iff \mathcal{U}(n), v \models \varphi_u \text{ (by Theorem 3.2.6)} \\
&\iff \varphi_u \in Cn(\varphi_v) \text{ (by (3.4))} \\
&\iff Cn(\varphi_u) \subseteq Cn(\varphi_v) \\
&\iff f(u)R'f(v).
\end{aligned}$$

Now, suppose $u \neq v$. W.l.o.g. we may assume that $\neg uRv$, which by Theorem 3.2.6 means that $\mathcal{U}(n), u \not\models \varphi_v$. Thus, $\varphi_v \notin Cn(\varphi_u)$ by (3.4), and so $f(u) = Cn(\varphi_u) \neq Cn(\varphi_v) = f(v)$. Hence, f is injective.

It remains to show that f is surjective. That is to show that for any $\Gamma \in \mathcal{H}(n)_{Cn(\varphi_w)}$ (i.e. any theory $\Gamma \supseteq Cn(\varphi_w)$ with the disjunction property) there exists $v \in \mathcal{U}(n)_w$ such that $\Gamma = Cn(\varphi_v)$. We show by induction on $d(u)$ that for any theory Γ with the disjunction property,

if $Cn(\varphi_u) \subseteq \Gamma$, then $\Gamma = Cn(\varphi_v)$ for some $v \in U(n)$ with uRv .

$d(u) = 1$. It suffices to show that if $Cn(\varphi_u) \subseteq \Gamma$, then $\Gamma = Cn(\varphi_u)$. Suppose $Cn(\varphi_u) \subsetneq \Gamma$. Then there exists $\theta \in \Gamma$ such that $\theta \notin Cn(\varphi_u)$. So we have that $\varphi_u \not\models \theta$. Then by Theorem 3.2.4, $\mathcal{U}(n) \not\models \varphi_u \rightarrow \theta$, which by Theorem 3.2.6 means that $u \not\models \theta$. Since $d(u) = 1$, we have that

$$u \models \neg\theta, \text{ and so } \mathcal{U}(n) \models \varphi_u \rightarrow \neg\theta,$$

by Theorem 3.2.6. Thus, by Theorem 3.2.4, $\vdash_{\mathbf{IPC}} \varphi_u \rightarrow \neg\theta$. Therefore $\neg\theta \in Cn(\varphi_u)$, and since $Cn(\varphi_u) \subset \Gamma$, we have that $\neg\theta \in \Gamma$, which is impossible since $\theta \in \Gamma$ and Γ is consistent.

$d(u) = k + 1$. Let $\{u_1, \dots, u_m\}$ be the set of all immediate successors of u . Suppose $Cn(\varphi_u) \subseteq \Gamma$. If $Cn(\varphi_{u_i}) \subseteq \Gamma$ for some $1 \leq i \leq m$, then by induction hypothesis, $\Gamma = Cn(\varphi_v)$ for some $v \in R(u_i)$, i.e. $v \in R(u)$. Now suppose

$Cn(\varphi_{u_i}) \not\subseteq \Gamma$ for all $1 \leq i \leq m$. Thus $\Gamma \not\vdash \varphi_{u_i}$ for each $1 \leq i \leq m$. Then for every $\theta \in \Gamma$, we have that

$$\theta \not\vdash \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

Since $Cn(\varphi_u) \subseteq \Gamma$, $\varphi_u \in \Gamma$ and $\theta \wedge \varphi_u \in \Gamma$, which implies that

$$\theta \wedge \varphi_u \not\vdash \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}.$$

Then there exists a finite rooted model \mathfrak{M} such that $\mathfrak{M} \models \theta \wedge \varphi_u$ and $\mathfrak{M} \not\models \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}$. By Theorem 3.2.3, there exists a unique $u' \in U(n)$ such that $\mathcal{U}(n)_{u'}$ is a p-morphic image of \mathfrak{M} . Thus we have that

$$\mathcal{U}(n), u' \models \varphi_u \text{ and } \mathcal{U}(n), u' \not\models \varphi_{u_1} \vee \cdots \vee \varphi_{u_m}. \quad (3.5)$$

By the definition, the former of the above means that

$$\mathcal{U}(n), u' \models \bigwedge \text{prop}(u) \wedge \left(\bigvee \text{newprop}(u) \vee \bigvee_{i=1}^m \psi_{u_i} \rightarrow \bigvee_{i=1}^m \varphi_{u_i} \right).$$

It then follows that

$$\mathcal{U}(n), u' \models \bigwedge \text{prop}(u), \quad (3.6)$$

$$\mathcal{U}(n), u' \models \bigvee \text{newprop}(u) \vee \bigvee_{i=1}^m \psi_{u_i} \rightarrow \bigvee_{i=1}^m \varphi_{u_i}. \quad (3.7)$$

In view of the second formula of (3.5), we must have that

$$\mathcal{U}(n), u' \not\models \bigvee \text{newprop}(u) \text{ and } \mathcal{U}(n), u' \not\models \bigvee_{i=1}^m \psi_{u_i}. \quad (3.8)$$

By Theorem 3.2.6 the second formula of (3.8) implies that $u' \notin U(n) \setminus R^{-1}(u_i)$ for all $1 \leq i \leq m$. Thus, we have that

$$u' \in \bigcap_{i=1}^m R^{-1}(u_i), \quad (3.9)$$

and so

$$\text{notprop}(u') \supseteq \bigcup_{i=1}^m \text{notprop}(u_i). \quad (3.10)$$

Then since

$$\text{notprop}(u) = \text{newprop}(u) \cup \bigcup_{i=1}^m \text{notprop}(u_i),$$

by the first formula of (3.8) we have that

$$\text{notprop}(u') \supseteq \text{notprop}(u).$$

Together with (3.6) we obtain

$$\text{col}(u') = \text{col}(u). \quad (3.11)$$

Next, claim that

$$u' \prec \{u_1, \dots, u_m\}, \quad (3.12)$$

i.e. u_1, \dots, u_m are the only immediate successors of u' . Suppose otherwise. Then there exists $v \neq u_i$ for all $1 \leq i \leq m$ such that v is an immediate successor of u' . Observe that $\{u_1, \dots, u_m, v\}$ is an anti-chain. Thus we have that

$$v \notin R^{-1}(u_i) \text{ and } v \notin R(u_i),$$

for each $1 \leq i \leq m$. Then by Theorem 3.2.6,

$$\mathcal{U}(n), v \models \bigvee_{i=1}^m \psi_{u_i} \text{ and } \mathcal{U}(n), v \not\models \bigvee_{i=1}^m \varphi_{u_i},$$

thus

$$\mathcal{U}(n), v \not\models \bigvee \text{newprop}(u) \vee \bigvee_{i=1}^m \psi_{u_i} \rightarrow \bigvee_{i=1}^m \varphi_{u_i},$$

which contradict (3.7) since $u' R v$.

Note that by the definition, there exists a unique point in $\mathcal{U}(n)$ which satisfies (3.9), (3.11) and (3.12). So $u' = u$, which implies that $\mathcal{U}(n), u \models \theta$. Thus by Theorem 3.2.6, $\mathcal{U}(n) \models \varphi_u \rightarrow \theta$, which by Theorem 3.2.4 implies that $\varphi_u \vdash \theta$, i.e. $\theta \in Cn(\varphi_u)$. Therefore $\Gamma = Cn(\varphi_u)$. \square

Now, we are in a position to prove the main theorem of this section.

Theorem 3.2.11. *Upper($\mathcal{H}(n)$) is isomorphic to $\mathcal{U}(n)$.*

Proof. Let $\mathcal{H}(n) = \langle H(n), R', V' \rangle$ and $\mathcal{U}(n) = \langle U(n), R, V' \rangle$. For any $x \in \text{Upper}(\mathcal{H}(n))$, by Theorem 3.2.3, there exists a unique w_x such that $\mathcal{U}(n)_{w_x}$ is a p-morphic image of $\mathcal{H}(n)_x$ via some surjective function h . Note that for any $y, z \in H(n)_x$,

$$y \neq z \implies Th(y) \neq Th(z), \quad (3.13)$$

thus since h is a p-morphism, $h(y) \neq h(z)$, therefore h is injective, and so by Remark 3.2.8

$$\mathcal{H}(n)_x \cong \mathcal{U}(n)_{w_x}.$$

Define $f : \text{Upper}(\mathcal{H}(n)) \rightarrow \mathcal{U}(n)$ by taking $f(x) = w_x$. We show that f is an isomorphism. Clearly, $\text{col}(x) = \text{col}(w_x)$. Suppose $xR'y$ and g is the unique isomorphism from $\mathcal{H}(n)_x$ onto $\mathcal{U}(n)_{f(x)}$. By Lemma 3.2.9 $\mathcal{H}(n)_y \cong \mathcal{U}(n)_{g(y)}$, thus by uniqueness, we must have that $g(y) = f(y)$. Thus, $f(x)Rf(y)$ since $f(x)Rg(y)$.

Suppose $f(x)Rf(y)$ for some $x, y \in U(n)$. Let g be the unique isomorphism from $\mathcal{H}(n)_x$ onto $\mathcal{U}(n)_{f(x)}$. Then there exists $z \in R'(x)$ such that $g(z) = f(y)$. Observe that

$$\text{col}(z) = \text{col}(f(y)) = \text{col}(y).$$

Since in $\mathcal{H}(n)$, distinct points have distinct colors, we conclude that $z = y$ and so $xR'y$.

From (3.13) it follows that for any $y, z \in H(n)$, $y \neq z$ implies $\mathcal{H}(n)_y \not\cong \mathcal{H}(n)_z$. Thus,

$$\mathcal{U}(n)_{w_y} \cong \mathcal{H}(n)_y \not\cong \mathcal{H}(n)_z \cong \mathcal{U}(n)_{w_z},$$

and so $w_y \neq w_z$, which means that f is injective.

It remains to show that f is surjective. For any $w \in U(n)$, consider the de Jongh formula φ_w . By Lemma 3.2.10, $\mathcal{H}(n)_{Cn(\varphi_w)} \cong \mathcal{U}(n)_w$, so $f(Cn(\varphi_w)) = w$. \square

We end this section by a corollary which follows from the correspondence between $\mathcal{H}(n)$ and $\mathcal{U}(n)$. For any n -models \mathfrak{M} and any point x in \mathfrak{M} , let $Th_n(\mathfrak{M}, x) = \{\varphi : \mathfrak{M}, x \models \varphi\}$.

Corollary 3.2.12. *Let \mathfrak{M} be any model and w be a point in $\mathcal{U}(n) = \langle W, R, V \rangle$. For any point x in \mathfrak{M} , if $\mathfrak{M}, x \models \varphi_w$, then there exists a unique $v \in R(w)$, such that*

$$\mathfrak{M}, x \models \varphi_v, \mathfrak{M}, x \not\models \varphi_{v_1}, \dots, \mathfrak{M}, x \not\models \varphi_{v_m}, \quad (3.14)$$

where $v \prec \{v_1, \dots, v_m\}$.

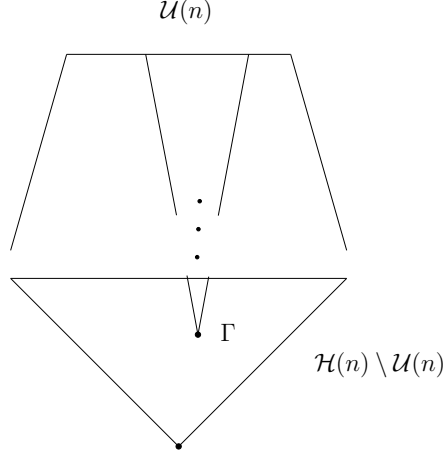


Figure 3.2: The n -Henkin model

Proof. Note that $Th_n(\mathfrak{M}, x)$ is a point in $\mathcal{H}(n) = \langle W', R', V' \rangle$. $\mathfrak{M}, x \models \varphi_w$ implies that $\vdash_{\mathbf{IPC}} Th_n(\mathfrak{M}, x) \rightarrow \varphi_w$ and $Th_n(\varphi_w)R'Th_n(\mathfrak{M}, x)$. Thus, by Lemma 3.2.10, $Th_n(\mathfrak{M}, x) = Th_n(\varphi_v)$ for a unique $v \in R(w)$. So $\mathfrak{M}, x \models \varphi_v$.

By Theorem 3.2.6, we have that $\mathcal{U}(n) \not\models \varphi_v \rightarrow \varphi_{v_i}$ for all $1 \leq i \leq m$. Thus $\not\vdash_{\mathbf{IPC}} \varphi_v \rightarrow \varphi_{v_i}$ and $\varphi_{v_i} \notin Th_n(\varphi_v) = Th_n(\mathfrak{M}, x)$, so $\mathfrak{M}, x \not\models \varphi_{v_i}$. \square

3.3 Some properties of $\mathcal{U}(n)$ and $\mathcal{H}(n)$

In view of Theorem 3.2.11, the n -universal model and n -Henkin model can be drawn in the way of Figure 3.2. In this section, we show that these two models are “connected” in the sense that every infinite upset of $\mathcal{H}(n)$ has an infinite intersection in $\mathcal{U}(n)$ (see also Figure 3.2).

Let $\mathfrak{F} = \langle W, R \rangle$ be a finite rooted frame. We introduce a new propositional variable p_w for every point w in W , and define a valuation V by letting $V(p_w) = R(w)$. Put $n = |W|$. By Theorem 3.2.3, there exists a p-morphism f from the model $\langle \mathfrak{F}, V \rangle$ onto a generated submodel $\mathcal{U}(n)_w$. By the construction, we know that different points of $\langle \mathfrak{F}, V \rangle$ have different colors, thus f is injective, which by Remark 3.2.8 means that f is an isomorphism, i.e. $\langle \mathfrak{F}, V \rangle$ is isomorphic to $\mathcal{U}(n)_w$.

The Kripke frame of the n -universal model $\mathcal{U}(n) = \langle W, R, V \rangle$ can be viewed as a general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ where $\mathcal{P} = Up(W)$, which is clearly refined. For every point w in $\mathcal{U}(n)$, the generated submodel $\mathcal{U}(n)_w$ is a finite

model, thus the underlying generated subframe \mathfrak{F}_w is also finite, which is then clearly compact. So the general frame \mathfrak{F}_w of $\mathcal{U}(n)_w$ is a descriptive frame.

An earlier version of the following theorem was proved by Jankov in [18]. De Jongh proved in [8] the same theorem algebraically by using de Jongh formulas. Here we prove the next theorem from the frame-theoretic point of view inspired by the algebraic proof in [8]. In the next section, we will prove Jankov's Theorem on **KC** (Theorem 3.4.9), where a similar idea as in the following proof will be used in Lemma 3.4.8. The proof of the next theorem is set up in a way to be easily generalized to the proof of Lemma 3.4.8.

Theorem 3.3.1 (Jankov-de Jongh). *For every finite rooted frame \mathfrak{F} , let ψ_w be the de Jongh formula of the point w in the model $\mathcal{U}(n)_w$ described above. Then for every descriptive frame \mathfrak{G} ,*

$$\mathfrak{G} \not\models \psi_w \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Proof. Let $\mathcal{U}(n)_w = \langle W, R, \mathcal{P}, V \rangle$. Suppose \mathfrak{F} is a p-morphic image of a generated subframe of \mathfrak{G} . By Theorem 3.2.6, $\mathcal{U}(n)_w \not\models \psi_w$, thus $\mathfrak{F} \not\models \psi_w$. So by applying Theorem 2.3.9, $\mathfrak{G} \not\models \psi_w$ is obtained.

Suppose $\mathfrak{G} \not\models \psi_w$. Then there exists a model \mathfrak{N} on \mathfrak{G} such that

$$\mathfrak{N} \not\models \varphi_w \rightarrow \varphi_{w_1} \vee \cdots \vee \varphi_{w_m}, \quad (3.15)$$

where $w \prec \{w_1, \dots, w_m\}$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi_w)} = \langle W', R', \mathcal{P}', V' \rangle$ of \mathfrak{N} . Note that since $V'(\varphi_w)$ is admissible, by Corollary 2.6.13, $\langle W', R', \mathcal{P}' \rangle$ is a descriptive frame. Define a map $f : W' \rightarrow W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi_v, \mathfrak{N}', x \not\models \varphi_{v_1}, \dots, \mathfrak{N}', x \not\models \varphi_{v_k}, \quad (3.16)$$

where $v \prec \{v_1, \dots, v_k\}$.

Note that for every $x \in W'$, $\mathfrak{N}', x \models \varphi_w$, thus by Corollary 3.2.12, there exists a unique $v \in R(w)$ satisfying (3.16). So f is well-defined.

We show that f is a surjective frame p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Suppose $x, y \in W'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi_v$, we have that $\mathfrak{N}', y \models \varphi_v$. By Corollary 3.2.12, there exists a unique $u' \in R(v)$ such that u' and y satisfy (3.16). So, since u and y also satisfy (3.16), by the uniqueness, $u' = u$ and vRu .

Next, suppose $x \in W'$ and $v, u \in U(n)_w$ such that $f(x) = v$ and vRu . Since x and v satisfy (3.16), by the definition of φ_v , we must have that

$$\mathfrak{N}', x \not\models \psi_{v_i}, \quad (3.17)$$

for $1 \leq i \leq k$. We now show by induction on $d(u)$ that there exists $y \in W'$ such that $f(y) = u$ and $xR'y$.

$d(u) = d(v) - 1$. Then u is an immediate successor of v and x and u satisfies (3.17). So, by the definition of ψ_u , we have that there exists $y \in R'(x)$ such that y and u satisfy (3.16), thus $f(y) = u$.

$d(u) < d(v) - 1$. Then there exists an immediate successor v_{i_0} of v such that $v_{i_0}Ru$. By the result of the basic step of the induction, there exists $y \in W'$ such that $xR'y$ and $f(y) = v_{i_0}$. Since $d(v_{i_0}) < d(v)$, by the induction hypothesis, there exists $z \in W'$ such that $yR'z$ and $f(z) = u$. And clearly, $xR'z$.

For any upset $X \in \mathcal{P}$, we have that $X = \bigcup_{v \in X} R(v)$. First, we show that for every $v \in X$,

$$f^{-1}(R(v)) = V'(\varphi_v). \quad (3.18)$$

For every $x \in f^{-1}(R(v))$, $f(x) \in R(v)$, which means that there exists $u \in R(v)$ such that $f(x) = u$ and so $\mathfrak{N}', x \models \varphi_u$. Note that by Theorem 3.2.6 and Theorem 3.2.4, we have that

$$\vdash_{\mathbf{IPC}} \varphi_u \rightarrow \varphi_v.$$

Thus $\mathfrak{N}', x \models \varphi_v$ and so $x \in V'(\varphi_v)$. On the other hand, for every $x \in V'(\varphi_v)$, by Corollary 3.2.12, there exists a unique $u \in R(v)$ such that $f(x) = u$, thus $x \in f^{-1}(R(v))$.

Now, by (3.18), we have that

$$f^{-1}(X) = f^{-1}\left(\bigcup_{v \in X} R(v)\right) = \bigcup_{v \in X} f^{-1}(R(v)) = \bigcup_{v \in X} V'(\varphi_v).$$

Since X is finite, we obtain $f^{-1}(X) \in \mathcal{P}'$.

Lastly, we show that f is surjective. First, it follows from (3.15) that there exists $x \in W'$ such that (3.16) holds for x and w , i.e. $f(x) = w$. Then, for every point $v \in U(n)_w$, we have that wRv . Since f is a p-morphism, there exists $y \in R'(x) \subseteq W'$ such that $f(y) = v$.

Hence f is a surjective frame p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P} \rangle$, \mathfrak{F} is a p-morphic image of $\langle W', R', \mathcal{P}' \rangle$, which is a generated subframe of \mathfrak{G} . \square

Next, we will show that if an upset U generated by a point in the n -Henkin model has a finite intersection with its upper part, the n -universal model, then U totally lies in $\mathcal{U}(n)$. First, we need a definition.

We call $w \in U$ a *border point* of an upset U of $\mathcal{U}(n)$, if $w \notin U$ and all the successors v of w with $v \neq w$ are in U . Denote the set of all border points of U by $B(U)$. Note that all endpoints which are not in U are in $B(U)$. For more details on border points, one may refer to [4].

Fact 3.3.2. *If U is finite, then $B(U)$ is also finite.*

Proof. Since U is finite, there exists $k \in \omega$ such that $U \subseteq U(n)^k$. Observe that $B(U) \subseteq U(n)^{k+1}$, which means that $B(U)$ is finite, since $U(n)^{k+1}$ is finite. \square

The next lemma shows the connection of upsets and their border points.

Lemma 3.3.3. *If $X = \{v_1, \dots, v_k\}$ is a finite anti-chain in $\mathcal{U}(n)$ and $B(\mathcal{U}(n)_X) = \{w_1, \dots, w_m\}$, then $\vdash_{\text{IPC}} (\varphi_{v_1} \vee \dots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \dots \wedge \psi_{w_m})$.*

Proof. In view of Theorem 3.2.4, it is sufficient to show that $\mathcal{U}(n) \models (\varphi_{v_1} \vee \dots \vee \varphi_{v_k}) \leftrightarrow (\psi_{w_1} \wedge \dots \wedge \psi_{w_m})$. By Theorem 3.2.6, it is then sufficient to show that

$$x \in R(v_1) \cup \dots \cup R(v_k) \text{ iff } x \notin R^{-1}(w_1) \cup \dots \cup R^{-1}(w_m).$$

For “ \Rightarrow ”: Suppose $x \in R(v_1) \cup \dots \cup R(v_k) = \mathcal{U}(n)_X$. If $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$, then since $\mathcal{U}(n)_X$ is upward closed, we have that $w_i \in \mathcal{U}(n)_X$, which contradicts the definition of $B(\mathcal{U}(n)_X)$.

For “ \Leftarrow ”: Suppose $x \notin R(v_1) \cup \dots \cup R(v_k) = \mathcal{U}(n)_X$. We show by induction on $d(x)$ that $x \in R^{-1}(w_i)$ for some $1 \leq i \leq m$.

$d(x) = 1$. Then x is an endpoint, which is a border point. Thus, $x = w_i$ for some $1 \leq i \leq m$ and so $x \in R^{-1}(w_i)$.

$d(x) > 1$. The result holds trivially if x is a border point. Now suppose there exists $y \in R(x)$ such that $y \notin \mathcal{U}(n)_X$. Since $d(y) < d(x)$, by the induction hypothesis, there exists $1 \leq i \leq m$ such that $y \in R^{-1}(w_i)$. Thus, $x \in R^{-1}(w_i)$. \square

Theorem 3.3.4. *Let Γ be a point in $\mathcal{H}(n)$, i.e. Γ is a theory with the disjunction property. If $R(\Gamma) \cap \mathcal{U}(n)$ is finite, then $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$.*

Proof. Suppose $X = R(\Gamma) \cap \mathcal{U}(n)$ is finite. Then the set $B(X)$ of border points of X is finite. Let $B(X) = \{w_1, \dots, w_m\}$. Suppose $\Gamma \not\vdash \psi_{w_i}$ for some $1 \leq i \leq m$. Then there exists a descriptive frame \mathfrak{G} such that $\mathfrak{G} \models \Gamma$ and $\mathfrak{G} \not\models \psi_{w_i}$. Since the underlying frame \mathfrak{F} of $\mathcal{U}(n)_{w_i}$ is finite rooted, by

Theorem 3.3.1, the latter implies that \mathfrak{F} is a p-morphic image of a generated submodel of \mathfrak{G} . Thus, by Theorem 2.3.9, $\mathfrak{F} \models \Gamma$ and so $\mathcal{U}(n)_{w_i} \models \Gamma$, which is impossible since $w_i \in B(X)$ and $w_i \notin R(\Gamma) \cap \mathcal{U}(n)$.

Hence, we conclude that $\Gamma \vdash \psi_{w_i}$ for all $1 \leq i \leq m$. Let Y be the anti-chain consisting of all least points of X . Then by Lemma 3.3.3, $\Gamma \vdash \varphi_w$ for some $w \in Y$, which by Theorem 3.2.6 means that $\Gamma \in R(w)$, so $\Gamma \in \mathcal{U}(n)$, therefore $R(\Gamma) = R(\Gamma) \cap \mathcal{U}(n)$. \square

The method of the above theorem gives more results.

3.4 An alternative proof of Jankov's Theorem on KC

KC is the intermediate logic axiomatized by $\neg\varphi \vee \neg\neg\varphi$. **KC** is complete with respect to finite rooted frames with unique top points. It is known that **KC** proves exactly the same negation-free formulas as **IPC**. That is for any negation-free formula φ , **KC** $\vdash \varphi$ iff **IPC** $\vdash \varphi$. Jankov proved in [19] that **KC** is the strongest intermediate logic that has this property. In this section, we give a frame theoretic alternative proof of Jankov's Theorem. The basic idea of the proof comes from Theorem 3.3.1 in the previous section and the next theorem. For more details on the next theorem, one may refer to [8], [9].

Theorem 3.4.1. *If L is an intermediate logic strictly extending **IPC**, i.e. $\mathbf{IPC} \subset L \subseteq \mathbf{CPC}$, then for some $n \in \omega$, $L \vdash \psi_w$ for some w in $\mathcal{U}(n)$.*

Proof. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{IPC} \not\vdash \chi.$$

Then there exists a finite rooted frame \mathfrak{F} such that $\mathfrak{F} \not\models \chi$. Introduce a new propositional variable p_w for every point w in W , and define a valuation V by letting $V(p_w) = R(w)$. Put $n = |\mathfrak{F}|$. By Theorem 3.2.3, there exists a generated submodel $\mathcal{U}(n)_w$ such that $\mathcal{U}(n)_w$ is a p-morphic image of \mathfrak{F} . By the construction, we know that different points of $\langle \mathfrak{F}, V \rangle$ have different colors, thus $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$.

Consider the de Jongh formula ψ_w . Suppose $L \not\vdash \psi_w$. Then there exists a descriptive frame \mathfrak{G} of L such that $\mathfrak{G} \not\models \psi_w$. By Theorem 3.3.1, \mathfrak{F} is a

p -morphic image of a generated subframe of \mathfrak{G} . Thus, by Theorem 2.3.9, \mathfrak{F} is an L model. Since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which leads to a contradiction. \square

Now, we define formulas φ'_w and ψ'_w , which are negation-free modifications of de Jongh formulas. They do a similar job for **KC**-frames as de Jongh formulas do for all frames. First, we introduce some terminologies.

For any finite set X of formulas with $|X| > 1$, let

$$\Delta X = \bigwedge \{\varphi \leftrightarrow \psi : \varphi, \psi \in X\}.$$

For the case that $|X| = 1$ or 0 , we stipulate $\Delta X = \top$.

Let $\mathcal{U}(n)_{w_0} = \langle W, R, V \rangle$ be a generated submodel with a largest element t of $\mathcal{U}(n)$ such that

- $t \models p_1 \wedge \cdots \wedge p_n$;
- $col_V(w) \neq col_V(v)$ for all $w, v \in W$ such that $w \neq v$.

Let r be a new propositional variable.

Definition 3.4.2. We inductively define the formulas φ'_w and ψ'_w for every $w \in W$.

If $d(w) = 1$,

$$\varphi'_w = p_1 \wedge \cdots \wedge p_n,$$

$$\psi'_w = \varphi'_w \rightarrow r.$$

If $d(w) = 2$, let q be the propositional letter in $notprop(w)$ with the least index. Define

$$\varphi'_w = \bigwedge prop(w) \wedge \Delta notprop(w) \wedge ((q \rightarrow r) \rightarrow q)^2,$$

$$\psi'_w = \varphi'_w \rightarrow q.$$

If $d(w) > 2$ and $w \prec \{w_1, \dots, w_m\}$, then let

$$\varphi'_w := \bigwedge prop(w) \wedge \left(\bigvee newprop(w) \vee \bigvee_{i=1}^m \psi'_{w_i} \rightarrow \bigvee_{i=1}^m \varphi'_{w_i} \right)$$

²Note that in the definition, it does not matter which $q \in notprop(w)$ is chosen. For simplicity, here we stipulate $q = q_1$ to be the one that has the least index.

and

$$\psi'_w := \varphi'_w \rightarrow \bigvee_{i=1}^m \varphi'_{w_i}.$$

We will prove for the φ'_w and ψ'_w formulas a lemma (Lemma 3.4.8) which is an analogy of Theorem 3.3.1 for the φ_w and ψ_w formulas.

Lemma 3.4.3. *Let V be a valuation on a frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and V' is a valuation on \mathfrak{F} defined by*

$$V'(p) = \begin{cases} V(p), & p \neq r; \\ \emptyset, & p = r. \end{cases}$$

Then for any formula φ ,

$$\langle \mathfrak{F}, V \rangle \models \varphi \text{ iff } \langle \mathfrak{F}, V' \rangle \models \varphi[\perp/r]^3.$$

Proof. We prove the lemma by induction on φ that for any $w \in W$,

$$\langle \mathfrak{F}, V \rangle, w \models \varphi \text{ iff } \langle \mathfrak{F}, V' \rangle, w \models \varphi[\perp/r].$$

$\varphi = \perp$. Then $w = t$, $\langle \mathfrak{F}, V \rangle, w \not\models \perp$ and $\langle \mathfrak{F}, V' \rangle, w \not\models r$.

The induction steps that $\varphi = \psi \wedge \chi$, $\psi \vee \chi$ and $\varphi = \psi \rightarrow \chi$ are proved easily. \square

Lemma 3.4.4. *Let $\mathcal{U}(n)_{w_0}$ be a model described above with a largest element t . Then for any $w \in W$, we have that*

$$(i) \vdash_{\text{IPC}} \varphi_w[\perp/r] \leftrightarrow \varphi'_w;$$

$$(ii) \vdash_{\text{IPC}} \psi_w[\perp/r] \leftrightarrow \psi'_w.$$

Proof. We prove the lemma by induction on $d(w)$.

$d(w) = 1$. Then $w = t$, $\varphi_w[\perp/r] = \varphi'_w$ and $\psi_w[\perp/r] = \psi'_w$, so the lemma holds trivially.

³We write $\varphi[p/\psi]$ for the formula obtained by substituting all occurrences of p in φ by ψ .

$d(w) = 2$. Then we have

$$\begin{aligned}
\vdash \varphi_w &\leftrightarrow \bigwedge \text{prop}(w) \wedge (\bigvee \text{notprop}(w) \vee \neg(p_1 \wedge \cdots \wedge p_n) \rightarrow p_1 \wedge \cdots \wedge p_n) \\
\vdash \varphi_w &\leftrightarrow \bigwedge \text{prop}(w) \wedge (\bigvee \text{notprop}(w) \vee \neg \bigwedge \text{notprop}(w) \rightarrow \bigwedge \text{notprop}(w)) \\
\vdash \varphi_w &\leftrightarrow \bigwedge \text{prop}(w) \wedge (\bigvee \text{notprop}(w) \rightarrow \bigwedge \text{notprop}(w)) \\
&\quad \wedge (\neg \bigwedge \text{notprop}(w) \rightarrow \bigwedge \text{notprop}(w)) \\
\vdash \varphi_w &\leftrightarrow \bigwedge \text{prop}(w) \wedge \bigwedge_{s \in \text{notprop}(w)} (s \rightarrow \bigwedge \text{notprop}(w)) \\
&\quad \wedge (\neg \bigwedge \text{notprop}(w) \rightarrow \bigwedge \text{notprop}(w)) \\
\vdash \varphi_w &\leftrightarrow \bigwedge \text{prop}(w) \wedge \Delta \text{notprop}(w) \wedge (\neg q \rightarrow q).
\end{aligned}$$

Thus, $\vdash \varphi_w[\perp/r] \leftrightarrow \varphi'_w$ and

$$\begin{aligned}
\vdash \psi_w[\perp/r] &\leftrightarrow (\varphi_w[\perp/r] \rightarrow \varphi_t[\perp/r]) \\
\vdash \psi_w[\perp/r] &\leftrightarrow (\varphi'_w \rightarrow \varphi'_t) \\
\vdash \psi_w[\perp/r] &\leftrightarrow \psi'_w.
\end{aligned}$$

The induction step that $d(w) > 2$ is proved easily by applying the induction hypothesis. \square

For any w in $\mathcal{U}(n)_{w_0}$ described above, by Theorem 3.2.6, $\mathcal{U}(n)_{w_0} \not\models \psi_w$ for each $w \in R(w_0)$. Thus, by the Lemma 3.4.3 and Lemma 3.4.4, the underlying frame of $\mathcal{U}(n)_{w_0}$ falsifies ψ'_w . Hence $\not\models_{\mathbf{IPC}} \psi'_w$ for each ψ'_w , where $w \in R(w_0)$. We will use this fact later in the proof of Theorem 3.4.9.

For any w, v with wRv in the n -universal model $\mathcal{U}(n)$, by Theorem 3.2.6 and Theorem 3.2.4, it is easy to prove that $\vdash_{\mathbf{IPC}} \varphi_v \rightarrow \varphi_w$. The next lemma shows that the φ_w and φ_v formulas have the same property. Note that Theorem 3.2.4 is not applicable for the φ'_w and ψ'_w formulas. So here we prove the next theorem directly from the construction of the φ'_w and ψ'_w formulas.

Lemma 3.4.5. *Let $\mathcal{U}(n)_{w_0} = \langle W, R, V \rangle$ be a model described above and let w, v be two points in W with wRv . Then we have that $\vdash_{\mathbf{IPC}} \varphi'_v \rightarrow \varphi'_w$.*

Proof. For any finite rooted model $\mathfrak{M} = \langle W', R', V' \rangle$ with the root r and some point $x \in R'(r)$, suppose $\mathfrak{M}, x \models \varphi'_v$. We show that

$$\mathfrak{M}, x \models \varphi'_w. \tag{3.19}$$

If $d(v) = 1$, then

$$\mathfrak{M}, x \models p_1 \wedge \cdots \wedge p_n. \quad (3.20)$$

Clearly,

$$\mathfrak{M}, x \models \bigwedge \text{prop}(w). \quad (3.21)$$

We show (3.19) by induction on $d(w)$.

$d(w) = d(v) + 1 = 2$. Then clearly (3.20) implies that $\mathfrak{M}, x \models q$, which implies that

$$\mathfrak{M}, x \models (q \rightarrow r) \rightarrow q.$$

Thus, together with (3.21), (3.19) is obtained.

$d(w) > d(v) + 1$. For any $w_i \in S_w$, since $d(w_i) < d(w)$, by the induction hypothesis, $\mathfrak{M}, x \models \varphi'_{w_i}$, thus $\mathfrak{M}, x \models \bigvee_{w_i \in S_w} \varphi'_{w_i}$ and

$$\mathfrak{M}, x \models \bigvee \text{newprop}(w) \vee \bigvee_{w_i \in S_w} \psi'_{w_i} \rightarrow \bigvee_{w_i \in S_w} \varphi'_{w_i}. \quad (3.22)$$

Hence, together with (3.21), (3.19) is obtained.

If $d(v) = 2$, then clearly (3.21) holds. We show (3.19) by induction on $d(w)$. $d(w) = d(v) + 1$. Then v is an immediate successor of w , and (3.22) follows from $\mathfrak{M}, x \models \varphi'_v$. Thus, together with (3.21), (3.19) is obtained.

$d(w) > d(v) + 1$. For any $w_i \in S_w$, since $d(w_i) < d(w)$, by the induction hypothesis, we have that $\mathfrak{M}, x \models \varphi'_{w_i}$ and so (3.22) holds. Thus, together with (3.21), (3.19) is obtained.

If $d(v) > 2$, then clearly $\mathfrak{M}, x \models \varphi'_v$ implies (3.21). By a similar argument as above, we can show that (3.22) holds, thus, (3.19) is obtained. \square

A similar result to the next lemma for the de Jongh formula φ_w can also be obtained by a similar argument.

Lemma 3.4.6. *Let $\mathfrak{M} = \langle W', R', V' \rangle$ be any model and $\mathcal{U}(n)_{w_0} = \langle W, R, V \rangle$ be a model described above. Put $V'' = V' \upharpoonright \{p_1, \dots, p_n\}$. For any point w in $\mathcal{U}(n)_{w_0}$ and any point x in \mathfrak{M} , if*

$$\mathfrak{M}, x \models \varphi'_w, \quad \mathfrak{M}, x \not\models \varphi'_{w_1}, \dots, \mathfrak{M}, x \not\models \varphi'_{w_m}, \quad (3.23)$$

where $w \prec \{w_1, \dots, w_m\}$, then $\text{col}_{V''}(x) = \text{col}_V(w)$.

Proof. We prove the lemma by induction on $d(w)$.

$d(w) = 1$. Then (3.23) means that $\mathfrak{M}, x \models p_1 \wedge \cdots \wedge p_n$. Note that $w = t$ also satisfies $\mathcal{U}(n)_{w_0}, t \models p_1 \wedge \cdots \wedge p_n$. So $col_{V''}(x) = col_V(w)$.

$d(w) = 2$. Then (3.23) implies that

$$\mathfrak{M}, x \models \bigwedge prop(w), \quad (3.24)$$

$$\mathfrak{M}, x \models \Delta notprop(w), \quad (3.25)$$

$$\mathfrak{M}, x \not\models p_1 \wedge \cdots \wedge p_n. \quad (3.26)$$

First, from (3.24), it follows that

$$prop(x) \cap \{p_1, \dots, p_n\} \supseteq prop(w). \quad (3.27)$$

Next, it follows from (3.26) that there exists p_i ($1 \leq i \leq n$) such that $p_i \notin prop(x)$, which by (3.27) implies that $p_i \in notprop(w)$. Thus, by (3.25), $\mathfrak{M}, x \not\models \bigvee notprop(w)$ and so

$$notprop(x) \cap \{p_1, \dots, p_n\} \supseteq notprop(w).$$

Together with (3.27), we obtain $col_{V''}(x) = col_V(w)$.

$d(w) > 2$. Then (3.23) implies (3.24) and

$$\mathfrak{M}, x \not\models \bigvee newprop(w), \quad (3.28)$$

$$\mathfrak{M}, x \not\models \psi'_{w_i}, \quad (3.29)$$

for all $w_i \in S_w$. From (3.24), we obtain (3.27). From (3.28), we obtain

$$notprop(x) \cap \{p_1, \dots, p_n\} \supseteq newprop(w). \quad (3.30)$$

It follows from (3.29) that for each $w_i \in S_w$, there exists $y \in R'(x)$ such that y and w_i satisfy (3.23). Since $d(w_i) < d(w)$, by the induction hypothesis, we have that $col_{V''}(y) = col_V(w_i)$, which implies that

$$notprop(x) \cap \{p_1, \dots, p_n\} \supseteq notprop(y) \cap \{p_1, \dots, p_n\} = notprop(w_i).$$

Together with (3.30), we obtain

$$notprop(x) \cap \{p_1, \dots, p_n\} \supseteq newprop(w) \cup \bigcup_{w_i \in S_w} notprop(w_i) = notprop(w).$$

The above and (3.27) proves $col_{V''}(x) = col_V(w)$. \square

The next lemma is crucial in the proof of Lemma 3.4.8. For the φ_w formulas, a similar lemma (Corollary 3.2.12) is obtained as a corollary of the results on the connection of $\mathcal{H}(n)$ and $\mathcal{U}(n)$. However, this method cannot be generalized to the φ'_w formulas. Here we prove the next lemma directly.

Lemma 3.4.7. *Let \mathfrak{M} and $\mathcal{U}(n)_{w_0}$ be models described above. For any point w in $\mathcal{U}(n)_{w_0}$ and any point x in \mathfrak{M} , if $\mathfrak{M}, x \models \varphi'_w$, then there exists a unique $v \in R(w)$ such that*

$$\mathfrak{M}, x \models \varphi'_v, \mathfrak{M}, x \not\models \varphi'_{v_1}, \dots, \mathfrak{M}, x \not\models \varphi'_{v_m}, \quad (3.31)$$

where $v \prec \{v_1, \dots, v_m\}$.

Proof. Suppose $\mathfrak{M}, x \models \varphi'_w$. If for all $w_i \in S_w$, $\mathfrak{M}, x \not\models \varphi'_{w_i}$, then w satisfies (3.31). Now suppose that for some $w_{i_0} \in S_w$, $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$. We show that there exists $v \in R(w)$ satisfying (3.31) by induction on $d(w)$.

$d(w) = 1$. Then trivially $v = w$ satisfies (3.31).

$d(w) > 1$. Since $\mathfrak{M}, x \models \varphi'_{w_{i_0}}$ and $d(w_{i_0}) < d(w)$, by the induction hypothesis, there exists $v \in W$, such that $w_{i_0}Rv$ and v satisfies (3.31). And clearly, wRv .

Next, suppose $v' \in R(w)$ also satisfies (3.31). By Lemma 3.4.6,

$$\text{col}_V(v') = \text{col}_{V''}(x) = \text{col}_V(v),$$

which by the property of $\mathcal{U}(n)_{w_0}$ means that $v' = v$. \square

Let \mathfrak{F} be a finite rooted frame with a largest element x_0 . For every point x in \mathfrak{F} , we introduce a new propositional variable p_x and define a valuation V on \mathfrak{F} by letting $V(p_x) = R(x)$. Let $n = |\mathfrak{F}|$. By Theorem 3.2.3, there exists a generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$ such that $\mathcal{U}(n)_w$ is a p-morphic image of $\langle \mathfrak{F}, V \rangle$. Since different points in $\langle \mathfrak{F}, V \rangle$ have different colors, the p-morphism is injective and so $\langle \mathfrak{F}, V \rangle \cong \mathcal{U}(n)_w$. Note that $\mathcal{U}(n)_w$ has a top point t and $t \models p_1 \wedge \dots \wedge p_n$.

The next lemma is a modification of the Jankov-de Jongh Theorem (Theorem 3.3.1) proved in the previous section. Both the statement of the lemma and the proof are generalized from those of Theorem 3.3.1.

Lemma 3.4.8. *For every finite rooted frame \mathfrak{F} with a largest element, let $\mathcal{U}(n)_w$ be the model described above. Then for every descriptive frame \mathfrak{G} ,*

$$\mathfrak{G} \not\models \psi'_w \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Proof. Let $\mathcal{U}(n)_w = \langle W, R, \mathcal{P}, V \rangle$. Suppose \mathfrak{F} is a p-morphic image of a generated subframe of \mathfrak{G} . By Theorem 3.2.6, $\mathcal{U}(n)_w \not\models \psi_w$, thus $\mathfrak{F} \not\models \psi_w$. By Lemma 3.4.3 and Lemma 3.4.4, we know that $\mathfrak{F} \not\models \psi'_w$. By applying Theorem 2.3.9, $\mathfrak{G} \not\models \psi'_w$ is obtained.

Suppose $\mathfrak{G} \not\models \psi'_w$. Then there exists a model \mathfrak{N} on \mathfrak{G} such that $\mathfrak{N} \not\models \psi'_w$. Consider the generated submodel $\mathfrak{N}' = \mathfrak{N}_{V'(\varphi'_w)} = \langle W', R', \mathcal{P}', V' \rangle$ of \mathfrak{N} . Since $V'(\varphi'_w)$ is admissible, by Lemma 2.6.13, \mathfrak{N}' is admissible. Define a map $f : W' \rightarrow W$ by taking $f(x) = v$ iff

$$\mathfrak{N}', x \models \varphi'_v, \mathfrak{N}', x \not\models \varphi'_{v_1}, \dots, \mathfrak{N}', x \not\models \varphi'_{v_k}, \quad (3.32)$$

where $v \prec \{v_1, \dots, v_k\}$.

Note that for every $x \in \mathfrak{N}'$, $\mathfrak{N}', x \models \varphi_w$, thus by Lemma 3.4.7, there exists a unique $v \in R(w)$ satisfying (3.32). So f is well-defined.

We show that f is a surjective frame p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Suppose $x, y \in \mathfrak{N}'$ with $xR'y$, $f(x) = v$ and $f(y) = u$. Since $\mathfrak{N}', x \models \varphi'_v$, we have that $\mathfrak{N}', y \models \varphi'_v$. By Lemma 3.4.7, there exists a unique $u' \in R(v)$ such that u' and y satisfy (3.32). So, since u and y also satisfy (3.32), by the uniqueness, $u' = u$ and vRu .

Next, suppose $x \in \mathfrak{N}'$ and $v, u \in W$ such that $f(x) = v$ and vRu . We show that

$$\text{there exists } y \in \mathfrak{N}' \text{ such that } f(y) = u \text{ and } xR'y. \quad (3.33)$$

If $d(v) = 1$, then $u = v$, so trivially $xR'x$ and $f(x) = v = u$.

If $d(v) = 2$, then if $u = v$, we have that (3.33) trivially holds. Now suppose $u = t$. Since $f(x) = v$, v and x satisfy (3.32), so

$$\mathfrak{N}', x \models \bigwedge \text{prop}(v) \wedge \Delta \text{notprop}(v) \wedge ((q \rightarrow r) \rightarrow q). \quad (3.34)$$

It then follows that $\mathfrak{N}', x \models (q \rightarrow r) \rightarrow q$. Note that

$$\vdash_{\mathbf{IPC}} ((q \rightarrow r) \rightarrow q) \rightarrow \neg\neg q.$$

Thus, $\mathfrak{N}', x \models \neg\neg q$, which means that there exists $y \in W'$ such that $xR'y$ and $\mathfrak{N}', y \models q$. Since

$$\mathfrak{N}', y \models \bigwedge \text{prop}(v) \wedge \Delta \text{notprop}(v),$$

we have that $\mathfrak{N}', y \models p_1 \wedge \dots \wedge p_n$, i.e. $f(y) = u$.

If $d(v) > 2$, then since x and v satisfy (3.32), by the definition of φ'_v , we must have that

$$\mathfrak{N}', x \not\models \psi'_{v_i}, \quad (3.35)$$

for $1 \leq i \leq k$. We now show by induction on $d(u)$ that (3.33) holds.

$d(u) = d(v) - 1$. Then u is an immediate successor of v and u satisfies (3.35). There are two cases. Case 1: $d(u) = 2$. Then it follows from $\mathfrak{N}', x \not\models \psi'_u$ that there exists $y \in R'(x)$ such that $\mathfrak{N}', y \models \varphi'_u$ and $\mathfrak{N}', y \not\models q$. The latter implies that $\mathfrak{N}, y \not\models p_1 \wedge \cdots \wedge p_n$, i.e. $\mathfrak{N}, y \not\models \varphi'_t$. Thus y and u satisfy (3.32), and $f(y) = u$.

Case 2: $d(u) > 2$. Then by the definition of ψ'_u , $\mathfrak{N}', x \not\models \psi'_u$ implies that there exists $y \in R'(x)$ such that y and u satisfy (3.32), thus $f(y) = u$.

$d(u) < d(v) - 1$. Then there exists an immediate successor v_{i_0} of v such that $v_{i_0} R u$. By the basic step of the induction, there exists $y \in W'$ such that $x R' y$ and $f(y) = v_{i_0}$. Since $d(v_{i_0}) < d(v)$, by the induction hypothesis, there exists $z \in W'$ such that $y R' z$ and $f(z) = u$. And clearly, $x R' z$.

For any upset $X \in \mathcal{P}$, we have that $X = \bigcup_{v \in X} R(v)$. By applying Lemma 3.4.5, Lemma 3.4.7 and using a same argument as that in the proof of Theorem 3.3.1, we can show that for every $v \in X$,

$$f^{-1}(R(v)) = V'(\varphi'_v).$$

So,

$$f^{-1}(X) = f^{-1}\left(\bigcup_{v \in X} R(v)\right) = \bigcup_{v \in X} f^{-1}(R(v)) = \bigcup_{v \in X} V'(\varphi'_v).$$

Since X is finite, we obtain $f^{-1}(X) \in \mathcal{P}'$.

Lastly, we show that f is surjective. First, by a similar argument as above, we can show that for $w \in W$, there exists $x \in W'$ such that $f(x) = w$. Next, for every $v \in W$, we have that $w R v$. So by (3.33), there exists $y \in R'(x) \subseteq W'$, such that $f(y) = v$.

Hence, f is a surjective frame p-morphism of $\langle W', R', \mathcal{P}' \rangle$ onto $\langle W, R, \mathcal{P} \rangle$. Then since $\mathfrak{F} \cong \langle W, R, \mathcal{P} \rangle$, \mathfrak{F} is a p-morphic image of $\langle W', R', \mathcal{P}' \rangle$, which is a generated subframe of \mathfrak{G} . \square

Now we are ready to prove Jankov's theorem on **KC**.

Theorem 3.4.9 (Jankov). *If L is an intermediate logic such that $L \not\subseteq \mathbf{KC}$, then $L \vdash \theta$ and $\mathbf{IPC} \not\vdash \theta$ for some negation-free formula θ .*

Proof. Suppose χ is a formula satisfying

$$L \vdash \chi \text{ and } \mathbf{KC} \not\vdash \chi.$$

Then there exists a finite rooted \mathbf{KC} -frame \mathfrak{F} with a largest element such that $\mathfrak{F} \not\models \chi$. For every point w in \mathfrak{F} , we introduce a new propositional variable p_w and define a valuation V on \mathfrak{F} by letting $V(p_w) = R(w)$. Let $n = |\mathfrak{F}|$. By Theorem 3.2.3, there exists a generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$ which is a p-morphic image of $\langle \mathfrak{F}, V \rangle$. Note that $\mathcal{U}(n)_w$ has a largest element t , $t \models p_1 \wedge \cdots \wedge p_n$ and $col_V(v) \neq col_V(u)$ for all v, u in $\mathcal{U}(n)_w$.

Consider the formula ψ'_w . Suppose $L \not\vdash \psi'_w$. Then there exists a descriptive frame \mathfrak{G} of L such that $\mathfrak{G} \not\models \psi'_w$. By Lemma 3.4.8, \mathfrak{F} is a p-morphic image of a generated subframe of \mathfrak{G} . Thus, by Theorem 2.3.9, \mathfrak{F} is an L -frame. Thus, since $L \vdash \chi$, we have that $\mathfrak{F} \models \chi$, which leads to a contradiction.

Hence, $L \vdash \psi'_w$. Note that $\mathbf{IPC} \not\vdash \psi'_w$ and ψ'_w is negation-free, thus $\theta = \psi'_w$ is the required formula. \square

Chapter 4

Subframe logics and subframe formulas

In this chapter, we summarize classic and recent results on subframe logics and subframe formulas. Subframe logics are intermediate logics that are characterized by a class of frames closed under subframes. The study of intuitionistic subframe logics was first inspired by the related results on modal subframe logics, where Fine [14] and Zakharyashev [28] defined the subframe formulas and proved the finite model property of subframe logics. In [27], [29] (see also [6]) Zakharyashev defined subframe formulas for intermediate logics, which are $[\wedge, \rightarrow]$ -formulas. It then follows from Zakharyashev [29], [30] (see also [6]) that subframe logics are exactly those logics axiomatized by $[\wedge, \rightarrow]$ -formulas. N. Bezhanishvili proved in [3] that subframe logics can also be axiomatized by *NNIL*-formulas. G. Bezhanishvili and Ghilardi [2] gave an algebraic approach to subframe logics by using the tools of nuclei in topos theory and proved that a variety of Heyting algebras is nuclear iff it is a subframe variety. Also in [2], an alternative proof of the finite model property of subframe logics is given from the algebraic point of view.

In Section 4.1, we give definitions of subframes of intuitionistic general frames and subframe logics. In Section 4.2, we give the definition of subframe formulas in *NNIL*-form, which was done in [3]. In Section 4.3, we provide a frame-based proof of the property that subreductions preserve $[\wedge, \rightarrow]$ -formulas. In Section 4.4, we state the equivalent characterizations of subframe logics.

4.1 Subframe logics

In this section, we recall the important notions related to subframe logics. For more details on subframe logics, one may refer to [6].

Definition 4.1.1. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ be intuitionistic general frames. A partial map f from W onto V is called a *subreduction* of \mathfrak{F} to \mathfrak{G} if it satisfies the following

- (R1') For any $w, v \in \text{dom}(f)$, wRv implies $f(w)Sf(v)$;
- (R2') $f(w)Sv'$ implies $\exists v \in \text{dom}(f)$, wRv and $f(v) = v'$;
- (R3') $\forall X \in \overline{\mathcal{Q}}$, $f^{-1}(X) \downarrow \in \overline{\mathcal{P}}$.

Remark 4.1.2. *If a subreduction is total, then (R3') is equivalent to (R3). This means that any reduction is also a subreduction and any total subreduction is also a reduction.*

An intuitionistic general frame $\mathfrak{G} = \langle V, S, \mathcal{Q} \rangle$ is called a *subframe* of an intuitionistic general frame $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$, if $\langle V, S \rangle$ is a subframe of $\langle W, R \rangle$ and the inclusion map is a subreduction, i.e.,

$$U \in \overline{\mathcal{Q}} \text{ implies } R^{-1}(U) \in \overline{\mathcal{P}}.$$

Alternatively, we can define subframe in topological terminology. A Heyting space $\mathcal{Y} = \langle Y, \nu, S \rangle$ is called a *subframe* of a Heyting space $\mathcal{X} = \langle X, \tau, R \rangle$ if $\langle Y, S \rangle$ is a subframe of $\langle X, R \rangle$, $\langle Y, \nu \rangle$ is a subspace of $\langle X, \tau \rangle$, and

$$U \text{ is a clopen of } \mathcal{Y} \text{ implies that } R^{-1}(U) \text{ is a clopen of } \mathcal{X}.$$

Also, a correspondence between subframe and nuclei can be found in [2].

Remark 4.1.3. *If \mathfrak{G} is a subreduct of \mathfrak{F} , then \mathfrak{G} is a reduct (p -morphic image) of a subframe of \mathfrak{F} .*

There are many ways of to define a subframe logic. In subsequent sections, we will see that these characterizations are equivalent.

Definition 4.1.4. An intermediate logic L is called a *subframe logic*, if it is characterized by a class of frames that is closed under subframes (i.e. every subframe of an L -frame is also an L -frame).

4.2 Subframe formulas

Subframe formulas are formulas axiomatizing subframe logics. In [3] N. Bezhanishvili defined subframe formulas in the *NNIL*-form. In this section, we spell out these results.

Let $\mathfrak{F} = \langle W, R \rangle$ be a finite (rooted) frame. For each point w in \mathfrak{F} , we introduce a new propositional variable p_w and define a valuation V on \mathfrak{F} by taking $V(p_w) = R(w)$.

Definition 4.2.1. We inductively define the subframe formula $\beta(\mathfrak{F})$. If $d(w) = 1$, then let

$$\beta(w) := \bigwedge \text{prop}(w) \rightarrow \bigvee \text{notprop}(w).$$

If $d(w) > 1$, let w_1, \dots, w_k be all the immediate successors of w . Define

$$\beta(w) := \bigwedge \text{prop}(w) \rightarrow \left(\bigvee \text{notprop}(w) \vee \bigvee_{i=1}^k \beta(w_i) \right).$$

Let r be the root of \mathfrak{F} . We define the *subframe formula* $\beta(\mathfrak{F})$ by

$$\beta(\mathfrak{F}) := \beta(r).$$

Note 4.2.2. *It is easy to see that $\mathfrak{F} \not\models \beta(\mathfrak{F})$.*

For proofs of the next two theorems, see Theorem 3.3.16 and Corollary 3.4.16 in [3].

Theorem 4.2.3. *Let \mathfrak{G} be a descriptive frame and \mathfrak{F} be a finite rooted frame. Then*

$$\mathfrak{G} \models \beta(\mathfrak{F}) \text{ iff } \mathfrak{G} \text{ is subreducible to } \mathfrak{F}.$$

Theorem 4.2.4. *Let L be an intermediate logic. Then L is axiomatized by subframe formulas iff L is a subframe logic.*

4.3 $[\wedge, \rightarrow]$ -formulas

In this section, we state the result that L is a subframe logic iff L is axiomatized by $[\wedge, \rightarrow]$ -formulas. This follows from [29], [30] (see also [6]).

For an algebraic proof of the next theorem, see Corollary 9.8 in [6]. Here we give a direct proof based on frames.

Theorem 4.3.1. *Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and $\mathfrak{G} = \langle W', R', \mathcal{P}' \rangle$ be general frames. If \mathfrak{F} is subreducible to \mathfrak{G} , then for any $\varphi \in [\wedge, \rightarrow]$,*

$$\mathfrak{F} \models \varphi \implies \mathfrak{G} \models \varphi.$$

Proof. Suppose f is a subreduction of \mathfrak{F} onto \mathfrak{G} and $\mathfrak{G} \not\models \varphi$. Then there exists a model $\mathfrak{N} = \langle \mathfrak{G}, V' \rangle$ such that $\mathfrak{N} \not\models \varphi$. We define a valuation V on \mathfrak{F} such that $\mathfrak{M} = \langle \mathfrak{F}, V \rangle \not\models \varphi$.

For any $w \in W$ and any $p \in \text{PROP}$, let

$$w \in V(p) \iff \forall v \in \text{dom}(f)(wRv \implies f(v) \in V'(p)). \quad (4.1)$$

We show that the valuation V is an admissible set in \mathcal{P} . That is to show that for any $p \in \text{PROP}$, $V(p) \in \mathcal{P}$. Note that $W' - V'(p) \in \overline{\mathcal{Q}}$, so by (R3'), we have $f^{-1}(W' - V'(p)) \downarrow \in \overline{\mathcal{P}}$, thus it suffices to show that

$$V(p) = W - f^{-1}(W' - V'(p)) \downarrow.$$

By (4.1), we have

$$\begin{aligned} w \in V(p) &\iff \forall v \in \text{dom}(f)(f(v) \notin V'(p) \implies \neg wRv) \\ &\iff \forall v \in \text{dom}(f)(v \in f^{-1}(W' - V'(p)) \implies \neg wRv) \\ &\iff w \notin f^{-1}(W' - V'(p)) \downarrow \\ &\iff w \in W - f^{-1}(W' - V'(p)) \downarrow. \end{aligned}$$

Now, we show that (4.1) holds for any $w \in \text{dom}(f) \uparrow$ and any formula $\varphi \in [\wedge, \rightarrow]$, and that for any $x \in \text{dom}(f)$,

$$\mathfrak{M}, x \models \varphi \iff \mathfrak{N}, f(x) \models \varphi. \quad (4.2)$$

We prove these two claims simultaneously by induction on φ .

Obviously, (4.1) holds for propositional letter, and (4.1) clearly implies that (4.2) holds for propositional letter as well. The case that $\varphi = \psi \wedge \chi$ can be proved easily. For the case that $\varphi = \psi \rightarrow \chi$, where $\psi, \chi \in [\wedge, \rightarrow]$, we first show that (4.1) holds.

For any $w \in \text{dom}(f) \uparrow$, we have that

$$\begin{aligned}
\mathfrak{M}, w \not\models \psi \rightarrow \chi &\implies \exists v \in R(w) \text{ s.t. } \mathfrak{M}, v \models \psi \text{ and } \mathfrak{M}, v \not\models \chi \\
&\implies \exists v \in R(w) \text{ s.t. } \forall u \in \text{dom}(f)(vRu \implies \mathfrak{N}, f(u) \models \psi) \\
&\quad \text{and } \exists u_v \in \text{dom}(f)(vRu_v \text{ and } \mathfrak{N}, f(u_v) \not\models \chi) \\
&\quad \text{(by the induction hypothesis of (4.1))} \\
&\implies \exists v \in R(w), \exists u_v \in \text{dom}(f) \text{ s.t. } vRu_v, \\
&\quad \mathfrak{N}, f(u_v) \models \psi \text{ and } \mathfrak{N}, f(u_v) \not\models \chi \\
&\implies \exists u \in \text{dom}(f)(wRu \text{ and } \mathfrak{N}, f(u) \not\models \psi \rightarrow \chi),
\end{aligned}$$

and that

$$\begin{aligned}
&\exists v \in \text{dom}(f)(wRv \text{ and } \mathfrak{N}, f(v) \not\models \psi \rightarrow \chi) \\
&\implies \exists v \in \text{dom}(f)(wRv, \exists u' \in S(f(v))(\mathfrak{N}, u' \models \psi \text{ and } \mathfrak{N}, u' \not\models \chi)) \\
&\implies \exists v \in \text{dom}(f)(wRv, \exists u \in \text{dom}(f) \cap R(v)(\mathfrak{N}, f(u) \models \psi \text{ and } \mathfrak{N}, f(u) \not\models \chi)) \\
&\quad \text{(by (R2'))} \\
&\implies \exists v \in \text{dom}(f)(wRv, \exists u \in \text{dom}(f) \cap R(v)(\mathfrak{M}, u \models \psi \text{ and } \mathfrak{M}, u \not\models \chi)) \\
&\quad \text{(by the induction hypothesis of (4.2))} \\
&\implies \exists u \in R(w) \text{ s.t. } \mathfrak{M}, u \models \psi \text{ and } \mathfrak{M}, u \not\models \chi \\
&\implies \mathfrak{M}, w \not\models \psi \rightarrow \chi.
\end{aligned}$$

Therefore (4.1) is obtained. To prove (4.2), for any $x \in \text{dom}(f)$, we have that

$$\begin{aligned}
\mathfrak{M}, x \not\models \psi \rightarrow \chi &\implies \exists y \in R(x) \text{ s.t. } \mathfrak{M}, y \models \psi \text{ and } \mathfrak{M}, y \not\models \chi \\
&\implies \exists y \in R(x) \text{ s.t. } \forall z \in \text{dom}(f)(yRz \implies \mathfrak{N}, f(z) \models \psi) \\
&\quad \text{and } \exists z_y \in \text{dom}(f)(yRz_y \text{ and } \mathfrak{N}, f(z_y) \not\models \chi) \\
&\quad \text{(by the induction hypothesis of (4.1))} \\
&\implies \exists y \in R(x), \exists z_y \in \text{dom}(f) \text{ s.t. } yRz_y, \\
&\quad \mathfrak{N}, f(z_y) \models \psi \text{ and } \mathfrak{N}, f(z_y) \not\models \chi \\
&\implies \exists z_y \in \text{dom}(f) \cap R(x) \text{ s.t. } \mathfrak{N}, f(z_y) \models \psi \text{ and } \mathfrak{N}, f(z_y) \not\models \chi \\
&\implies \exists f(z_y) \in V, \text{ s.t. } f(x)Sf(z_y), \mathfrak{N}, f(z_y) \models \psi \\
&\quad \text{and } \mathfrak{N}, f(z_y) \not\models \chi \text{ (by (R1'))} \\
&\implies \mathfrak{N}, f(x) \not\models \psi \rightarrow \chi,
\end{aligned}$$

and that

$$\begin{aligned}
\mathfrak{N}, f(x) \not\models \psi \rightarrow \chi &\implies \exists y' \in S(f(x)), \text{ s.t. } \mathfrak{N}, y' \models \psi \text{ and } \mathfrak{N}, y' \not\models \chi \\
&\implies \exists y \in R(x) \cap \text{dom}(f) \text{ s.t. } \mathfrak{N}, f(y) \models \psi \text{ and } \mathfrak{N}, f(y) \not\models \chi \\
&\quad (\text{by (R2')}) \\
&\implies \exists y \in R(x) \text{ s.t. } \mathfrak{M}, y \models \psi \text{ and } \mathfrak{M}, y \not\models \chi \\
&\quad (\text{by the induction hypothesis of (4.2)}) \\
&\implies \mathfrak{M}, x \not\models \psi \rightarrow \chi.
\end{aligned}$$

Thus, (4.2) is obtained. Therefore, since $\mathfrak{N}, f(x) \not\models \varphi$ for some $x \in \text{dom}(f)$, by (4.2) we conclude that $\mathfrak{M}, x \not\models \varphi$. This finishes the proof. \square

Theorem 4.3.2. *Let L be an intermediate logic. Then L is axiomatized by $[\wedge, \rightarrow]$ -formulas iff L is a subframe logic.*

Proof. Follows from Theorem 4.3.1 and [29], [30] (see also [6]). \square

4.4 Equivalent characterizations of subframe logics

Theorem 4.4.1. *The following are equivalent:*

1. L is a subframe logic;
2. L is axiomatized by $[\wedge, \rightarrow]$ -formulas;
3. L is axiomatized by NNIL-formulas;
4. there exists a set F of finite frame such that every formula in L refutes any frame in F ;
5. L is nuclear.

Proof. Follows from Theorem 4.2.3, 4.2.4, 4.3.2 and [2]. \square

Chapter 5

The $[\wedge, \rightarrow]$ -fragment of **IPC**

Denote the fragment of **IPC** consisting of formulas that have only \wedge and \rightarrow as connectives by $[\wedge, \rightarrow]$. In this chapter, we discuss properties of the $[\wedge, \rightarrow]$ -fragment of **IPC**. It is mentioned in Chapter 4 that $[\wedge, \rightarrow]$ -formulas axiomatize subframe logics.

Diego proved in [12] that the variety of Hilbert algebras is locally finite. Let $[\wedge, \rightarrow]^n$ and $[\rightarrow]^n$ denote the subfragment of $[\wedge, \rightarrow]$ and $[\rightarrow]$ for which propositional variables are only among the set $\{p_1, \dots, p_n\}$ respectively. Then dually, the $[\rightarrow]^n$ -fragment of **IPC** is finite. As a consequence, we know that the $[\wedge, \rightarrow]^n$ -fragment of **IPC** is finite. This fact is essential in the proof that subframe logics have the finite model property.

Hendriks defined in [16] (also in [11]), a finite exact model for the $[\wedge, \rightarrow]^n$ -fragment of **IPC**. As a matter of fact, the exact model acts as an n -universal model for $[\wedge, \rightarrow]^n$ -formulas. In Section 5.1, we reformulate this definition by making some modifications to the n -universal model of **IPC**. This enables us to prove properties of $[\wedge, \rightarrow]$ -formulas in an uniform manner. In particular, we give a detailed proof of Theorem 5.1.8 (corresponding to Lemma 3.2.1 in [11]) of the exactness of the n -universal model of $[\wedge, \rightarrow]^n$ -formulas which was too sketchy in [11]. In Section 5.2, we spell out some important results on the algebras of the $[\wedge, \rightarrow]$ -fragment of **IPC**, Brouwerian semilattices, including the congruence extension property.

5.1 n -universal models of $[\wedge, \rightarrow]$ -formulas

In [16], Hendriks gave a characterization property of $[\wedge, \rightarrow]$ -formulas, and showed that the $[\wedge, \rightarrow]$ -fragment of **IPC** is complete with respect to all \mathfrak{M}^- models. In this section, we reformulate these results and redefine the exact model in [16] by modifying the n -universal model of **IPC**.

We first start with some definitions.

Definition 5.1.1. Let $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ be a frame. Let $V : \{p_1, \dots, p_n\} \rightarrow \wp(W)$ be a valuation on \mathfrak{F} , w be a point in W .

- We call V *maximal* on a non-endpoint w if and only if

$$\text{prop}(w) = \bigcap_{v \in R(w)} \text{prop}(v).$$

- We call V *total* on w if and only if $|\text{prop}(w)| = n$.

A point w is called an *intersection point* in a model \mathfrak{M} if it is not an endpoint and the valuation on it is maximal. w is called *total* if the valuation on it is total.

Lemma 5.1.2. *Given a model $\mathfrak{M} = \langle W, R, V \rangle$. If w is a total point in W , then $\mathfrak{M}, w \models \varphi$, for every $\varphi \in [\wedge, \rightarrow]$.*

Proof. By induction on $d(w)$ and on φ . □

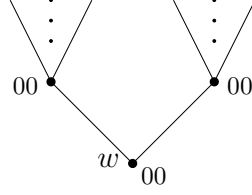
We now show that the truth values of $[\wedge, \rightarrow]$ -formulas are determined only by non-intersection and non-total points. Given an n -model \mathfrak{M} , let \mathfrak{M}^- denote the submodel obtained from \mathfrak{M} by eliminating all the intersection and total points. For the proof of the next theorem, one may also refer to Lemma 3.6.0.3 in [16].

Theorem 5.1.3. *Let $\mathfrak{M} = \langle W, R, V \rangle$ be a finite model and w be a non-total point in W . Then for any $\varphi \in [\wedge, \rightarrow]$,*

$$\mathfrak{M}, w \models \varphi \iff (\mathfrak{M}_w)^- \models \varphi.$$

Proof. Suppose $\mathfrak{M}, w \models \varphi$. We show by induction on $d(w)$ that $(\mathfrak{M}_w)^- \models \varphi$. If $d(w) = 1$, then $(\mathfrak{M}_w)^- = \mathfrak{M}_w$ since w is not total, so the theorem holds trivially.

Figure 5.1:



$d(w) = k > 1$. We prove by induction on φ . $\varphi = p$. Clearly, $\mathfrak{M}_w \models p$ implies $(\mathfrak{M}_w)^- \models p$. Now suppose $(\mathfrak{M}_w)^- \models p$. Observe that every point $v \in (\mathfrak{M} \setminus (\mathfrak{M}_w)^-)$ is either a total or an intersection point. So $v \in V(p)$, which means that $\mathfrak{M}_w \models p$.

The case that $\varphi = \psi \wedge \chi$ holds trivially.

Now consider the case that $\varphi = \psi \rightarrow \chi$. Suppose $\mathfrak{M}, w \not\models \psi \rightarrow \chi$. Then there exists $v \in R(w)$ such that $\mathfrak{M}, v \models \psi$ and $\mathfrak{M}, v \not\models \chi$. From the latter, since $\chi \in [\wedge, \rightarrow]$, in view of Lemma 5.1.2, we conclude that v is not a total point. Thus since $d(v) < k$, by the induction hypothesis, $(\mathfrak{M}_v)^- \models \psi$ and $(\mathfrak{M}_v)^- \not\models \chi$. It then follows that $(\mathfrak{M}_w)^- \not\models \psi \rightarrow \chi$.

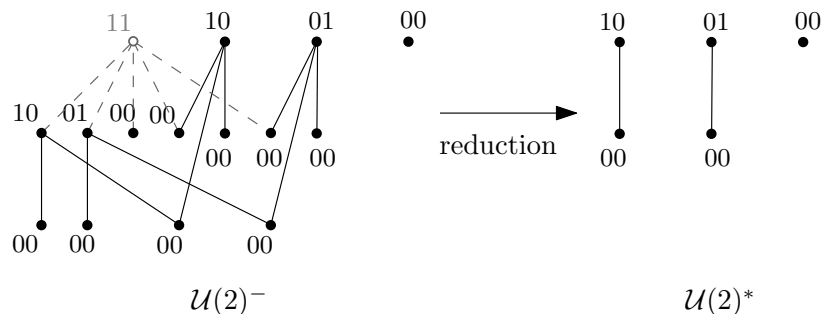
Suppose $(\mathfrak{M}_w)^- \not\models \psi \rightarrow \chi$. Then there exists $v \in (\mathfrak{M}_w)^-$ such that $(\mathfrak{M}_w)^-, v \models \psi$ and $(\mathfrak{M}_w)^-, v \not\models \chi$. It follows that $(\mathfrak{M}_v)^- \models \psi$ and $(\mathfrak{M}_v)^- \not\models \chi$. Then, by the induction hypothesis, $\mathfrak{M}, v \models \psi$ and $\mathfrak{M}, v \not\models \chi$. So $\mathfrak{M}, w \not\models \psi \rightarrow \chi$. \square

Corollary 5.1.4. *The $[\wedge, \rightarrow]$ -fragment of IPC is complete with respect to all \mathfrak{M}^- models.*

Proof. By the finite model property of $[\wedge, \rightarrow]$ -logics and Theorem 5.1.3. \square

In order to show the converse of Theorem 5.1.3, we consider the model $\mathcal{U}(n)^-$ of n -universal model with intersection and total points eliminated. Observe for example that in the 2-universal model $\mathcal{U}(2)$, for every upset of the form in Figure 5.1, the point w will be eliminated in $\mathcal{U}(2)^-$ since it is an intersection point: For this reason, it is not hard to see that for every $n \in \omega$, $\mathcal{U}(n)^-$ is finite. Then by Theorem 3.2.3, $\mathcal{U}(n)^-$ can be mapped p-morphically onto a generated submodel $\mathcal{U}(n)^*$ of $\mathcal{U}(n)$. Clearly, $\mathcal{U}(n)^*$ is also finite. $\mathcal{U}(2)^*$ is depicted in Figure 5.2.

Figure 5.2:



Void dots and dashed lines stand for points and relations that are deleted from $\mathcal{U}(n)$ respectively.

The model $\mathcal{U}(n)^*$ is isomorphic to the *exact model* in [16] for the $[\wedge, \rightarrow]^n$ -fragment of **IPC** (see also [11]). Next, we show that $\mathcal{U}(n)^*$ acts as the n -universal model for the $[\wedge, \rightarrow]^n$ -fragment of **IPC**. First, we show that $\mathcal{U}(n)^*$ is complete with respect to the $[\wedge, \rightarrow]^n$ -fragment of **IPC**.

Theorem 5.1.5. *For every formula $\varphi \in [\wedge, \rightarrow]^n$, we have that $\mathcal{U}(n)^* \models \varphi$ iff $\vdash_{\mathbf{IPC}} \varphi$.*

Proof. “ \Leftarrow ” holds trivially. Suppose $\mathcal{U}(n)^* \models \varphi$, for a formula $\varphi \in [\wedge, \rightarrow]^n$. Since $\mathcal{U}(n)^*$ is a p-morphic image of $\mathcal{U}(n)^-$, by Theorem 2.3.9, $\mathcal{U}(n)^- \models \varphi$.

Now, for every finite model \mathfrak{M} , we know by Theorem 3.2.3 that there exists a generated submodel $\mathcal{U}(n)_w$ of $\mathcal{U}(n)$ such that $\mathcal{U}(n)_w$ is a p-morphic image of \mathfrak{M} . It is easy to see that $(\mathcal{U}(n)_w)^-$ is a generated submodel of $\mathcal{U}(n)^-$. Thus, by Theorem 2.3.9 again, we have that $(\mathcal{U}(n)_w)^- \models \varphi$.

Noting that $\mathcal{U}(n)_w$ is a finite model, we apply Theorem 5.1.3 and obtain that $\mathcal{U}(n)_w \models \varphi$. So by using Theorem 2.3.9 a third time, we obtain that $\mathfrak{M} \models \varphi$, which gives $\vdash_{\mathbf{IPC}} \varphi$ by the finite model property of the $[\wedge, \rightarrow]^n$ -fragment of **IPC**. \square

Define a relation \equiv on a set Θ of formulas by

$$\varphi \equiv \psi \text{ iff } \vdash_{\mathbf{IPC}} \varphi \leftrightarrow \psi.$$

Clearly, \equiv is an equivalence relation.

The next theorem is due to Diego [12]. Here we give an alternative proof based on Theorem 5.1.5.

Theorem 5.1.6 (Diego). *There are only finitely many provably non-equivalent formulas in $[\wedge, \rightarrow]^n$.*

Proof. Note that $\mathcal{U}(n)^*$ is finite. So there are only finitely many upsets in $\mathcal{U}(n)^*$. Enumerate all upsets in $\mathcal{U}(n)^*$ and let $Up(\mathcal{U}(n)^*) = \{X_1, \dots, X_k\}$. Define a function $\sigma : [\wedge, \rightarrow]^n \rightarrow \mathbf{C}^k$ by taking $\sigma(\varphi) = c_1 \cdots c_k$, where for each $1 \leq i \leq k$

$$c_i = \begin{cases} 1, & (\mathcal{U}(n)^*)_{X_i} \models \varphi; \\ 0, & (\mathcal{U}(n)^*)_{X_i} \not\models \varphi. \end{cases}$$

Consider the equivalence relation \equiv on the set of all $[\wedge, \rightarrow]^n$ -formulas. For any $\varphi, \psi \in [\wedge, \rightarrow]^n$, we have that

$$\begin{aligned} \varphi \equiv \psi & \text{ iff } \vdash_{\mathbf{IPC}} \varphi \leftrightarrow \psi \\ & \text{ iff } \mathcal{U}(n)^* \models \varphi \leftrightarrow \psi \text{ (by Theorem 5.1.5)} \\ & \text{ iff } \sigma(\varphi) = \sigma(\psi). \end{aligned}$$

Put $[\wedge, \rightarrow]_{\equiv}^n = \{[\varphi] : \varphi \in [\wedge, \rightarrow]^n\}$. We have that

$$|[\wedge, \rightarrow]_{\equiv}^n| = |\sigma([\wedge, \rightarrow]^n)| \leq |\mathbf{C}^k| = 2^k,$$

which means that there are up to provable equivalence only finitely many $[\wedge, \rightarrow]^n$ -formulas. \square

Next, for every point w in $\mathcal{U}(n)^*$, we define two formulas $\varphi_w^*, \psi_w^* \in [\wedge, \rightarrow]^n$ which do the same job in $\mathcal{U}(n)^*$ for the $[\wedge, \rightarrow]^n$ -fragment of \mathbf{IPC} as the de Jongh formulas do in $\mathcal{U}(n)$ for the whole \mathbf{IPC} . These two formulas are given in [11].

Let w be a point in an n -model \mathfrak{M} . We define $cl(w)$ the *level of color* of w in \mathfrak{M} as

$$cl(w) = n - |prop(w)|.$$

Observe that since $\mathcal{U}(n)^*$ does not contain total points, points w in $\mathcal{U}(n)^*$ with $cl(w) = 1$ are endpoints. On the other hand, endpoints w in $\mathcal{U}(n)^*$ do not necessary satisfy that $cl(w) = 1$.

Definition 5.1.7. We define φ_w^* and ψ_w^* for every point w in $\mathcal{U}(n)^* = \langle W, R, V \rangle$ inductively on $d(w)$. Let q be the propositional letter in $newprop(w)$ with the least index.

If $d(w) = 1$, then let

$$\varphi_w^* := \bigwedge prop(w) \wedge \Delta notprop(w)^1,$$

and

$$\psi_w^* := \varphi_w^* \rightarrow q^2.$$

If $d(w) > 1$, we define φ_w^* and ψ_w^* by induction on $cl(w)$. Assume $w \prec S_w = \{w_1, \dots, w_m\}$. Note that in this case $cl(w) > 1$. If $cl(w) = 2$, then let

$$\begin{aligned} \varphi_w^* := & \bigwedge prop(w) \wedge \Delta newprop(w) \wedge \bigwedge \{\psi_{w_i}^* \rightarrow q : w_i \in S_w\} \wedge \\ & \bigwedge \{\psi_v^* : v \notin R(w) \text{ and } prop(v) \supseteq \bigcap \{prop(w_i) : w_i \in S_w\}\}, \end{aligned}$$

and

$$\psi_w^* := \varphi_w^* \rightarrow q.$$

Note that points w_i in the third conjunct and v in the fourth conjunct of the above φ_w^* satisfy $d(w_i) = d(v) = 1$, therefore the above definition is sound.

If $cl(w) > 2$, φ_w^* and ψ_w^* are defined in the same way as that of the case that $cl(w) = 2$. Note that the w_i in the third conjunct and v in the fourth conjunct of φ_w^* in this case satisfy $cl(w_i) < cl(w)$ and $cl(v) < cl(w)$, therefore this definition is sound.

Note that the φ_w^* and ψ_w^* formulas defined above are in $[\wedge, \rightarrow]^n$. The next theorem is analogous to Theorem 3.2.6. It corresponds to Lemma 3.2.1 in [11], but the proof in [11] was too sketchy. Here we give a detailed proof.

Theorem 5.1.8. *For each point v in $\mathcal{U}(n)^* = \langle W, R, V \rangle$, we have that*

- (i) $v \models \varphi_w^*$, iff wRv ;
- (ii) $v \not\models \psi_w^*$, iff vRw .

¹In this case, $newprop(w) = notprop(w)$.

²Note that by the definition of φ_w^* , we have that $\vdash_{\mathbf{IPC}} (\varphi_w^* \rightarrow q) \leftrightarrow (\varphi_w^* \rightarrow q')$ for any $q, q' \in newprop(w)$. So it does not really matter which $q \in newprop(w)$ is chosen in the definition of ψ_w^* . For simplicity, here we stipulate $q = p_i$ to be the one that has the least index.

Proof. We prove (i) and (ii) by simultaneous induction on $d(w)$.

$d(w) = 1$. For (i), suppose wRv . Then $v = w$. Clearly, we have that $w \models \varphi_w^*$, which means that $v \models \varphi_w^*$. Now suppose $v \models \varphi_w^*$. It follows from $v \models \bigwedge \text{prop}(w)$ that

$$\text{prop}(v) \supseteq \text{prop}(w); \quad (5.1)$$

and from $v \models \Delta \text{notprop}(w)$ that

$$v \models \bigwedge \text{notprop}(w) \quad (5.2)$$

or

$$v \not\models \bigvee \text{notprop}(w). \quad (5.3)$$

Observe that (5.1) together with (5.2) imply that v is a total point, which is impossible. Thus, we obtain (5.3), which means that

$$\text{col}(v) = \text{col}(w).$$

Suppose v is not an endpoint. Then for any $u \in R(v)$ such that $u \neq v$, since v is not an intersection point, $\text{col}(u) > \text{col}(v)$, i.e. there exists $r \in \text{notprop}(v)$ such that $u \models r$. Since $v \models \Delta \text{notprop}(w)$, then we must have that

$$u \models \bigwedge \text{notprop}(w),$$

which means that u is a total point; a contradiction. Hence, v is an endpoint. So, by the definition of $\mathcal{U}(n)^*$, $v = w$, thus wRv .

For (ii), suppose vRw . Clearly, $w \models \varphi_w^*$ and $w \not\models q$. Thus, $v \not\models \varphi_w^* \rightarrow q$. Now suppose $v \not\models \psi_w^*$. Then there exists $u \in R(v)$ such that

$$u \models \varphi_w^* \text{ and } u \not\models q.$$

By (i), the former of the above means that wRu . Then since w is an endpoint, $u = w$, thus vRw .

$d(w) > 1$. We show (i) and (ii) by induction on $cl(w)$. The cases that $cl(w) = 2$ and $cl(w) > 2$ can be proved by similar arguments. Here we prove the two cases at the same time. First, note that in the basic step that $cl(w) = 2$, we have $d(w) = 2$.

For (i), suppose wRv . We show that $w \models \varphi_w^*$, which implies that $v \models \varphi_w^*$. Clearly,

$$w \models \text{prop}(w) \text{ and } w \models \Delta \text{newprop}(w).$$

For any $w_i \in S_w$, if $cl(w) = 2$, then $d(w_i) = d(w) - 1 = 1$. If $cl(w) > 2$, then $cl(w_i) < cl(w)$. In both cases, either by (ii) of the basic step of the induction, or by (ii) of the induction step, we have that $w \not\models \psi_{w_i}^*$. For any $u \in R(w)$ with $u \neq w$, we have that $u \models q$, since $q \in newprop(w)$. Hence, we conclude that $w \models \psi_{w_i}^* \rightarrow q$.

Lastly, let u be a point in $\mathcal{U}(n)^*$ satisfying

$$\neg wRu \text{ and } prop(u) \supseteq \bigcap \{prop(w_i) : w_i \in S_w\}.$$

The latter of the above implies that $cl(u) < cl(w)$. If $cl(w) = 2$, then $d(u) = 1$. Therefore by the former of the above and (ii) of the basic step of the induction or (ii) of the induction step, we obtain that $w \models \psi_u^*$. Hence, $w \models \varphi_w^*$.

Now, suppose $v \models \varphi_w^*$. First, from $v \models \bigwedge prop(w)$, we obtain (5.1). Next, we distinguish two cases.

Case 1: $q \in prop(v)$. Then since $v \models \Delta newprop(w)$, we must have that

$$prop(v) \supseteq newprop(w). \quad (5.4)$$

Together with (5.1), we obtain

$$prop(v) \supseteq prop(w) \cup newprop(w) = \bigcap \{prop(w_i) : w_i \in S_w\}.$$

The above implies that $cl(v) < cl(w)$. If $cl(w) = 2$, then $d(v) = 1$. Thus either by (ii) of the basic step of the induction, or by (ii) of the induction step, $v \not\models \psi_v^*$. Thus, since

$$v \models \bigwedge \{\psi_u^* : u \notin R(w) \text{ and } prop(u) \supseteq \bigcap \{prop(w_i) : w_i \in S_w\}\} \quad (5.5)$$

we must have that wRv as required.

Case 2: $q \notin prop(v)$. Then since $v \models \psi_{w_i}^* \rightarrow q$ for every $w_i \in S_w$, we have that $v \not\models \psi_{w_i}^*$. Note that either $cl(w) = 2$ and $d(w_i) = 1$, or $cl(w) > 2$ and $cl(w_i) < cl(w)$. Thus by (ii) of the basic step of the induction, or by (ii) of the induction step, we have that vRw_i for every $w_i \in S_w$.

Furthermore, for every $w_i \in S_w$ we have that

$$notprop(v) \supseteq notprop(w_i).$$

In the meantime, since $v \models \Delta newprop(w)$ and $v \not\models q$,

$$notprop(v) \supseteq newprop(w).$$

Hence,

$$\text{notprop}(v) \supseteq \text{newprop}(w) \cup \bigcup_{w_i \in S_w} \text{notprop}(w_i) = \text{notprop}(w).$$

Then, together with (5.1), we obtain

$$\text{col}(v) = \text{col}(w).$$

Next, we show that w_i is an immediate successor of v . Suppose there exists an immediate successor u of v in $\mathcal{U}(n)^*$ such that uRw_i . Clearly, $\neg wRu$, since wRu contradicts the fact that w is an immediate successor of w_i . We distinguish two subcases.

Subcase 1: $\text{prop}(u) \supseteq \bigcap \{\text{prop}(w_i) : w_i \in S_w\}$. Then by (5.5), $v \models \psi_u^*$. Note that either $\text{cl}(w) = 2$ and $d(u) = 1$, or $\text{cl}(w) > 2$ and $\text{cl}(u) < \text{cl}(w)$. Thus by (ii) of the basic step of the induction, or (ii) of the induction step, $\neg vRu$; a contradiction.

Subcase 2: there exists $r \in \bigcap \{\text{prop}(w_i) : w_i \in S_w\}$ such that $r \notin \text{prop}(u)$. Since vRu , we also have that $r \notin \text{prop}(v) = \text{prop}(w)$, thus,

$$r \in \text{newprop}(w).$$

On the other hand, since u is not an intersection point, $\text{col}(u) > \text{col}(v)$, thus there exists s such that

$$s \in \text{prop}(u) \text{ and } s \notin \text{prop}(v).$$

If $s \in \text{newprop}(w)$, then $v \not\models s \rightarrow r$ will contradict $v \models \Delta \text{newprop}(w)$.

Thus we have that $s \notin \text{newprop}(w)$. Then there exists $w_j \in S_w$ such that

$$\neg uRw_j \text{ and } s \notin \text{prop}(w_j).$$

Since either $\text{cl}(w) = 2$ and $d(w_j) = 1$, or $\text{cl}(w) > 2$ and $\text{cl}(w_j) < \text{cl}(w)$, by (ii) of the basic step of the induction, or (ii) of the induction step, the former of the above implies that $u \models \psi_{w_j}^*$. Thus we have that

$$v \not\models \psi_{w_j}^* \rightarrow r,$$

which contradicts

$$v \models \bigwedge \{\psi_{w_i}^* \rightarrow q : w_i \in S_w\}. \quad (5.6)$$

Hence, we have proved that every $w_i \in S_w$ is an immediate successor of v .

Now, we show that every immediate successor of v is an immediate successor of w . Suppose u is an immediate successor of v and $\neg wRu$. First, for every $w_i \in S_w$ note that either $cl(w) = 2$ and $d(w_i) = 1$, or $cl(w) > 2$ and $cl(w_i) < cl(w)$. Thus since $\neg uRw_i$, by (ii) of the basic step of the induction, or (ii) of the induction step, we have that $u \models \psi_{w_i}^*$. Then since $v \models \psi_{w_i}^* \rightarrow q$, we must have that $u \models q$. Together with the fact that $v \models \Delta newprop(w)$, we obtain that $prop(u) \supseteq newprop(w)$. On the other hand, since vRu , we have that $prop(u) \supseteq prop(w) = prop(w)$. Thus,

$$prop(u) \supseteq newprop(w) \cup prop(w) = \bigcap \{prop(w_i) : w_i \in S_w\}.$$

Since $\neg wRu$, by (5.5) means that $v \models \psi_u^*$. However, since $d(u) < d(w)$ and vRu , this contradicts the induction hypothesis.

Now suppose there exists u' such that u' is an immediate successor of w and $u'Ru$. Clearly, $\neg vRu'$, since vRu' contradicts the fact that u is an immediate successor of v . Since either $cl(w) = 2$ and $d(u') = 1$, or $cl(w) > 2$ and $cl(u') < cl(w)$, by (ii) of the basic step of the induction, or (ii) of the induction step, $v \models \psi_{u'}^*$. Hence, since $col(v) = col(w)$, $v \not\models \psi_{u'}^* \rightarrow q$, which contradicts (5.6).

Hence, we have proved that the set of immediate successors of w is the same as the set of immediate successors of v . Together with the fact that $col(w) = col(v)$, by the definition of $\mathcal{U}(n)^*$, we conclude that $w = v$, thus wRv .

For (ii), “ \Leftarrow ” is proved by the same argument as in the case that $d(w) = 1$. For “ \Rightarrow ”, suppose $v \not\models \psi_w^*$. By a similar argument to that in the case that $d(w) = 1$, we can prove that there exists $u \in R(v)$ such that wRu and $u \not\models q$. Since $q \in newprop(w)$, we must have that $w = u$. Thus, in both cases that $cl(w) = 2$ and $cl(w) > 2$, we have that vRw . \square

Next, we show that every upset of $\mathcal{U}(n)^*$ is defined by a $[\wedge, \rightarrow]^n$ formula. Since we do not allow \vee in the $[\wedge, \rightarrow]^n$ -fragment, we cannot simply take the formula $\bigvee_{w \in X} \varphi_w^*$ as the formula that defines an upset X .

By Theorem 5.1.6, for any point w in $\mathcal{U}(n)^*$, there are only finitely many $[\wedge, \rightarrow]^n$ -formulas in $Th^n(w)$. For each upset X of $\mathcal{U}(n)^*$, we define a formula $\theta^n(X) \in [\wedge, \rightarrow]^n$ with finite length as

$$\theta^n(X) := \bigwedge \{ \varphi \in [\wedge, \rightarrow]^n : \varphi \in \bigcap_{w \in X} Th^n(w) \}.$$

In the case that $X = \{w\}$, we will only write $\theta^n(w)$ instead of $\theta^n(\{w\})$.

The following lemma is adapted from Theorem 3.6.0.12 in [16].

Lemma 5.1.9. *Let X be an upset of $\mathcal{U}(n)^* = \langle W, R, V \rangle$. We have that $X = V(\theta^n(X))$.*

Proof. Clearly, by the definition $X \subseteq V(\theta^n(X))$. For each $w \in V(\theta^n(X))$, we show by induction on $d(w)$ that $w \in X$.

$d(w) = 1$. Consider the formula

$$\begin{aligned} \eta_w &= \bigwedge \text{prop}(w) \wedge \bigwedge \{q \rightarrow \bigwedge \{p_1, \dots, p_n\} : q \in \text{notprop}(w)\} \\ &\rightarrow \bigwedge \{p_1, \dots, p_n\}. \end{aligned}$$

Clearly, $\eta_w \notin Th^n(w)$. Thus, since $w \models \theta^n(X)$, we may conclude that there exists $v \in X$ such that $v \not\models \eta_w$, which means that there exists $v' \in R(v)$, such that

$$v' \models \bigwedge \text{prop}(w) \wedge \bigwedge \{q \rightarrow \bigwedge \{p_1, \dots, p_n\} : q \in \text{notprop}(w)\}. \quad (5.7)$$

v' is an endpoint. Suppose otherwise, i.e. there exists v'' such that $v'' \neq v'$ and $v' R v''$. Note that by the definition of $\mathcal{U}(n)^*$, $\text{col}(v') \leq \text{col}(v'')$. Thus there exists $q \in \text{prop}(v'') \cap \text{notprop}(v')$. It then follows that $q \in \text{notprop}(w)$ and

$$v' \not\models q \rightarrow \bigwedge \{p_1, \dots, p_n\}.$$

These contradict (5.7).

(5.7) implies that $\text{col}(v') = \text{col}(v)$. Thus since in $\mathcal{U}(n)$ distinct endpoints have distinct colors, we conclude that $v' = w$, and hence $v R w$. Thus $w \in X$ since X is upward closed.

$d(w) > 1$. Assume $w \notin X$. Since w is not maximal, there exists $q \in \text{newprop}(w)$. Consider the formula $\theta^n(w) \rightarrow q$. For each $v \in X$, suppose $v' \in R(v)$ and $v' \models \theta^n(w)$. From the former, since X is upward closed, it follows that $v' \in X$ and so $v' \neq w$. Since φ_w is a conjunct in $\theta^n(w)$, by Theorem 5.1.8, the latter implies that $w R v'$. Observe that $v' \models \theta^n(w) \rightarrow q$, which implies that $v' \models q$, therefore we conclude that $v \models \theta^n(w) \rightarrow q$. This means that $\theta^n(w) \rightarrow q$ is a conjunct of $\theta^n(X)$. So, since $w \models \theta^n(X)$, we have that $w \models \theta^n(w) \rightarrow q$, which is impossible. \square

Having Theorem 5.1.5 and Lemma 5.1.9 at hand, we can view $\mathcal{U}(n)^*$ as an n -universal model for the $[\wedge, \rightarrow]^n$ -fragment of **IPC**. Hereafter, we will use the symbol $\mathcal{U}(n)^{[\wedge, \rightarrow]}$ instead of $\mathcal{U}(n)^*$.

We finish this section by showing the converse of Theorem 5.1.3. For the proof of the next theorem, one may also refer to [16].

Theorem 5.1.10. *Let φ be a formula. If for every finite model $\mathfrak{M} = \langle W, R, V \rangle$ and every total point $w \in W$, $\mathfrak{M}, w \models \varphi$ holds, and for every non-total point $w \in W$,*

$$\mathfrak{M}_w \models \varphi \iff (\mathfrak{M}_w)^- \models \varphi,$$

then φ is provably equivalent to a formula $\varphi' \in [\wedge, \rightarrow]$.

Proof. Let φ be the formula described in the theorem. Consider the model $\mathcal{U}(n)^{[\wedge, \rightarrow]} = \langle W, R, V \rangle$. Let $\varphi' = \theta^n(V(\varphi)) \in [\wedge, \rightarrow]^n$.

For every w in $\mathcal{U}(n)$, if w is total, then $\mathcal{U}(n)_w$ is a singleton. By the assumption and Lemma 5.1.2, we have that

$$\mathcal{U}(n), w \models \varphi \iff \mathcal{U}(n), w \models \varphi'. \quad (5.8)$$

If w is non-total, then we apply the assumption to the finite model $\mathcal{U}(n)_w$ and obtain

$$\mathcal{U}(n)_w \models \varphi \iff (\mathcal{U}(n)_w)^- \models \varphi. \quad (5.9)$$

Suppose w is not maximal. Note that $(\mathcal{U}(n)_w)^- \cong (\mathcal{U}(n)^-)_w$ and there exists a p-morphism f of $\mathcal{U}(n)^-$ onto $\mathcal{U}(n)^{[\wedge, \rightarrow]}$. Since $w \in \mathcal{U}(n)^-$, and thus by Theorem 2.3.9 and Lemma 5.1.9,

$$\begin{aligned} (\mathcal{U}(n)_w)^- \models \varphi &\iff (\mathcal{U}(n)^{[\wedge, \rightarrow]})_{f(w)} \models \varphi \\ &\iff (\mathcal{U}(n)^{[\wedge, \rightarrow]})_{f(w)} \models \varphi' \\ &\iff (\mathcal{U}(n)_w)^- \models \varphi'. \end{aligned}$$

Observe that $f(w)$ is not total. So since $\varphi' \in [\wedge, \rightarrow]$ and $\mathcal{U}(n)_w$ is finite, by Theorem 5.1.3,

$$(\mathcal{U}(n)_w)^- \models \varphi' \iff \mathcal{U}(n)_w \models \varphi'.$$

Therefore, we obtain (5.8) for all $w \in \mathcal{U}(n)$.

Now, suppose w is maximal. Then we have that $(\mathcal{U}(n)_w)^- \cong \mathcal{U}(n)^-$. Since $\mathcal{U}(n)^*$ is a p-morphic image of $\mathcal{U}(n)^-$, by Lemma 5.1.9,

$$\mathcal{U}(n)^- \models \varphi \iff \mathcal{U}(n)^{[\wedge, \rightarrow]} \models \varphi \iff \mathcal{U}(n)^{[\wedge, \rightarrow]} \models \varphi' \iff \mathcal{U}(n)^- \models \varphi'.$$

Since $(\mathcal{U}(n)_w)^- \cong \mathcal{U}(n)^-$, $\mathcal{U}(n)_w$ is finite, $\varphi' \in [\wedge, \rightarrow]$ and w is not total, by Theorem 5.1.3,

$$\mathcal{U}(n)^- \models \varphi' \iff \mathcal{U}(n)_w \models \varphi'.$$

Therefore, together with (5.9), we obtain (5.8) for all $w \in \mathcal{U}(n)$.

By Theorem 3.2.4, (5.8) implies that $\vdash_{\mathbf{IPC}} \varphi \leftrightarrow \varphi'$. □

5.2 Brouwerian semilattices

In this section, we spell out the algebraic characterization of the $[\wedge, \rightarrow]$ -fragment of \mathbf{IPC} . The Lindenbaum algebra of the $[\wedge, \rightarrow]$ -fragment of \mathbf{IPC} is the free Brouwerian semilattice on ω many generators (i.e. $\mathbf{F}_{\mathbf{BS}}(\omega)$). For more details on Brouwerian semilattices, one may refer to [20], [23] and [24].

Definition 5.2.1. A *Brouwerian semilattice* (also known as *Implicative semilattice*) $\mathfrak{A} = \langle A, \wedge, \rightarrow, 1 \rangle$ is a meet-semilattice $\langle A, \wedge, 1 \rangle$ with a binary operator \rightarrow , defined as:

$$c \leq a \rightarrow b \text{ iff } a \wedge c \leq b.$$

The following result was first observed by Monteiro [22].

Theorem 5.2.2. *The class of all Brouwerian semilattices forms a variety, denoted by \mathbf{BS} .*

It follows from Theorem 5.1.6 that \mathbf{BS} is locally finite.

Definition 5.2.3. An algebra is *locally finite* if every finitely generated sub-algebra is finite.

Theorem 5.2.4. *\mathbf{BS} is locally finite.*

Proof. Follows from Lemma 5.1.6. □

For a detailed proof of the following lemma, see e.g. Theorem 3.2 in [23].

Lemma 5.2.5. *Let $\mathfrak{A} \in \mathbf{BS}$. Then the filters of \mathfrak{A} are in 1-1 correspondence with congruences of \mathfrak{A} .*

Proof. (sketch) Let F be a filter of \mathfrak{A} . It can be showed that the relation θ_F defined as follows is a congruence of \mathfrak{A} :

$$a\theta_F b \text{ iff } (a \rightarrow b) \wedge (b \rightarrow a) \in F.$$

Let $\theta \in \text{Con}\mathfrak{A}$. Then the natural map $f : \mathfrak{A} \rightarrow \mathfrak{A}/\theta$, defined by $f(a) = a/\theta$, is a surjective homomorphism. It is easy to check that $F_\theta = f^{-1}(1/\theta)$ is a filter of \mathfrak{A} . \square

Remark 5.2.6. *We have that $F_{\theta_F} = F$ and $\theta_{F_\theta} = \theta$.*

Now we prove the congruence extension property of **BS**.

Theorem 5.2.7. ***BS** has the congruence extension property.*

Proof. Let $\mathfrak{A}, \mathfrak{B} \in \mathbf{BS}$ and \mathfrak{B} is a subalgebra of \mathfrak{A} . For every $\theta \in \text{Con}\mathfrak{B}$, the F_θ defined in Lemma 5.2.5 is a filter of \mathfrak{B} . Let F' be the filter of \mathfrak{A} generated by F_θ . The $\theta_{F'}$ defined in Lemma 5.2.5 is a congruence of \mathfrak{A} . Clearly, $F_\theta = F' \cap B$. It then follows from Remark 5.2.6 that $\theta = \theta_{F'} \cap B^2$. \square

Corollary 5.2.8. *For any $\mathfrak{A} \in \mathbf{BS}$, $\mathbf{HS}(\mathfrak{A}) = \mathbf{SH}(\mathfrak{A})$.*

Proof. By Theorem 2.7.7. \square

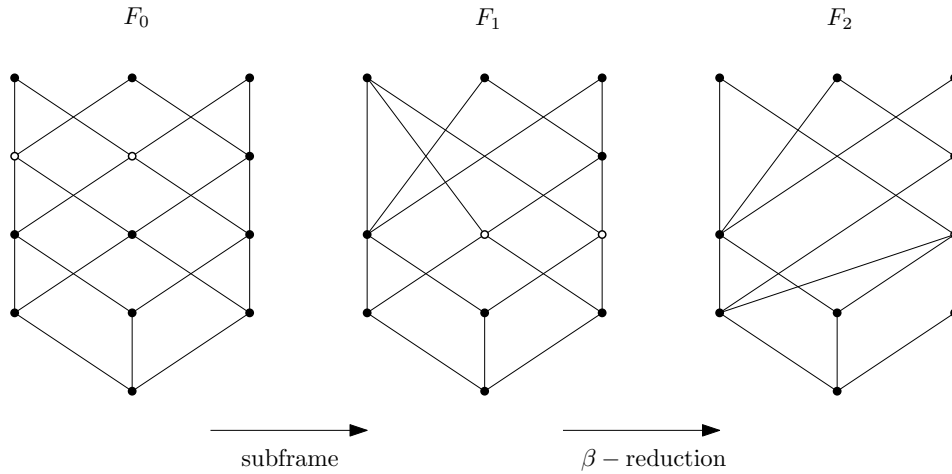
It is worthwhile to point out that Brouwerian semilattices turn out to coincide with subalgebras of $[\wedge, \rightarrow, 1]$ -reducts of Heyting algebras. Let $\mathbf{H}^{[\wedge, \rightarrow]}$ be the operation of taking Heyting homomorphisms that preserve \wedge and \rightarrow only. Let $\mathbf{S}^{[\wedge, \rightarrow]}$ be the operation of taking subalgebras with respect to \wedge and \rightarrow only. Then Corollary 5.2.8 implies that for any Heyting algebra $\mathfrak{A} \in \mathbf{HA}$, $\mathbf{H}^{[\wedge, \rightarrow]}\mathbf{S}^{[\wedge, \rightarrow]}(\mathfrak{A}) = \mathbf{S}^{[\wedge, \rightarrow]}\mathbf{H}^{[\wedge, \rightarrow]}(\mathfrak{A})$.

One may wonder whether the operations $\mathbf{H}^{[\wedge, \rightarrow]}$ and \mathbf{S} on Heyting algebras also commute. We next give a counterexample to it. First, observe that the operation $\mathbf{H}^{[\wedge, \rightarrow]}$ corresponds to taking subframes of intuitionistic frames. The next example shows that there exists a Kripke frame \mathfrak{F} which is a p-morphic image of a subframe of a Kripke frame \mathfrak{G} , such that \mathfrak{F} is not a subframe of any p-morphic image of \mathfrak{G} . Dually, this means that not for every Heyting algebra \mathfrak{A} , $\mathbf{H}^{[\wedge, \rightarrow]}\mathbf{S}(\mathfrak{A}) \subseteq \mathbf{SH}^{[\wedge, \rightarrow]}(\mathfrak{A})$.

Example 5.2.9. *In Figure 5.3, the frame F_2 is a p-morphic image of a subframe of the frame F_0 . However, F_2 is not a subframe of a p-morphic image of F_0 . This is because the only p-morphic image one can get from F_0*

(the only way is to apply β -reductions) is F_0 itself, or a frame obtained by identifying two or three of the end points. However, F_2 is not a subframe of any of these frames.

Figure 5.3: Example 5.2.9



- Void dots in F_0 stand for the points that are taken out in F_1 .
- Void dots in F_1 stand for the points that are identified by the β -reduction.
- The void dot in F_2 stands for the point obtained by identifying the two void dots in F_1 by the β -reduction.

Chapter 6

NNIL-formulas

In this chapter, we discuss properties of *NNIL*-formulas. *NNIL*-formulas were studied by Visser, de Jongh, van Benthem and Renardel de Lavalette in [25]. They are formulas that have **no nesting of implications to the left**. Like the set of $[\wedge, \rightarrow]$ -formulas, the set of *NNIL*-formulas is (modulo provable equivalence) finite. This was sketched in [25]. In Section 6.1, we give a detailed proof of this fact by introducing the normal form of *NNIL*-formulas. In Section 6.2, we introduce the notion of subsimulation of [25], and show that *NNIL*-formulas are preserved under subsimulations, and therefore preserved under taking submodels. As a consequence, *NNIL*-formulas are preserved under subframes as well. As it is mentioned in Chapter 4, it then follows from [3] that *NNIL*-formulas and $[\wedge, \rightarrow]$ -formulas define the same subframe logics. In Section 6.3, we give an algorithm to translate every *NNIL*-formula to a $[\wedge, \rightarrow]$ -formula in such a way that they are equivalent on frames. This indicates the fact that every subframe logic defined by *NNIL*-formulas is equivalent to a logic defined by $[\wedge, \rightarrow]$ -formulas.

It follows from [25] that if two models totally subsimulate each other, then they satisfy the same *NNIL*-formulas. In Section 6.4, we develop this idea and give a construction of suitable representative models of equivalence classes of rooted generated models of $\mathcal{U}(n)$ induced by two-way subsimulations. In section 6.5, we give n -universal models $\mathcal{U}(n)^{NNIL}$ of *NNIL*-formulas with n variables. The n -universal models of *NNIL*-formulas come from the n -universal model of **IPC**. This enables us to prove properties of *NNIL*-formulas in an easy way. In particular, the theorem prove in [25] that formulas preserved under subsimulations are equivalent to *NNIL*-formulas becomes a natural consequence of the properties of $\mathcal{U}(n)^{NNIL}$ of *NNIL*-formulas.

For comparison, note that each point in the n -universal model $\mathcal{U}(n)$ of **IPC** can be viewed also as the generated submodel of $\mathcal{U}(n)$ generated by that point, and the relation on points of $\mathcal{U}(n)$ can be viewed also as the generated submodel ordering. The essential difference between $\mathcal{U}(n)^{NNIL}$ and $\mathcal{U}(n)$ is that in $\mathcal{U}(n)^{NNIL}$, the generated submodel relation no longer plays a central role; instead, we consider the subsimulation relation. Each point in $\mathcal{U}(n)^{NNIL}$ is labeled by a representative model of a equivalence class of models induced by two-way subsimulations. The relation on points in $\mathcal{U}(n)^{NNIL}$ is the subsimulation ordering on the representative models. The generated submodel generated by a point in $\mathcal{U}(n)^{NNIL}$ is generally not isomorphic to the representative model of this point, however, these two models are equivalent up to the equivalence relation induced by two-way subsimulations. The author realizes and apologizes that the method to construct the model $\mathcal{U}(n)^{NNIL}$ and prove its properties is very cumbersome. The author hope to obtain simpler proofs in the future.

The representative models of points in $\mathcal{U}(n)^{NNIL}$ may reasonably give more results in the study of *NNIL*-formulas and subframe logics. In section 6.6, we discuss the connection of the model $\mathcal{U}(2)^{NNIL}$ and subframe logics defined by two-variable *NNIL*-formulas. We obtain characterization properties of frames that define subframe logics by using the structure of representative models in $\mathcal{U}(2)^{NNIL}$. This suggests a method for future work on subframe logics.

6.1 *NNIL*-formulas

In this section, we give a formal definition of *NNIL*-formulas and prove that there are only (module provable equivalence) finitely many *NNIL*-formulas.

Definition 6.1.1. The smallest class satisfying the following is called the class of *NNIL*-formulas:

- $p \in NNIL$, for any $p \in \text{PROP}$.
- $\perp \in NNIL$.
- If $\varphi, \psi \in NNIL$, then $\varphi \wedge \psi, \varphi \vee \psi \in NNIL$.
- If $\psi \in NNIL$, φ does not contain any implication or negation, then $\varphi \rightarrow \psi \in NNIL$.

For any *NNIL*-formula, conjunctions and disjunctions in front of implications can be removed using:

- $\vdash ((\varphi \wedge \psi) \rightarrow \chi) \leftrightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,
- $\vdash ((\varphi \vee \psi) \rightarrow \chi) \leftrightarrow ((\varphi \rightarrow \chi) \wedge (\psi \rightarrow \chi))$.

So, as it is mentioned in [25] that *NNIL* formulas can be translated modulo provable equivalence to *NNIL*₀ formulas which have only propositional variables to the left of implications.

Definition 6.1.2. *NNIL*₀ is the smallest class of formulas satisfying:

- $p \in \text{NNIL}_0$, for any $p \in \text{PROP}$.
- $\perp \in \text{NNIL}_0$.
- If $\varphi, \psi \in \text{NNIL}_0$, then $\varphi \wedge \psi, \varphi \vee \psi \in \text{NNIL}_0$.
- if $\varphi \in \text{NNIL}_0$, then $p \rightarrow \varphi \in \text{NNIL}_0$ for any $p \in \text{PROP}$.

We write *NNIL*(\vec{p}) to indicate all *NNIL*-formulas with variables among $\vec{p} = \{p_1, \dots, p_n\}$ and always assume the length $|\vec{p}| = n$. In view of the definition of *NNIL*₀, we can define a normal form for *NNIL*-formulas.

Definition 6.1.3. A *NNIL*(\vec{p})-formula is in *normal form* (NF) if it is a conjunction of disjunction of atoms and formulas of the form $p \rightarrow \psi$, where $\psi \in \text{NNIL}_0(\vec{p} \setminus \{p\})$ and ψ contains no conjunction to the right of implications.

Proposition 6.1.4. *Every NNIL*(\vec{p})-*formula* φ *is equivalent to some formula* φ' *in NF.*

Proof. We start from the *NNIL*₀(\vec{p})-formulas and prove the proposition by induction on the complexity of $\varphi \in \text{NNIL}_0(\vec{p})$.

If $\varphi = p$ or \perp or \top , then the proposition holds trivially.

If $\varphi = \psi \wedge \chi$, then by the induction hypothesis, ψ and χ are in NF, thus φ is in NF.

If $\varphi = \psi \vee \chi$, then by the induction hypothesis, $\psi = \bigwedge_{i=1}^n (\bigvee_{j=1}^m \alpha_{i,j})$ and $\chi = \bigwedge_{i=1}^k (\bigvee_{j=1}^l \beta_{i,j})$ are in NF. By using the distributive law

$$\vdash ((\varphi \wedge \psi) \vee \chi) \leftrightarrow ((\varphi \vee \chi) \wedge (\psi \vee \chi)),$$

φ can be rewritten as $\chi = \bigwedge_{i=1}^s (\bigvee_{j=1}^t \gamma_{i,j})$ for some $\gamma_{i,j}$'s.

If $\varphi = p \rightarrow \psi$, then by the induction hypothesis, $\psi = \bigwedge_{i=1}^n (\bigvee_{j=1}^m \chi_{i,j})$ is in NF. Next, note that $p \rightarrow \bigwedge_{i=1}^n (\bigvee_{j=1}^m \chi_{i,j})$ is equivalent modulo provable equivalence to $p \rightarrow \bigwedge_{i=1}^n \bigvee_{j=1}^m (\chi_{i,j}[p/\top])$. Lastly, by using inductively

$$\vdash (\alpha \rightarrow \beta \wedge \gamma) \leftrightarrow (\alpha \rightarrow \beta) \wedge (\alpha \rightarrow \gamma),$$

φ can be rewritten as $\bigwedge_{i=1}^r (p \rightarrow \bigvee_{j=1}^t (\chi_{i,j}[p/\top]))$, which is in NF. \square

For every *NNIL*-formula we define the complexity σ of right-nesting of \rightarrow as follows:

- $\sigma(p) := \sigma(\perp) := \sigma(\top) := 0$
- $\sigma(\alpha \wedge \beta) := \sigma(\alpha \vee \beta) := \max(\sigma(\alpha), \sigma(\beta))$
- $\sigma(\alpha \rightarrow \beta) := \max(\sigma(\alpha), \sigma(\beta) + 1)$

Proposition 6.1.5. *For every $\text{NNIL}(\vec{p})$ -formula φ in NF,*

$$\sigma(\varphi) \leq |\vec{p}|.$$

Proof. We prove the proposition by induction on $|\vec{p}|$. Note that Boolean operators do not increase right-nesting complexity. We only show that for each formula $p \rightarrow \psi$ in NF, we have

$$\sigma(p \rightarrow \psi) \leq |\vec{p}|.$$

If $|\vec{p}| = 1$, then the only $\text{NNIL}_0(\vec{p})$ formula that is in NF is $p \rightarrow \perp$. Clearly $\sigma(p \rightarrow \perp) = 1$.

If $|\vec{p}| = n + 1$, then for each formula $p \rightarrow \psi$ in NF, by the induction hypothesis, $\sigma(\psi) \leq n$, so $\sigma(p \rightarrow \psi) \leq n + 1$. \square

The basic idea of a proof of the following proposition was sketched in the proof of Theorem 2.2 in [25]. Here we give a detailed calculation.

Proposition 6.1.6. *There are only finitely many $\text{NNIL}(\vec{p})$ -formulas in NF that are of the form $p \rightarrow \psi$.*

Proof. Let $|\vec{p}| = n$. We show by induction on $\sigma(p \rightarrow \psi)$.

If $\sigma(p \rightarrow \psi) = 1$, then the set of formulas in NF of the form

$$p \rightarrow \perp$$

has the cardinality n . The set of formulas in NF of the form

$$p \rightarrow q_1 \vee q_2 \vee \cdots \vee q_k,$$

where $k \leq n - 1$, has the cardinality $n \cdot C_{n-1}^k$. Thus, the cardinality $sum(1)$ of the set of formulas in NF of the form $p \rightarrow \psi$ with $\sigma(p \rightarrow \psi) = 1$ is

$$sum(1) = n + \sum_{k \leq n-1} n \cdot C_{n-1}^k.$$

Suppose $\sigma(p \rightarrow \psi) = m + 1$. Then by Proposition 6.1.5, $m + 1 \leq n$. Let

$$sum(\leq m) = \sum_{i=1}^m sum(i).$$

All formulas $p \rightarrow \psi$ in NF with $\sigma(p \rightarrow \psi) = m + 1$ are of the form

$$p \rightarrow \chi_0 \vee \bigvee_{i=1}^k \chi_i,$$

where $\sigma(\chi_0) = m$ and $0 \leq k \leq sum(\leq m) - 1$. Thus

$$sum(m + 1) = n \cdot sum(m) \cdot \left(\sum_{i=1}^{sum(\leq m)-1} C_{sum(\leq m)-1}^i + 1 \right).$$

Hence, there are in total $\sum_{m=1}^n sum(m)$ many $NNIL(\vec{p})$ -formulas in NF that are of the form $p \rightarrow \psi$. □

Corollary 6.1.7. *There are (modulo provable equivalence) only finitely many $NNIL(\vec{p})$ -formulas.*

Proof. Since by using \wedge and \vee , only finitely many new non-equivalent formulas can be produced, by Proposition 6.1.6, there are (modulo provable equivalence) only finitely many $NNIL(\vec{p})$ -formulas that are in NF. So by Proposition 6.1.4, there are (modulo provable equivalence) only finitely many $NNIL(\vec{p})$ -formulas. □

Consider the equivalence relation \equiv on $NNIL(\vec{p})$. By Corollary 6.1.7, the set $NNIL(\vec{p})_{\equiv} = \{[\varphi] : \varphi \in NNIL(\vec{p})\}$ is finite.

6.2 NNIL-formulas and subsimulations

This section is based on [25]. By the standard translation, we can translate *NNIL*-formulas to first-order formulas. These first-order formulas are Π_1 -formulas. We know that Π_1 -formulas are preserved in substructures (see e.g. [17]). Thus, *NNIL*-formulas are preserved under taking submodels. In this section, we introduce subsimulations defined in [25] and spell out the result that *NNIL*-formulas are exactly those formulas preserved under subsimulations.

Definition 6.2.1. Let $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{N} = \langle W', R', V' \rangle$ be n -models. A relation Z on $W' \times W$ is a *subsimulation* if it satisfies the following conditions.

(S1) If vZw , then $\mathfrak{N}, v \models p_i$ iff $\mathfrak{M}, w \models p_i$, for each $1 \leq i \leq n$.

(S2) If vZw and $vR'v'$, then there exists $w' \in W$ such that $v'Zw'$ and wRw' .

Z is total if for any $v \in W'$, there exists $w \in W$ such that vZw . We denote $\mathfrak{N} \preceq \mathfrak{M}$ and say that “ \mathfrak{M} subsimulates \mathfrak{N} ”, if there exists a total subsimulation Z of \mathfrak{N} in \mathfrak{M} .

Remark 6.2.2. If \mathfrak{N} is a submodel of \mathfrak{M} , then the inclusion map is a subsimulation.

Remark 6.2.3. If f is a subreduction between two models (a subreduction preserving propositional variables), then f^{-1} is a subsimulation.

The proof of the next theorem is adapted from Theorem 6.4 in [25].

Theorem 6.2.4. Let $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{N} = \langle W', R', V' \rangle$ be n -models, and $Z : \mathfrak{N} \preceq \mathfrak{M}$. If vZw , then for any *NNIL*(\vec{p})-formula φ , $\mathfrak{M}, w \models \varphi$ implies that $\mathfrak{N}, v \models \varphi$.

Proof. The theorem is proved by induction on the complexity of a *NNIL* $_0(\vec{p})$ formula φ . The only non-trivial case is that $\varphi = p \rightarrow \psi$ for some $\psi \in \text{NNIL}_0(\vec{p})$. Suppose vZw and $\mathfrak{N}, v \not\models p \rightarrow \psi$. Then there exists $v' \in W'$ such that $vR'v'$,

$$\mathfrak{N}, v' \models p \text{ and } \mathfrak{N}, v' \not\models \psi.$$

By (S2), we have that there exists $w' \in W$ such that wRw' and $v'Zw'$. Thus, by (S1), $\mathfrak{M}, w' \models p$. By the induction hypothesis, $\mathfrak{M}, w' \not\models \psi$. Since wRw' , we conclude that $\mathfrak{M}, w \not\models p \rightarrow \psi$. \square

Corollary 6.2.5. *If $\mathfrak{N} \preceq \mathfrak{M}$, then $Th_{NNIL(\vec{p})}(\mathfrak{M}) \subseteq Th_{NNIL(\vec{p})}(\mathfrak{N})$. In particular, NNIL formulas are preserved under taking submodels.*

The next theorem is proved in [25]. We will prove it at the end of this chapter in Theorem 6.5.14 as a corollary of the properties of the n -universal of $NNIL(\vec{p})$ formulas.

Theorem 6.2.6. *If φ satisfies that for any models \mathfrak{M} and \mathfrak{N} with $\mathfrak{N} \preceq \mathfrak{M}$,*

$$\mathfrak{M} \models \varphi \Rightarrow \mathfrak{N} \models \varphi,$$

then φ is provably equivalent to a formula $\varphi' \in NNIL(\vec{p})$.

Theorem 6.2.7. *The following are equivalent:*

- (1) *φ is provably equivalent to a NNIL-formula;*
- (2) *if $\mathfrak{N} \preceq \mathfrak{M}$, then $\mathfrak{M} \models \varphi$ implies $\mathfrak{N} \models \varphi$;*

6.3 NNIL-formulas and $[\wedge, \rightarrow]$ -formulas

In this section, we give an algorithm to translate every NNIL-formula to a $[\wedge, \rightarrow]$ -formula in such a way that they are equivalent on frames. This shows that every subframe logic defined by NNIL-formulas is equivalent to a subframe logic defined by $[\wedge, \rightarrow]$ -formulas.

Lemma 6.3.1. *For any rooted Kripke model $\mathfrak{M} = \langle W, R, V \rangle$, any formulas φ, ψ ,*

$$\mathfrak{M} \models \varphi \vee \psi \iff \mathfrak{M} \models \varphi \text{ or } \mathfrak{M} \models \psi$$

Proof. “ \Leftarrow ” holds trivially. For “ \Rightarrow ”, suppose $\mathfrak{M} \models \varphi \vee \psi$. Then, for the root $r \in W$, we have $r \models \varphi \vee \psi$, which means either $r \models \varphi$ or $r \models \psi$. It then follows from the persistency of V that either for any $w \in W$, $w \models \varphi$, or for any $w \in W$, $w \models \psi$. That is either $\mathfrak{M} \models \varphi$ or $\mathfrak{M} \models \psi$. \square

By substituting \perp by a new propositional variable, we can turn every NNIL-formula into a negation-free NNIL-formula in such a way that they are equivalent on frames.

Lemma 6.3.2. *For any $\varphi \in NNIL$, there exists a negation-free formula $\varphi_0 \in NNIL$ such that for any general frame \mathfrak{F} ,*

$$\mathfrak{F} \models \varphi \iff \mathfrak{F} \models \varphi_0. \tag{6.1}$$

Proof. Since $NNIL$ -formulas can be translated into $NNIL_0$ -formulas, it suffices to prove the lemma for $NNIL_0$ formulas. For any $\varphi \in NNIL_0$, let $\varphi_0 = \varphi[\perp/q]$, where q is a new variable. Note that φ_0 is still in $NNIL_0$.

First, by a similar argument to the proof of Lemma 3.4.3, we can prove that for any $\varphi \in NNIL$, any $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and any $w \in W$,

$$\langle \mathfrak{F}, V \rangle, w \not\models \varphi \iff \langle \mathfrak{F}, V' \rangle, w \not\models \varphi_0, \quad (6.2)$$

where

$$V'(p) = \begin{cases} V(p), & p \neq q; \\ \emptyset, & p = q. \end{cases}$$

(Note that $V'(q) = \emptyset \in \mathcal{P}$.)

Next, we show by induction on $\varphi \in NNIL$ that for any $\mathfrak{F} = \langle W, R, \mathcal{P} \rangle$ and any $w \in W$,

$$\langle \mathfrak{F}, V \rangle, w \not\models \varphi_0 \implies \langle \mathfrak{F}, V \rangle, w \not\models \varphi, \quad (6.3)$$

The cases that $\varphi = p$ or \perp or $\psi \wedge \chi$ or $\psi \vee \chi$ are trivial.

$\varphi = p \rightarrow \psi$. Suppose $\langle \mathfrak{F}, V \rangle, w \not\models p \rightarrow \psi_0$. Then there exists $v \in R(w)$ such that $\langle \mathfrak{F}, V \rangle, v \models p$ and $\langle \mathfrak{F}, V \rangle, v \not\models \psi_0$. By the induction hypothesis, we have that $\langle \mathfrak{F}, V \rangle, v \not\models \psi$. Thus $\langle \mathfrak{F}, V \rangle, w \not\models p \rightarrow \psi$.

Lastly, observe that the “ \iff ” of (6.1) follows from (6.2), and the “ \implies ” of (6.1) follows from (6.3). This finishes the proof. \square

Theorem 6.3.3. *For any $\varphi \in NNIL$, there exists a formula $\varphi' \in [\wedge, \rightarrow]$ such that for any general frame \mathfrak{F} ,*

$$\mathfrak{F} \models \varphi \iff \mathfrak{F} \models \varphi'. \quad (6.4)$$

Proof. First, by Lemma 6.3.2, for any $NNIL_0$ -formula φ , there exists a negation-free formula φ_0 which is also in $NNIL_0$ such that (6.1) holds.

Next, for any $\varphi_0 \in NNIL_0$, we define a formula $\varphi' \in [\wedge, \rightarrow]$ as follows:

- If $\varphi_0 = p$, let $\varphi' = \varphi_0$.
- If $\varphi_0 = \psi \vee \chi$, let $\varphi' = (\psi' \rightarrow q) \wedge (\chi' \rightarrow q) \rightarrow q$, where q does not occur in ψ' or χ' .
- If $\varphi_0 = \psi \wedge \chi$, let $\varphi' = \psi' \wedge \chi'$, where we do not introduce the same new variable for ψ' and χ' .

- If $\varphi_0 = p \rightarrow \psi$, let $\varphi' = p \rightarrow \psi'$, where we do not introduce p as a new variable for ψ' .

We show “ \implies ” of (6.4) by showing by induction a stronger result that for any model $\mathfrak{M} = \langle W, R, V \rangle$,

$$\mathfrak{M} \models \varphi_0 \implies \mathfrak{M} \models \varphi'. \quad (6.5)$$

The cases that $\varphi_0 = p$ or $\psi \wedge \chi$ are trivial.

If $\varphi_0 = p \rightarrow \psi$, suppose $\mathfrak{M} \models p \rightarrow \psi$. Then for each $w \in W$ such that $\mathfrak{M}, w \models p$, we have that $\mathfrak{M}, w \models \psi$. It follows that $\mathfrak{M}_w \models \psi$, which by the induction hypothesis implies that $\mathfrak{M}_w \models \psi'$. Thus $\mathfrak{M}, w \models \psi'$ and $\mathfrak{M} \models p \rightarrow \psi'$.

If $\varphi_0 = \psi \vee \chi$, suppose $\mathfrak{M} \models \psi \vee \chi$. Note that $\mathfrak{M} = \mathfrak{M}_X$ for some set of points $X \subseteq W$. Then we have $\mathfrak{M}_x \models \psi \vee \chi$ for each $x \in X$. By Lemma 6.3.1, we have $\mathfrak{M}_x \models \psi$ or $\mathfrak{M}_x \models \chi$. Without loss of generality, we may assume that $\mathfrak{M}_x \models \psi$. Then, by the induction hypothesis, $\mathfrak{M}_x \models \psi'$. For any $w \in R(x)$ with

$$\mathfrak{M}_x, w \models (\psi' \rightarrow q) \wedge (\chi' \rightarrow q),$$

we have that $\mathfrak{M}_x, w \models q$, which implies $\mathfrak{M}_x, w \models (\psi' \rightarrow q) \wedge (\chi' \rightarrow q) \rightarrow q$. Thus, $\mathfrak{M}, w \models (\psi' \rightarrow q) \wedge (\chi' \rightarrow q) \rightarrow q$.

To show the direction “ \impliedby ”, we first define a formula φ'' for every formula $\varphi_0 \in NNIL_0$ as follows:

- If $\varphi_0 = p$, let $\varphi'' = \varphi_0$.
- If $\varphi_0 = \psi \vee \chi$, let $\varphi'' = (\psi'' \rightarrow \psi'' \vee \chi'') \wedge (\chi'' \rightarrow \psi'' \vee \chi'') \rightarrow \psi'' \vee \chi''$.
- If $\varphi_0 = \psi \wedge \chi$, let $\varphi'' = \psi'' \wedge \chi''$.
- If $\varphi_0 = p \rightarrow \psi$, let $\varphi'' = p \rightarrow \psi''$.

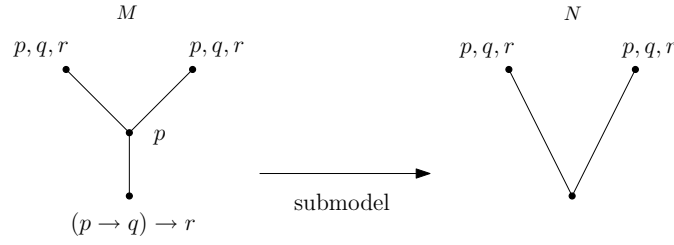
Next, we show by induction on φ_0 that for any model \mathfrak{M} ,

$$\mathfrak{M} \models \varphi'' \implies \mathfrak{M} \models \varphi_0. \quad (6.6)$$

By a similar argument to the one used in proving (6.5), we can prove (6.6) for the cases that $\varphi_0 = p$ or $\psi \wedge \chi$ or $p \rightarrow \psi$.

If $\varphi_0 = \psi \vee \chi$, suppose $\mathfrak{M} \models (\psi'' \rightarrow \psi'' \vee \chi'') \wedge (\chi'' \rightarrow \psi'' \vee \chi'') \rightarrow \psi'' \vee \chi''$. Since $\vdash_{\mathbf{IPC}} (\psi'' \rightarrow \psi'' \vee \chi'') \wedge (\chi'' \rightarrow \psi'' \vee \chi'')$, we must have that $\mathfrak{M} \models \psi'' \vee \chi''$.

Figure 6.1: Example 6.3.5



Propositional variables that are shown at a point are true at the point. Those that are not shown are false at the point.

Note that $\mathfrak{M} = \mathfrak{M}_X$ for some set of points $X \subseteq W$. Thus, for each $x \in X$, $\mathfrak{M}_x \models \psi'' \vee \chi''$, which by lemma 6.3.1 implies $\mathfrak{M}_x \models \psi''$ or $\mathfrak{M}_x \models \chi''$. Then by the induction hypothesis, $\mathfrak{M}_x \models \psi$ or $\mathfrak{M}_x \models \chi$, so by lemma 6.3.1 again, $\mathfrak{M}_x \models \psi \vee \chi$. Thus $\mathfrak{M} \models \psi \vee \chi$.

Now, suppose $\mathfrak{F} \models \varphi'$ for any general frame \mathfrak{F} and $\varphi_0 \in NNIL_0$. Note that φ'' is a formula obtained from φ' by replacing all occurrences of some propositional variables with some formulas. Thus, we have $\mathfrak{F} \models \varphi''$. Then, by (6.6), $\mathfrak{F} \models \varphi_0$, which by (6.1) means that $\mathfrak{F} \models \varphi$. \square

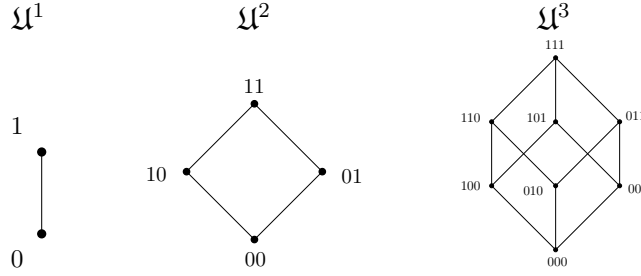
Corollary 6.3.4. *NNIL-formulas are preserved under subreductions.*

Proof. Let f be a subreduction from \mathfrak{F} onto \mathfrak{G} and $\varphi \in NNIL$. Suppose $\mathfrak{F} \models \varphi$. Let $\varphi' \in [\wedge, \rightarrow]$ be the formula in Theorem 6.3.3, then $\mathfrak{F} \models \varphi'$, which by Theorem 4.3.1 implies that $\mathfrak{G} \models \varphi'$. By Theorem 6.3.3 again, we obtain $\mathfrak{G} \models \varphi$. \square

We finish this section by giving an example showing that subsimulations do not preserve $[\wedge, \rightarrow]$ -formulas. Thus, it follows from Theorem 6.2.7 that $[\wedge, \rightarrow]$ -formulas are not provably equivalent to NNIL-formulas.

Example 6.3.5. *The formula $(p \rightarrow q) \rightarrow r$ is in $[\wedge, \rightarrow]$, but not in NNIL. Consider the two models shown in Figure 6.1. The model N is a submodel of the model M . We have $M \models (p \rightarrow q) \rightarrow r$, however $N \not\models (p \rightarrow q) \rightarrow r$.*

Figure 6.2: \mathfrak{U}^n



6.4 Equivalence of models under subsimulations

In this section, we discuss equivalence relations on models under subsimulations.

For every $n \in \omega$, there exists a model \mathfrak{U}^n subsimulating every n -model. \mathfrak{U}^1 , \mathfrak{U}^2 and \mathfrak{U}^3 are shown in Figure 6.2. More generally, \mathfrak{U}^n is defined as follows:

Definition 6.4.1. Define $\mathfrak{U}^n = \langle W, R, V \rangle$ by taking

$$W = \mathbf{C}^n, R = \leq \text{ and } col_V(w) = w.$$

Now, we show that \mathfrak{U}^n subsimulates every n -model.

Theorem 6.4.2. Let $\mathfrak{M} = \langle W', R', V' \rangle$ be a n -model, and $\mathfrak{U}^n = \langle W, R, V \rangle$. There exists a total subsimulation $Z : \mathfrak{M} \preceq \mathfrak{U}^n$. In particular, $\mathcal{U}(n) \preceq \mathfrak{U}^n$.

Proof. Define a relation Z on $W' \times W$ by

$$Z = \bigcup_{s \in \mathbf{C}^n} Z_s,$$

where for every $s \in \mathbf{C}^n$,

$$Z_s = \{(w', w) : w' \in W', w \in W, col_{V'}(w') = col_V(w) = s\}.$$

Clearly, Z is total. We show that Z is a subsimulation. Clearly, (S1) is satisfied. For (S2), for any $w', v' \in W'$ and $w \in W$ such that $w'Zw$ and $w'Rv'$, observe that

$$col_{V'}(v') \geq col_{V'}(w').$$

By the definition of \mathfrak{U}^n , for the color $s = col_{V'}(v') \geq col_{V'}(w') = col_V(w)$ there exists a point $v \in W$ such that wRv and $col_V(v) = s$. Thus, by the definition of Z , $v'Zv$. \square

Theorem 6.4.3. *Let $\mathfrak{M}_0 = \langle W_0, R_0, V_0 \rangle$, $\mathfrak{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathfrak{M}_2 = \langle W_2, R_2, V_2 \rangle$ be n -models. Suppose $Z_0 : \mathfrak{M}_0 \preceq \mathfrak{M}_1$ and $Z_1 : \mathfrak{M}_1 \preceq \mathfrak{M}_2$. Then $Z_1 \circ Z_0 : \mathfrak{M}_0 \preceq \mathfrak{M}_2$.*

Proof. Clearly, $Z_1 \circ Z_0$ is total. (S1) is trivially satisfied. For (S2), suppose $w_0Z_1 \circ Z_0w_2$ and $w_0R_0v_0$. Then there exists $w_1 \in W_1$ such that $w_0Z_0w_1$ and $w_1Z_1w_2$. By (S2) of Z_0 , there exists $v_1 \in W_1$ such that $v_0Z_0v_1$ and $w_1R_1v_1$. Next, by (S2) of Z_1 , there exists $v_2 \in W_2$ such that $v_1Z_1v_2$ and $w_2R_1v_2$. Thus $v_0Z_1 \circ Z_0v_2$. \square

Corollary 6.4.4. *The relation \preceq is transitive.*

As in [25], we define a relation \equiv between two models by

$$\mathfrak{M} \equiv \mathfrak{N} \text{ iff } \mathfrak{M} \preceq \mathfrak{N} \text{ and } \mathfrak{N} \preceq \mathfrak{M}.$$

Clearly, \equiv is reflexive and symmetric. By Corollary 6.4.4, \equiv is also transitive. So \equiv is an equivalence relation.

By the definition of \mathfrak{U}^n , it is easy to see that \mathfrak{U}^n is a generated submodel of $\mathcal{U}(n)$, thus $\mathfrak{U}^n \preceq \mathcal{U}(n)$. So together with Theorem 6.4.2, we obtain that $\mathfrak{U}^n \equiv \mathcal{U}(n)$.

Theorem 6.4.5. *If $\mathfrak{M} \equiv \mathfrak{N}$, then \mathfrak{M} and \mathfrak{N} satisfy the same NNIL(\vec{p})-formulas.*

Proof. By Corollary 6.2.5. \square

Let $\mathbf{Sub}(\mathcal{U}(n))$ be the set of all finite rooted generated submodels of $\mathcal{U}(n)$. Consider the set

$$\mathbf{Sub}(\mathcal{U}(n))_{\equiv} = \{[\mathfrak{M}] : \mathfrak{M} \in \mathbf{Sub}(\mathcal{U}(n))\}.$$

For each equivalence class $[\mathfrak{M}] \in \mathbf{Sub}(\mathcal{U}(n))_{\equiv}$, we want to choose a “small” enough model as its representative. The model \mathfrak{U}^n is an example of such a “small” model. We next give an algorithm to construct the “small” representative \mathfrak{M}_0 of each $[\mathfrak{M}] \in \mathbf{Sub}(\mathcal{U}(n))_{\equiv}$. First, we need to introduce some notations.

For any two models $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{N} = \langle W', R', V' \rangle$. Put

$$\mathfrak{M}' = \langle W, R, V \upharpoonright P \rangle,$$

$$\mathfrak{N}' = \langle W', R', V' \upharpoonright P \rangle,$$

where P is a set of propositional variables and $|P| = m$.

- If $\mathfrak{M}' \cong \mathfrak{N}'$, then we say that \mathfrak{M} and \mathfrak{N} are *m-isomorphic*, in symbols $\mathfrak{M} \cong^m \mathfrak{N}$.
- If $\mathfrak{M}' \equiv \mathfrak{N}'$, then we denote $\mathfrak{M} \equiv^m \mathfrak{N}$.

Note that when we use the symbols \cong^m and \equiv^m , the elements of the set P will always be clear from the context.

For each equivalence class $[\mathfrak{N}]$ in $\mathbf{Sub}(\mathcal{U}(n))_{\equiv}$, we are going to construct the representative model \mathfrak{M}_0 inductively in four steps. The four steps involve four kinds of reductions and the technique of unraveling. Take a model \mathfrak{M} in $[\mathfrak{N}]$: by unraveling and applying each kind of reduction, we reduce \mathfrak{M} to a smaller model in such a way that the resulting model is still in $[\mathfrak{N}]$.

We now start with the first step. The first step involves the first kind of reduction. By transforming any finite rooted n -model \mathfrak{M} to $\overline{\mathfrak{M}}^s$, we identify all points of \mathfrak{M} with the same color s as the root.

To be precise, for every finite rooted n -model $\mathfrak{M} = \langle W, R, V \rangle$ with the root r and $col_V(r) = s$, we define a model $\overline{\mathfrak{M}}^s = \langle W', R', V' \rangle$ (see Figure 6.3) by taking

$$\begin{aligned} W' &= W \setminus \{w \in W : col_V(w) = s\} \cup \{r'\}, \\ R' &= R \upharpoonright W' \cup \{(r', x) : (w, x) \in R \text{ and } col_V(w) = s\}, \\ col_{V'} &= col_V \upharpoonright W' \cup \{(r', s)\}. \end{aligned}$$

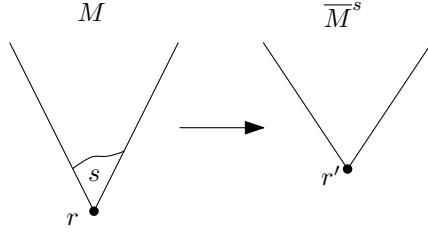
Lemma 6.4.6. *Let \mathfrak{M} and $\overline{\mathfrak{M}}^s$ be models described above. Then $\mathfrak{M} \equiv \overline{\mathfrak{M}}^s$.*

Proof. Note that from the definition of $\overline{\mathfrak{M}}^s$, we can easily see that there exists an isomorphism $f : W \setminus \{w \in W : col_V(w) = s\} \rightarrow W' \setminus \{r'\}$. To show that $\mathfrak{M} \preceq \overline{\mathfrak{M}}^s$, define a relation Z on $W \times W'$ by

$$Z = \{(w, r') : col_V(w) = s\} \cup \{(v, f(v)) : col_V(v) \neq s\}.$$

Clearly, (S1) holds. For (S2), suppose wZw' and wRv for $w, v \in W$ and $w' \in W'$. There are two cases. Case 1: $col_V(w) = s$. Then by the definition

Figure 6.3: $\overline{\mathfrak{M}}^s$



of Z , $w' = r'$. If $col_V(v) = s$, then we have that for $r' \in W'$, vZr' and $r'R'r'$. If $col_V(v) \neq s$, then we have that for $f(v) \in W'$, $vZf(v)$ and $r'R'f(v)$.

Case 2: $col_V(w) \neq s$. Then by the definition of Z , $w' = f(w)$. So we have that for $f(v) \in W'$, $vZf(v)$ and $f(w)R'f(v)$.

To show that $\overline{\mathfrak{M}}^s \preceq \mathfrak{M}$, define a relation Z on $W' \times W$ by

$$Z = \{(r', r)\} \cup \{(f(w), w) : col_V(w) \neq s\}.$$

It is easy to see that Z is total and both (S1) and (S2) hold. \square

In the second step, we unravel the n -model \mathfrak{M} obtained from the first step to a tree-like n -model \mathfrak{N} . It is easy to prove that $\mathfrak{M} \equiv \mathfrak{N}$.

Let \mathfrak{M} be a finite rooted tree-like model with the root r and $r \prec S_r$ obtained from the second step. In the third step, the second kind of reduction replaces each generated submodel \mathfrak{M}_w ($w \in S_r$) of \mathfrak{M} by a $n - |prop(w)|$ -equivalent representative $n - |prop(w)|$ -model. The resulting model is denoted by $\overrightarrow{\mathfrak{M}}^w$.

To be precise, let $\mathfrak{M} = \langle W, R, V \rangle$ be a finite rooted tree-like n -model and $w \in W$ be a point with $|prop(w)| = k$. Then there exists an $n - k$ -model \mathfrak{N} such that $\mathfrak{M}_w \cong^{n-k} \mathfrak{N}$ via an isomorphism f . Let $\mathfrak{N}_0 = \langle W_0, R_0, V_0 \rangle$ be the representative $n - k$ model in $[\mathfrak{N}]$. It is not hard to see by induction that there exists a surjective functional subsimulation g from \mathfrak{N} onto \mathfrak{N}_0 . Note that since \mathfrak{M} is a tree-like model, for any $v \in S_r$ such that $v \neq w$, $R(w) \cap R(v) = \emptyset$. Now, define an n -model $\overrightarrow{\mathfrak{M}}^w = \langle W', R', V' \rangle$ by taking

$$W' = (W \setminus R(w)) \cup W_0,$$

$$R' = R \upharpoonright W' \cup R_0 \cup \{(r, x) : x \in W_0\},$$

$$col_{V'}(x) = \begin{cases} col_V(x), & x \notin W_0; \\ col_V(f^{-1}(y)), & \text{where } y \in g^{-1}(x) \text{ } x \in W_0. \end{cases}$$

In the above definition of $col_{V'}$, note that for all $x \in W_0$, for any $y, z \in g^{-1}(x)$, $col_V(f^{-1}(y)) = col_V(f^{-1}(z))$, thus this definition is sound.

Lemma 6.4.7. *Let \mathfrak{M} and $\vec{\mathfrak{M}}^w$ be the models defined above. Then $\mathfrak{M} \equiv \vec{\mathfrak{M}}^w$.*

Proof. Note that from the definition of $\vec{\mathfrak{M}}^w$, we can easily see that there exists an isomorphism $h : W \setminus R(w) \rightarrow W' \setminus W_0$. To show that $\mathfrak{M} \preceq \vec{\mathfrak{M}}^w$, define a relation Z on $W \times W'$ by

$$Z = \{(x, h(x)) : x \notin R(w)\} \cup \{(y, g(f(y))) : y \in R(w)\}.$$

For (S1), clearly for $x \notin R(w)$, $col_V(x) = col_{V'}(h(x))$ since h is an isomorphism. If $x \in R(w)$, by the definition of $col_{V'}$, we have that

$$col_{V'}(g(f(x))) = col_V(f^{-1}(f(x))) = col_V(x).$$

For (S2), suppose xZx' and xRy for $x, y \in W$ and $x' \in W'$. There are two cases. Case 1: $y \notin R(w)$. Then $x \notin R(w)$ and $x' = h(x)$. So there exists $h(y) \in W'$ such that $h(x)R'h(y)$ and $yZh(y)$.

Case 2: $y \in R(w)$. Then either $x = r$ and $x' = h(r)$, or $x \in R(w)$ and $x' = g(f(x))$. So in both cases, by the definition of R' and Z , for $g(f(y)) \in W'$, we have that $x'R'g(f(y))$ and $yZg(f(y))$.

To show that $\vec{\mathfrak{M}}^w \preceq \mathfrak{M}$, consider the relation Z^{-1} on $W' \times W$. It is easy to see that Z^{-1} is a total subsimulation. \square

In the fourth step, for the finite rooted tree-like model \mathfrak{M} obtained from the third step, the third kind of reduction deletes every generated submodel of some point $v \in S_r$ that can be subsimulated by another generated submodel of some point $w \in S_r$.

To be precise, let $\mathfrak{M} = \langle W, R, V \rangle$ be a finite rooted tree-like model with the root r . Suppose $w, v \in S_r$, and \mathfrak{M}_w and \mathfrak{M}_v are two generated submodels of \mathfrak{M} with a subsimulation Z_0 such that $Z_0 : \mathfrak{M}_v \preceq \mathfrak{M}_w$. Note that by the construction of the third step, \mathfrak{M}_w and \mathfrak{M}_v can both be viewed as $n - k$ -models for some $k \in \omega$. It can be proved by induction that Z_0 is total and functional. Note also that since \mathfrak{M} is a tree-like model, $R(v) \cap R(w) = \emptyset$. Define a model $\vec{\mathfrak{M}}^{w,v} = \langle W', R', V' \rangle$ (see Figure 6.4.8) by taking

$$W' = W \setminus R(v), \quad R' = R \upharpoonright W', \quad col_{V'} = col_V \upharpoonright W'.$$

Figure 6.4: $\widetilde{\mathfrak{M}}^{w,v}$

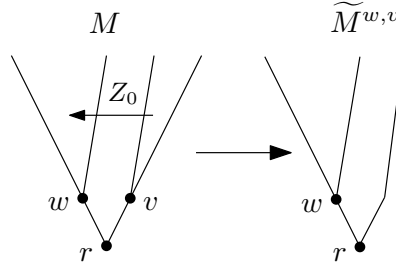
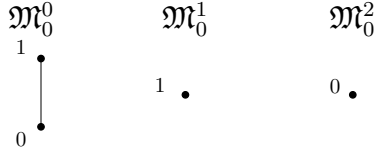


Figure 6.5: Representative models for 1-models



Lemma 6.4.8. *Let \mathfrak{M} and $\widetilde{\mathfrak{M}}^{w,v}$ be models defined above. Then $\mathfrak{M} \equiv \widetilde{\mathfrak{M}}^{w,v}$.*

Proof. Note that from the definition of $\widetilde{\mathfrak{M}}^{w,v}$, we can easily see that there exists an isomorphism $f : W \setminus R(v) \rightarrow W'$. To show that $\mathfrak{M} \preceq \widetilde{\mathfrak{M}}^{w,v}$, define a relation Z on $W \times W'$ by

$$Z = \{(x, f(x)) : x \notin R(v)\} \cup \{(y, f(Z_0(y))) : y \in R(v)\}.$$

Clearly, (S1) is satisfied. For (S2), suppose xZx' and xRy for $x, y \in W$ and $x' \in W'$. There are two cases. Case 1: $y \notin R(v)$. Then there exists $y' = f(y)$ in W' such that $x'R'y'$ and yZy' .

Case 2: $y \in R(v)$. Then either $x = r$ and $x' = f(r)$, or $x \in R(v)$ and $x' = f(Z_0(x))$. So by the definition of R' and Z , for $f(Z_0(y)) \in W'$, in both cases we have that $x'R'f(Z_0(y))$ and $yZf(Z_0(y))$.

To show that $\widetilde{\mathfrak{M}}^{w,v} \preceq \mathfrak{M}$, clearly, the isomorphism $f^{-1} : W' \rightarrow W \setminus R(v)$ is a total subsimulation from $\widetilde{\mathfrak{M}}^{w,v}$ to \mathfrak{M} . \square

In the last step, for the finite rooted tree-like model \mathfrak{M} obtained from the fourth step, the fourth kind of reduction identifies isomorphic subtrees.

To be precise, let $\mathfrak{M} = \langle W, R, V \rangle$ be a finite rooted tree-like model. Suppose $w, v \in W$, and \mathfrak{M}_w and \mathfrak{M}_v are two isomorphic subtrees. Define a model $\widehat{\mathfrak{M}}^{w,v} = \langle W', R', V' \rangle$ by taking

$$\begin{aligned} W' &= W \setminus R(v), \quad R' = R \upharpoonright W' \cup \{(x, w) : xRv\} \\ \text{col}_{V'} &= \text{col}_V \upharpoonright W'. \end{aligned}$$

Lemma 6.4.9. *Let \mathfrak{M} and $\widehat{\mathfrak{M}}^{w,v}$ be models defined above. Then $\mathfrak{M} \equiv \widehat{\mathfrak{M}}^{w,v}$.*

Proof. Note that from the definition of $\widehat{\mathfrak{M}}^{w,v}$, we can easily see that there exist isomorphisms $f : W \setminus R(v) \rightarrow W'$ and $g : R(v) \rightarrow R(w)$. Define a function $h : W \rightarrow W'$ by

$$h(x) = \begin{cases} f(x), & x \notin R(v); \\ f(g(x)), & x \in R(v). \end{cases}$$

It is easy to show that h is a surjective p-morphism from \mathfrak{M} onto $\widehat{\mathfrak{M}}^{w,v}$. Then by Remark 6.2.3, h^{-1} is a surjective subsimulation of $\widehat{\mathfrak{M}}^{w,v}$ in \mathfrak{M} , so $\widehat{\mathfrak{M}}^{w,v} \preceq \mathfrak{M}$. On the other hand, h clearly satisfies (S1). For (S2), suppose xRy . It is sufficient to show that $h(x)R'h(y)$.

Case 1: $y \notin R(v)$. Then $x \notin R(v)$ and by the definitions of h , $h(x) = f(x)$ and $h(y) = f(y)$. Since f is an isomorphism, we have that $f(x)R'f(y)$.

Case 2: $x, y \in R(v)$. Then by the definitions of h , $h(x) = f(g(x))$ and $h(y) = f(g(y))$. Since $f \circ g$ is an isomorphism, we have that $f(g(x))R'f(g(y))$.

Case 3: $x \notin R(v)$ and $y \in R(v)$. Then we must have that xRv and $h(x) = f(x)$, so by the definition of R' , $f(x)R'f(w)$, i.e. $f(x)R'f(g(v))$. By the definition of h , $h(y) = f(g(y))$. Since $f \circ g$ is an isomorphism, we have that $f(g(v))R'f(g(y))$, so by transitivity $f(x)R'f(g(y))$.

Hence, h is a surjective subsimulation of \mathfrak{M} in $\widehat{\mathfrak{M}}^{w,v}$, i.e. $\mathfrak{M} \preceq \widehat{\mathfrak{M}}^{w,v}$. \square

Now, we describe the construction of \mathfrak{M}_0 for each finite rooted generated n -submodel \mathfrak{M} of $\mathcal{U}(n)$ with the root r by induction on n precisely.

- For $n = 1$, if $\mathfrak{M} \models p$, then let \mathfrak{M}_0 be the model \mathfrak{M}_0^1 in Figure 6.5. If $\mathfrak{M} \not\models p$, then let \mathfrak{M}_0 be the model \mathfrak{M}_0^2 in Figure 6.5. For all other finite rooted 1-models \mathfrak{M} , let \mathfrak{M}_0 be the model \mathfrak{M}_0^0 in Figure 6.5.
- For $n > 1$, we construct \mathfrak{M}_0 by following the following steps:

Step 1 Let $s = \text{col}(r)$. Construct the model $\mathfrak{N}_1 = \overline{\mathfrak{M}}^s$.

Step 2 Unravel the model \mathfrak{N}_1 to a tree-like model \mathfrak{N}_2 .

Step 3 For every $w \in S_r$ such that $|prop(w)| = k$ and $(\mathfrak{N}_2)_w \not\cong^{n-k} \mathfrak{N}_0$ for all representative $k - m$ -model \mathfrak{N}_0 , construct the model $\mathfrak{N}_3 = \overrightarrow{\mathfrak{N}_2}^w$ as described in Lemma 6.4.7.

Step 4 For every $w, v \in S_r$ such that $(\mathfrak{N}_3)_v \preceq (\mathfrak{N}_3)_w$, construct the model $\mathfrak{N}_4 = \widetilde{\mathfrak{N}_3}^{w,v}$ as described in Lemma 6.4.8.

Step 5 For every $w, v \in \mathfrak{N}_4$ such that $(\mathfrak{N}_4)_v \cong (\mathfrak{N}_4)_w$, construct the model $\mathfrak{N}_5 = \widehat{\mathfrak{N}_4}^{w,v}$ as described in Lemma 6.4.9.

Denote the resulting model by \mathfrak{M}_0 . It is not hard to see by induction that \mathfrak{M}_0 is a finite rooted generated submodel of $\mathcal{U}(n)$. So indeed, $\mathfrak{M}_0 \in [\mathfrak{M}]$ for all $[\mathfrak{M}] \in \mathbf{Sub}(\mathcal{U}(n))$. We will use \mathfrak{M}_0 as representative of the equivalence class $[\mathfrak{M}]$.

Corollary 6.4.10. $\mathfrak{M} \equiv \mathfrak{M}_0$.

Proof. By Lemma 6.4.6, 6.4.8 and 6.4.7. □

Corollary 6.4.11. *Let \mathfrak{M}_0 be any representative n -model with the root r . For any point w in \mathfrak{M}_0 , the generated submodel $(\mathfrak{M}_0)_w$ is isomorphic to some representative n -model \mathfrak{N}_0 with $\mathfrak{N}_0 \preceq \mathfrak{M}_0$.*

Proof. For any point w in \mathfrak{M}_0 with $|prop(w)| = k$, by the above construction, $(\mathfrak{M}_0)_w$ is k -isomorphic to some representative k -model \mathfrak{N}'_0 . This means that the above 4 steps are not applicable on \mathfrak{N}'_0 , hence not applicable on $(\mathfrak{M}_0)_w$. So $(\mathfrak{M}_0)_w$ is isomorphic to a representative n -model \mathfrak{N}_0 . Clearly, we have that $\mathfrak{N}_0 \preceq \mathfrak{M}_0$. □

The union of two total subsimulations is again a subsimulation. Thus on a model \mathfrak{M} there exists a maximal subsimulation which is the union of all subsimulations on \mathfrak{M} . It is worthwhile to point out that \mathfrak{M}_0 is in fact the image of a maximal subsimulation on \mathfrak{M} . For more details on maximal subsimulations, one may refer to [25].

The above construction makes \mathfrak{M}_0 a “smallest” model in the equivalence class $[\mathfrak{M}]$, in the sense that it is irreducible with respect to the steps in the construction. The next lemma shows that such a “smallest” model is unique.

Lemma 6.4.12. *For any generated submodels \mathfrak{M} and \mathfrak{N} of $\mathcal{U}(n)$ with $\mathfrak{M} \equiv \mathfrak{N}$, we have that $\mathfrak{M}_0 \cong \mathfrak{N}_0$.*

Proof. We show the lemma by induction on n .

$n = 1$. All representative 1-models are displayed in Figure 6.5. By the construction, the lemma holds.

$n > 1$. Let \mathfrak{M} and \mathfrak{N} be two n -models with $\mathfrak{M} \equiv \mathfrak{N}$. Let r_0 be the root of \mathfrak{M}_0 and r_1 be the root of \mathfrak{N}_0 . Clearly

$$\text{col}(r_0) = \text{col}(r_1);$$

otherwise $\mathfrak{M}_0 \not\equiv \mathfrak{N}_0$.

Consider the sets S_{r_0} and S_{r_1} of immediate successors of r_0 and r_1 , respectively. For any $w \in S_{r_0}$, according to the construction of \mathfrak{M}_0 ,

$$(\mathfrak{M}_0)_w \cong^m \mathfrak{M}'_0,$$

where \mathfrak{M}'_0 is some representative m -model and $m = |\text{prop}(w)| < n$.

Assume for any $v \in S_{r_1}$ such that $\text{col}(v) = \text{col}(w)$, $(\mathfrak{N}_0)_v$ is not m -isomorphic to \mathfrak{M}'_0 . Since $m < n$, by the induction hypothesis we know that

$$(\mathfrak{M}_0)_w \not\equiv^m (\mathfrak{N}_0)_v.$$

Thus for any $v \in S_{r_1}$ such that $\text{col}(v) = \text{col}(w)$, we have $(\mathfrak{M}_0)_w \not\equiv (\mathfrak{N}_0)_v$. Without loss of generality, we may assume that $(\mathfrak{M}_0)_w \not\preceq (\mathfrak{N}_0)_v$.

Now suppose $Z : \mathfrak{M}_0 \preceq \mathfrak{N}_0$. Then there exists a total subsimulation $Z' \subseteq Z \upharpoonright (\mathfrak{M}_0)_w$ of $(\mathfrak{M}_0)_w$ in $(\mathfrak{N}_0)_v$ for some $v \in S_{r_1}$, which leads to a contradiction.

Thus we conclude that there exists some $v \in S_{r_1}$ with $\text{col}(v) = \text{col}(w)$ such that

$$(\mathfrak{N}_0)_v \cong^m \mathfrak{M}'_0 \cong^m (\mathfrak{M}_0)_w.$$

Then since $\text{prop}(v) = \text{prop}(w)$, we have that $(\mathfrak{N}_0)_v \cong (\mathfrak{M}_0)_w$. From the construction of \mathfrak{M}_0 , we know that distinct points in S_{r_0} generates non-isomorphic submodels. This also holds for \mathfrak{N}_0 . Since \mathfrak{M} and \mathfrak{N} are generated submodels of $\mathcal{U}(n)$, S_{r_0} and S_{r_1} are finite. Thus, we conclude that $|S_{r_0}| \leq |S_{r_1}|$.

By a similar argument, we can show that for any $v \in S_{r_1}$, there exists some $w \in S_{r_0}$ with $\text{col}(w) = \text{col}(v)$ such that $(\mathfrak{N}_0)_v \cong (\mathfrak{M}_0)_w$. And $|S_{r_1}| \leq |S_{r_0}|$. Thus, we have that $|S_{r_0}| = |S_{r_1}|$ and so $(\mathfrak{M}_0)_{S_{r_0}} \cong (\mathfrak{N}_0)_{S_{r_1}}$. Furthermore, since $\text{col}(r_0) = \text{col}(r_1)$, we finally obtain $\mathfrak{M}_0 \cong \mathfrak{N}_0$. \square

6.5 n -universal models of $NNIL(\vec{p})$ -formulas

In this section, we define a model $\mathcal{U}(n)^{NNIL}$ acting as an universal model for $NNIL(\vec{p})$ -formulas. Each $NNIL$ -formula defines a unique upset in $\mathcal{U}(n)^{NNIL}$. We have proved in Section 6.1 that the set $NNIL(\vec{p})_{\equiv}$ is finite, from which we know that $\mathcal{U}(n)^{NNIL}$ is finite.

Definition 6.5.1. Define a model $\mathcal{U}(n)^{NNIL} = \langle W, R, V \rangle$ by taking

- $W = \mathbf{Sub}(\mathcal{U}(n))_{\equiv}$;
- $[\mathfrak{M}]R[\mathfrak{N}]$ iff $\mathfrak{N}' \preceq \mathfrak{M}'$ for some $\mathfrak{M}' \in [\mathfrak{M}]$ and $\mathfrak{N}' \in [\mathfrak{N}]$;
- $col([\mathfrak{M}]) = col(t_0)$, where t_0 is the root of the representative model \mathfrak{M}_0 of $[\mathfrak{M}]$. $V = V_{col}$.

Remark 6.5.2. *The R in Definition 6.5.1 is really a partial order.*

Proof. Clearly, R is reflexive. For transitivity, suppose $[\mathfrak{M}_1]R[\mathfrak{M}_2]$ and $[\mathfrak{M}_2]R[\mathfrak{M}_3]$. Then there exists $\mathfrak{M}'_1 \in [\mathfrak{M}_1]$, $\mathfrak{M}'_2, \mathfrak{M}''_2 \in [\mathfrak{M}_2]$ and $\mathfrak{M}'_3 \in [\mathfrak{M}_3]$ such that

$$\mathfrak{M}'_2 \preceq \mathfrak{M}'_1 \text{ and } \mathfrak{M}'_3 \preceq \mathfrak{M}''_2.$$

Since

$$\mathfrak{M}''_2 \preceq \mathfrak{M}'_2,$$

by the transitivity of \preceq , $\mathfrak{M}'_3 \preceq \mathfrak{M}'_1$, which means that $[\mathfrak{M}_1]R[\mathfrak{M}_3]$.

For anti-symmetry, suppose $[\mathfrak{M}_1]R[\mathfrak{M}_2]$ and $[\mathfrak{M}_2]R[\mathfrak{M}_1]$. Then there exists $\mathfrak{M}'_1, \mathfrak{M}''_1 \in [\mathfrak{M}_1]$ and $\mathfrak{M}'_2, \mathfrak{M}''_2 \in [\mathfrak{M}_2]$ such that

$$\mathfrak{M}'_2 \preceq \mathfrak{M}'_1 \text{ and } \mathfrak{M}''_1 \preceq \mathfrak{M}''_2.$$

Note that we also have that

$$\mathfrak{M}'_1 \preceq \mathfrak{M}''_1 \text{ and } \mathfrak{M}''_2 \preceq \mathfrak{M}'_2.$$

Thus by the transitivity of \preceq , $\mathfrak{M}'_1 \preceq \mathfrak{M}'_2$ and so $\mathfrak{M}'_1 \equiv \mathfrak{M}'_2$, which means that $[\mathfrak{M}_1] = [\mathfrak{M}_2]$. \square

Remark 6.5.3. $\mathcal{U}(n)^{NNIL}$ is rooted.

Proof. It follows from Theorem 6.4.2 that the model \mathcal{U}^n subsimulates all finite rooted generated submodels of $\mathcal{U}(n)$. Thus, $[\mathcal{U}^n]R[\mathfrak{M}]$ for all $[\mathfrak{M}] \in \mathbf{Sub}(\mathcal{U}(n))_{\equiv}$. So $[\mathcal{U}^n]$ is the root of $\mathcal{U}(n)^{NNIL}$. \square

Figure 6.6: $\mathcal{U}(1)^{NNIL}$

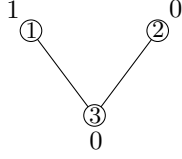
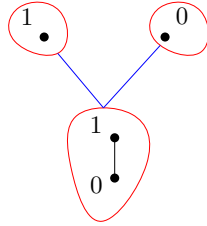


Figure 6.7: $\mathcal{U}(1)^{NNIL}$



We will prove soon that each rooted upset in $\mathcal{U}^\sharp(n)$ is defined by a *DP-NNIL*(\vec{p})-formula.

Definition 6.5.4. A formula φ is called a *formula with the disjunction property* (a *DP-formula* for short) iff for all formulas $\psi, \chi \in \text{FORM}$, the following holds:

$$\vdash \varphi \rightarrow \psi \vee \chi \Rightarrow \vdash \varphi \rightarrow \psi \text{ or } \vdash \varphi \rightarrow \chi.$$

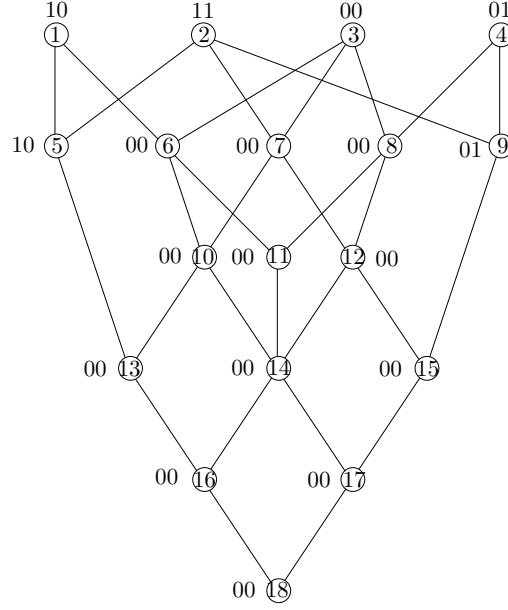
For more details on *DP*-formulas, one may refer to [4].

The model $\mathcal{U}(1)^{NNIL}$ of *NNIL*(p)-formulas is shown in Figure 6.6 (see also Figure 6.7). The representative models for all equivalence classes of models of the points in $\mathcal{U}(1)^{NNIL}$ have been added in Figure 6.7. The *DP-NNIL*(p)-formulas are (the number before a formula stands for the number of the root in Figure 6.6 of the upset defined by the formula):

$$1.p \quad 2.\neg p \quad 3.p \rightarrow p$$

The model $\mathcal{U}(2)^{NNIL}$ of *NNIL*(p, q)-formulas is shown in Figure 6.8 (see also Figure 6.9). The representative models for all equivalence classes of

Figure 6.8: $\mathcal{U}(2)^{NNIL}$



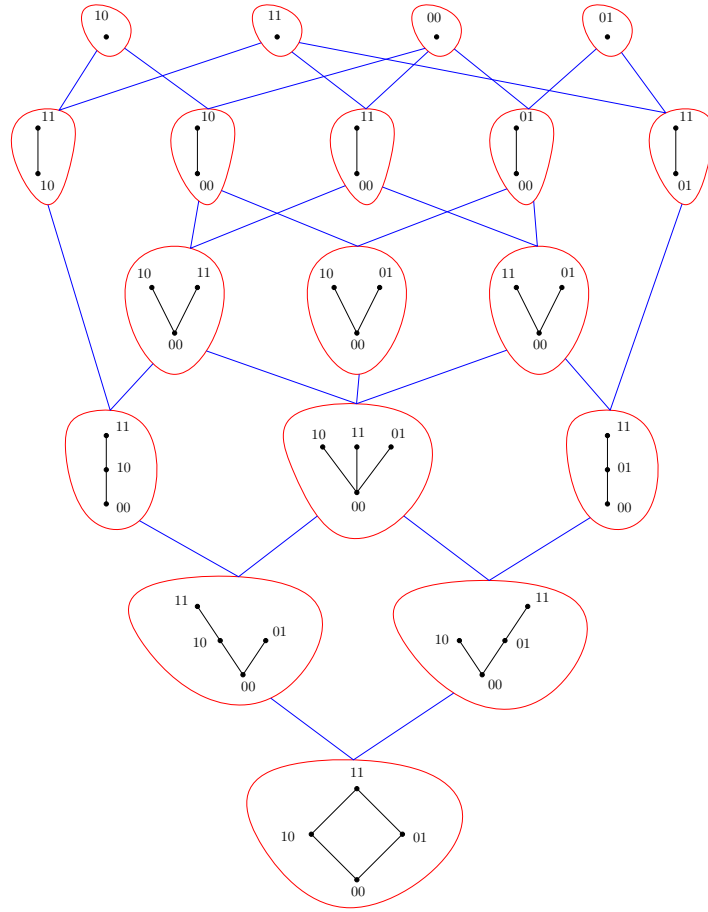
models of the points in $\mathcal{U}(2)^{NNIL}$ have been added in Figure 6.9. The DP - $NNIL(p, q)$ -formulas are (the number before a formula stands for the number of the root in Figure 6.8 of the upset defined by the formula):

- | | |
|---|--|
| 1. $p \wedge \neg q$ | 2. $p \wedge q$ |
| 3. $\neg p \wedge \neg q$ | 4. $\neg p \wedge q$ |
| 5. p | 6. $\neg q$ |
| 7. $(p \rightarrow q) \wedge (q \rightarrow p)$ | 8. $\neg p$ |
| 9. q | 10. $(q \rightarrow p) \wedge (p \rightarrow q \vee \neg q)$ |
| 11. $\neg(p \wedge q)$ | 12. $(p \rightarrow q) \wedge (q \rightarrow p \vee \neg p)$ |
| 13. $q \rightarrow p$ | 14. $(p \rightarrow q \vee \neg q) \wedge (q \rightarrow p \vee \neg p)$ |
| 15. $p \rightarrow q$ | 16. $q \rightarrow p \vee \neg p$ |
| 17. $p \rightarrow q \vee \neg q$ | 18. $p \rightarrow p$ |

We now prove that $\mathcal{U}(n)^{NNIL}$ is finite.

Theorem 6.5.5. *For every $n \in \omega$, $\mathbf{Sub}(\mathcal{U}(n))_{\equiv}$ is finite.*

Figure 6.9: $\mathcal{U}(2)^{NNIL}$



Proof. It suffices to show that for each $n \in \omega$, there are only finitely many rooted representative n -models \mathfrak{M}_0 for equivalence classes in $\mathbf{Sub}(\mathcal{U}(n))_{\equiv}$. We show this by induction on n .

$n = 1$. It is easy to see that there are only 3 rooted representative 1-models (see Figure 6.7).

Suppose the theorem hold for all $\mathbf{Sub}(\mathcal{U}(n))_{\equiv}$ with $n < k$. Now consider $\mathbf{Sub}(\mathcal{U}(k))_{\equiv}$. For any $[\mathfrak{M}]$ in $\mathbf{Sub}(\mathcal{U}(k))_{\equiv}$, by the construction we know that the representative k -model \mathfrak{M}_0 is rooted with the root r and $col(r) = s$. We know that there are only 2^k many k -colors. Thus, all \mathfrak{M}_0 have only 2^k many different colors for the roots.

By the construction, we also know that for every $w \in S_r$ with $|prop(w)| = m$, $(\mathfrak{M}_0)_w$ is $k - m$ -isomorphic to a representative $k - m$ -model \mathfrak{N}_0^w . Put $t = col(w)$. By the induction hypothesis, there are only finitely many representative $k - m$ -models, say l_t many.

Note that the set C_s of n -colors that are greater than s is finite. Now, fix a color s of the root of a representative k -model, the number of different \mathfrak{M}_0 's is equal to

$$\alpha_s = \sum_{i=1}^m C_m^i + 1,$$

where

$$m = \sum_{t \in C_s} l_t.$$

Hence, the number of all representative k -model is

$$\sum_{s \in C^k} \alpha_s,$$

which is finite. □

Corollary 6.5.6. *For every $n \in \omega$, $\mathcal{U}(n)^{NNIL}$ is finite.*

For every point $[\mathfrak{M}_0]$ in $\mathcal{U}(n)^{NNIL}$, the representative \mathfrak{M}_0 of $[\mathfrak{M}_0]$ is generally not isomorphic to the submodel of $\mathcal{U}(n)^{NNIL}$ generated by the point $[\mathfrak{M}_0]$. However, \mathfrak{M}_0 is \equiv -equivalent to $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]}$.

Lemma 6.5.7. *For every $[\mathfrak{M}_0] \in \mathcal{U}(n)^{NNIL}$, let $\mathfrak{M}_0 = \langle W', R', V' \rangle$ be the representative model with the root r in $[\mathfrak{M}_0]$. Let $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]} = \langle W, R, V \rangle$. Then $\mathfrak{M}_0 \equiv (\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]}$.*

Proof. We show by induction on $d([\mathfrak{M}_0])$ that

$$\exists Z \text{ s.t. } Z : (\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]} \preceq \mathfrak{M}_0 \text{ and } Z \text{ is root-preserving;} \quad (6.7)$$

$$\exists Z \text{ s.t. } Z : \mathfrak{M}_0 \preceq (\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]}. \quad (6.8)$$

$d([\mathfrak{M}_0]) = 1$. Then both $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{M}_0]}$ and \mathfrak{M}_0 are singletons and clearly (6.7) and (6.8) hold.

$d([\mathfrak{M}_0]) > 1$. We first show (6.7). For any proper successor $[\mathfrak{N}_0^i]$ ($1 \leq i \leq k$) of $[\mathfrak{M}_0]$, by the construction of $\mathcal{U}(n)^{NNIL}$, we know that there exists Z_0^i such that

$$Z_0^i : \mathfrak{N}_0 \preceq \mathfrak{M}_0. \quad (6.9)$$

Since $d([\mathfrak{N}_0^i]) < d([\mathfrak{M}_0])$, by the induction hypothesis of (6.7) we have that there exists a surjective Z_1^i such that

$$Z_1^i : (\mathcal{U}(n)^{NNIL})_{[\mathfrak{N}_0^i]} \preceq \mathfrak{N}_0^i. \quad (6.10)$$

By Lemma 6.4.3,

$$Z_0^i \circ Z_1^i : (\mathcal{U}(n)^{NNIL})_{[\mathfrak{N}_0^i]} \preceq \mathfrak{M}_0.$$

Define a relation Z on $W \times W'$ by

$$Z = \{([\mathfrak{M}_0], r)\} \cup \bigcup_{1 \leq i \leq k} Z_0^i \circ Z_1^i.$$

Clearly, Z is total and root-preserving. We show that Z is a subsimulation. For (S1), by the construction of $\mathcal{U}(n)^{NNIL}$, $col([\mathfrak{M}_0]) = col(r)$. For all $1 \leq i \leq k$ and all $(w, w') \in Z_0^i \circ Z_1^i$, since $Z_0^i \circ Z_1^i$ is a subsimulation, $col(w) = col(w')$.

For (S2), suppose wZw' and wRv . There are two cases.

Case 1: $w = [\mathfrak{M}_0]$. Then $w' = r$. If $v = w$, then clearly $w'R'w'$ and vZw' . Now suppose $v \neq w$. Then $v = [\mathfrak{N}_0]$ for some $[\mathfrak{N}_0] \in R([\mathfrak{M}_0])$. By the definition of R , we have that there exists a total subsimulation Z_0 satisfying (6.9), and so

$$r'Z_0v' \text{ for some } v' \in R'(r),$$

where r' is the root of \mathfrak{N}_0 . Since $d([\mathfrak{N}_0]) < d([\mathfrak{M}_0])$, by the induction hypothesis of (6.7), there exists a root-preserving Z_1 satisfying (6.10). Clearly,

$$[\mathfrak{N}_0]Z_1r'.$$

Thus, $[\mathfrak{N}_0]Z_0 \circ Z_1v'$, which means that $[\mathfrak{N}_0]Zv'$ and so (S2) is obtained.

Case 2: $w = [\mathfrak{N}_0^i]$ for some $[\mathfrak{N}_0^i] \in R([\mathfrak{M}_0])$ with $[\mathfrak{N}_0^i] \neq [\mathfrak{M}_0]$. Then we have that $(w, w') \in Z_0^i \circ Z_1^i$, so since $Z_0^i \circ Z_1^i$ is a subsimulation, there exists $v' \in R'(w')$ such that $vZ_0^i \circ Z_1^iv'$, i.e. vZv' .

Next, we show (6.8). For any point w' in \mathfrak{M}_0 with $w \neq r$, by Lemma 6.4.11, $(\mathfrak{M}_0)_{w'}$ is isomorphic to a proper successor of $[\mathfrak{M}_0]$. Since $d([\mathfrak{M}_0]_{w'}) < d([\mathfrak{M}_0])$, by the induction hypothesis of (6.8), there exists a total subsimulation $Z^{w'}$ satisfying

$$Z^w : (\mathfrak{M}_0)_{w'} \preceq (\mathcal{U}(n)^{NNIL})_{[(\mathfrak{M}_0)_{w'}]}. \quad (6.11)$$

Define a relation Z' on $W' \times W$ by

$$Z' = \{(r, [\mathfrak{M}_0])\} \cup \bigcup_{w' \in R'(r)} Z^{w'}.$$

Clearly, Z' is total. We show that Z' is a subsimulation. For (S1), by the construction of $\mathcal{U}(n)^{NNIL}$, $col(r) = col([\mathfrak{M}_0])$. For all $w' \in R'(r)$ and all $(w', w) \in Z^{w'}$, since $Z^{w'}$ is a subsimulation, $col(w') = col(w)$.

For (S2), suppose $w'Z'w$ and $w'R'v'$. We distinguish two cases.

Case 1: $w' = r$. Then $w = [\mathfrak{M}_0]$. If $v' = w'$, then clearly, wRw and $v'Zw$. Now suppose $v' \neq w'$. Then since $d([\mathfrak{M}_0]_{v'}) < d([\mathfrak{M}_0])$, by the induction hypothesis of (6.8), there exists a total subsimulation $Z^{v'}$ satisfying (6.11). Thus, there exists $v \in (\mathcal{U}(n)^{NNIL})_{[(\mathfrak{M}_0)_{v'}]}$ such that $v'Z^{v'}v$. Clearly, wRv .

Case 2: $w' \neq r$. Then $(w', w) \in Z^{w'}$ and $w \in (\mathcal{U}(n)^{NNIL})_{[(\mathfrak{M}_0)_{w'}]} = \langle W'', R \upharpoonright W'', V'' \rangle$. Since $Z^{w'} : (\mathfrak{M}_0)_{w'} \preceq (\mathcal{U}(n)^{NNIL})_{[(\mathfrak{M}_0)_{w'}]}$, there exists $v \in R \upharpoonright W''(w) \subseteq R(w)$ such that $v'Zv$. \square

The next theorem shows that $\mathcal{U}(n)^{NNIL}$ is a universal model for $NNIL(\vec{p})$ -formulas.

Theorem 6.5.8. *For every formula $\varphi \in NNIL(\vec{p})$, we have that $\mathcal{U}(n)^{NNIL} \models \varphi$ iff $\vdash_{\mathbf{IPC}} \varphi$.*

Proof. “ \Leftarrow ” holds trivially. Suppose $\mathcal{U}(n)^{NNIL} \models \varphi$. Then $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{U}^n]} \models \varphi$. By Lemma 6.5.7, we know that $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{U}^n]} \equiv \mathfrak{U}^n$. Note that $\mathfrak{U}^n \equiv \mathcal{U}(n)$. So $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{U}^n]} \equiv \mathcal{U}(n)$. By Lemma 6.4.5, we obtain that $\mathcal{U}(n) \models \varphi$. Then, by Theorem 3.2.4, $\vdash_{\mathbf{IPC}} \varphi$. \square

Corollary 6.5.9. *There are only finitely many provably non-equivalent formulas in $NNIL(\vec{p})$.*

Proof. By a same argument as the proof of Theorem 5.1.6. \square

Next, we show that $NNIL(\vec{p})$ -formulas define upsets in $\mathcal{U}(n)^{NNIL}$. We are going to define formulas $\varphi_{[\mathfrak{M}_0]}^\sharp$ for all points $[\mathfrak{M}_0]$ in $\mathcal{U}(n)^{NNIL}$. $\varphi_{[\mathfrak{M}_0]}^\sharp$ will share the same property as φ_w and φ_w^* in Chapter 3 and Chapter 5.

Consider the representative n -model \mathfrak{M}_0 with the root r of a point $[\mathfrak{M}_0]$ in $\mathcal{U}(n)^{NNIL}$. Let w be a point in \mathfrak{M}_0 with $\mathfrak{M}_0, w \models p_{i_0}$ ($1 \leq i_0 \leq n$). By the construction of \mathfrak{M}_0 , we know that there exists a representative $n - 1$ -model \mathfrak{N}_0^w such that $\mathfrak{N}_0^w \cong^{n-1} (\mathfrak{M}_0)_w$.

For any point $[\mathfrak{M}_0] \in \mathcal{U}(n)^{NNIL}$, let $\mathfrak{M}_0 = \langle W, R, V \rangle$ and r be the root of \mathfrak{M}_0 . Now we define a formula $\varphi_{[\mathfrak{M}_0]}^\sharp$.

Definition 6.5.10. We define $\varphi_{[\mathfrak{M}_0]}^\sharp$ for $\mathcal{U}(n)^{NNIL}$ by induction on n .

$n = 1$. Then $\mathcal{U}(1)^{NNIL}$ consists of three points $[\mathfrak{M}_0^0], [\mathfrak{M}_0^1], [\mathfrak{M}_0^2]$ (see Figure 6.5 and also Figure 6.7). Define

$$\begin{aligned}\varphi_{[\mathfrak{M}_0^0]}^\sharp &= p \rightarrow p, \\ \varphi_{[\mathfrak{M}_0^1]}^\sharp &= p, \\ \varphi_{[\mathfrak{M}_0^2]}^\sharp &= \neg p.\end{aligned}$$

$n > 1$. First, put

$$Q = \text{prop}(r) \cup \bigcap_{w \in W} \text{notprop}(w).$$

For each propositional letter p , let $X_p = V(p) \cap R(r)$. For each $w \in X_p$, let \mathfrak{N}_0^w be the representative $n - 1$ -model such that $\mathfrak{N}_0^w \cong^{n-1} (\mathfrak{M}_0)_w$. Define

$$\begin{aligned}\varphi_{[\mathfrak{M}_0]}^\sharp &= \bigwedge \text{prop}(r) \wedge \bigwedge \{ \neg q : q \in \bigcap_{w \in W} \text{notprop}(w) \} \\ &\quad \wedge \bigwedge \{ p \rightarrow \bigvee_{w \in X_p} \varphi_{[\mathfrak{N}_0^w]} : p \notin Q \}.\end{aligned}$$

Note that in the above definition, $p \notin PV(\varphi_{[\mathfrak{N}_0^w]})$.

Here for the following proofs, we do not need to define $\psi_{[\mathfrak{M}_0]}^\sharp$ formulas which satisfy $[\mathfrak{N}_0] \not\models \psi_{[\mathfrak{M}_0]}^\sharp$ iff $[\mathfrak{N}_0]R[\mathfrak{M}_0]$ for each point $[\mathfrak{M}_0]$ in $\mathcal{U}(n)^{NNIL}$. However, $\psi_{[\mathfrak{M}_0]}^\sharp$ formulas may have a close connection with the $\beta(w)$ formulas defined in Section 4.2 and this gives a topic for future work.

Theorem 6.5.11. *For each point $[\mathfrak{M}_0]$ in $\mathcal{U}(n)^{NNIL} = \langle W, R, V \rangle$, we have that $[\mathfrak{N}_0] \models \varphi_{[\mathfrak{M}_0]}^\#$ iff $[\mathfrak{M}_0]R[\mathfrak{N}_0]$.*

Proof. Let $\mathfrak{M}_0 = \langle W', R', V' \rangle$ and $\mathfrak{N}_0 = \langle W'', R'', V'' \rangle$. Let r be the root of \mathfrak{M}_0 . We show the theorem by induction on n .

$n = 1$. Clearly the theorem holds.

$n > 1$. For “ \Leftarrow ”, it suffices to show that $\mathcal{U}(n)^{NNIL}, [\mathfrak{M}_0] \models \varphi_{[\mathfrak{M}_0]}^\#$. By Lemma 6.5.7, it then suffices to show that

$$\mathfrak{M}_0, r \models \varphi_{[\mathfrak{M}_0]}^\#.$$

Clearly,

$$\mathfrak{M}_0, r \models \bigwedge \text{prop}(r) \wedge \bigwedge \{\neg q : q \in \bigcap_{w \in W'} \text{notprop}(w)\}.$$

For any $p \notin Q$, suppose $\mathfrak{M}_0, w \models p$ for some $w \in R(r)$. We know that there exists an representative $n - 1$ -model \mathfrak{N}_0^w such that $\mathfrak{N}_0^w \cong^{n-1} (\mathfrak{M}_0)_w$. By the induction hypothesis we have that

$$\mathfrak{N}_0^w \models \varphi_{[\mathfrak{N}_0^w]},$$

which implies that

$$\mathfrak{M}_0, w \models \varphi_{[\mathfrak{N}_0^w]}.$$

So for each such w ,

$$\mathfrak{M}_0, w \models \bigvee_{v \in X_p} \varphi_{[\mathfrak{N}_0^v]}.$$

Thus

$$\mathfrak{M}_0, r \models p \rightarrow \bigvee_{w \in X_p} \varphi_{[\mathfrak{N}_0^w]},$$

and $\mathfrak{M}_0, r \models \varphi_{[\mathfrak{M}_0]}^\#$.

For “ \Leftarrow ”, suppose $[\mathfrak{N}_0] \models \varphi_{[\mathfrak{M}_0]}^\#$. We show $\mathfrak{N}_0 \preceq \mathfrak{M}_0$, i.e. $[\mathfrak{M}_0]R[\mathfrak{N}_0]$. By the definition, $(\mathcal{U}(n)^{NNIL})_{[\mathfrak{N}_0]} \models \varphi_{[\mathfrak{M}_0]}^\#$. Thus by Lemma 6.5.7 and Theorem 6.4.5 we have that

$$\mathfrak{N}_0 \models \varphi_{[\mathfrak{M}_0]}^\#.$$

For any $p \notin Q$ and any point $v \in \mathfrak{N}_0$ such that $\mathfrak{N}_0, v \models p$, we have that $\mathfrak{N}_0, v \models \bigvee_{w \in X_p} \varphi_{[\mathfrak{N}_0^w]}$, thus there exists $w \in X_p$ such that

$$(\mathfrak{N}_0)_v \models \varphi_{[\mathfrak{N}_0^w]}.$$

There exists an $n - 1$ -model $((\mathfrak{N}_0)_v)_0$ such that $((\mathfrak{N}_0)_v)_0 \cong^{n-1} (\mathfrak{N}_0)_v$. Clearly

$$((\mathfrak{N}_0)_v)_0 \models \varphi_{[\mathfrak{N}_0^w]}.$$

By Lemma 6.5.7 and Theorem 6.4.5,

$$(\mathcal{U}(n - 1)^{NNIL})_{[(\mathfrak{N}_0)_v)_0]} \models \varphi_{[\mathfrak{N}_0^w]}.$$

Then since both $((\mathfrak{N}_0)_v)_0$ and \mathfrak{N}_0^w are $n - 1$ -models, by the induction hypothesis we have that there exists a total subsimulation Z^v such that

$$Z^v : ((\mathfrak{N}_0)_v)_0 \preceq \mathfrak{N}_0^w.$$

Note that there exist frame isomorphisms $f : \mathfrak{N}_0 \rightarrow ((\mathfrak{N}_0)_v)_0$ and $g : \mathfrak{N}_0^w \rightarrow (\mathfrak{M}_0)_w$. Consider the relation Z_0^v between \mathfrak{N}_0 and $(\mathfrak{M}_0)_w$ defined by

$$Z_0^v = \{(x, y) : (f(x), g(y)) \in Z^v\}.$$

Note that $x \models p$ and $y \models p$. It is then easy to see that Z_0^v is a total subsimulation.

Define a relation on $W'' \times W'$ by

$$Z = \{(u, r) : \mathfrak{N}_0, u \not\models p \text{ for all } p \notin Q\} \cup \bigcup_{\mathfrak{N}_0, v \models p \text{ for some } p \notin Q} Z_0^v.$$

Clearly, Z is total. We show that Z is a subsimulation.

For (S1), clearly, for any $(x, y) \in Z_0^v$ for some v satisfying $\mathfrak{N}_0, v \models p$ for some $p \notin Q$, since Z_0^v is a subsimulation $col(x) = col(y)$. For $(u, r) \in Z$ with $\mathfrak{N}_0, u \not\models p$ for all $p \notin Q$, we know that

$$\mathfrak{M}_0, r \not\models p \text{ for all } p \notin Q.$$

By $\mathfrak{N}_0, u \models \varphi_{[\mathfrak{M}_0]^\#}$, we know that

$$\mathfrak{N}_0, u \models \bigwedge prop(r) \wedge \bigwedge \{\neg q : q \in \bigcap_{w \in W} notprop(w)\}.$$

Thus,

$$\begin{aligned} prop(u) &\supseteq prop(r), \\ notprop(u) &\supseteq \bigcap_{w \in W} notprop(w) \cup (\{p_1, \dots, p_n\} \setminus Q) = notprop(r). \end{aligned}$$

Hence $col(u) = col(r)$.

For (S2), suppose xZx' and $xR''y$. We distinguish three cases.

Case 1: $\mathfrak{N}_0, x \models p$ for some $p \notin Q$. Then $(x, x') \in Z_0^x$. So since Z_0^x is a subsimulation, there exists $y' \in R''(x')$ such that $yZ_0^x y'$, i.e. yZy' .

Case 2: $\mathfrak{N}_0, x \not\models p$ and $\mathfrak{N}_0, y \not\models p$ for all $p \notin Q$. Then by the definition of Z , we must have that $x' = r$. Clearly, yZr and $rR''r$.

Case 3: $\mathfrak{N}_0, x \not\models p$ for all $p \notin Q$ and $\mathfrak{N}_0, y \models p$ for some $p \notin Q$. We have that $y \in dom(Z_0^y)$, so since Z_0^y is total, there exists $y' \in W'$ such that $yZ_0^y y'$, i.e. yZy' . And clearly, $rR''y'$. \square

For each upset X of $\mathcal{U}(n)^{NNIL}$, we define a formula $\theta^n(X) \in NNIL(\vec{p})$ as

$$\theta^n(X) := \bigvee_{w \in X} \varphi_w^\#.$$

In case that $X = \{w\}$, we will only write $\theta^n(w)$ instead of $\theta^n(\{w\})$.

Lemma 6.5.12. *For every upset X of $\mathcal{U}(n)^{NNIL}$, we have that $X = V(\theta^n(X))$.*

Proof. By Theorem 6.5.11. \square

We end this section by giving a characterization property for $NNIL$ -formulas.

Theorem 6.5.13. *Let φ be a formula. If φ satisfies that for any finite models \mathfrak{M} and \mathfrak{N} with $\mathfrak{M} \equiv \mathfrak{N}$,*

$$\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{N} \models \varphi,$$

then φ is provably equivalent to a formula $\varphi' \in NNIL(\vec{p})$.

Proof. Consider the model $\mathcal{U}(n)^{NNIL} = \langle W, R, V \rangle$. Let $\varphi' = \theta^n(V(\varphi))$.

For every point $w \in \mathcal{U}(n)$, we have that

$$\begin{aligned} \mathcal{U}(n), w \models \varphi &\iff \mathcal{U}(n)_w \models \varphi \\ &\iff (\mathcal{U}(n)^{NNIL})_{[\mathcal{U}(n)_w]} \models \varphi \text{ (by Theorem 6.5.7 and the property} \\ &\quad \text{of } \varphi) \\ &\iff (\mathcal{U}(n)^{NNIL})_{[\mathcal{U}(n)_w]} \models \varphi' \text{ (by the definition of } \varphi') \\ &\iff \mathcal{U}(n)_w \models \varphi' \text{ (by Theorem 6.5.7 and Theorem 6.4.5)} \\ &\iff \mathcal{U}(n), w \models \varphi'. \end{aligned}$$

Thus $\mathcal{U}(n) \models \varphi \leftrightarrow \varphi'$, so by Theorem 3.2.4, $\vdash_{\mathbf{IPC}} \varphi \leftrightarrow \varphi'$. \square

The next theorem was proved in Theorem 7.4.1 in [25]. We now show it as a corollary of Theorem 6.5.13.

Theorem 6.5.14. *Let φ be a formula. If φ satisfies that for any finite models \mathfrak{M} and \mathfrak{N} with $\mathfrak{N} \preceq \mathfrak{M}$,*

$$\mathfrak{M} \models \varphi \Rightarrow \mathfrak{N} \models \varphi,$$

then φ is provably equivalent to a formula $\varphi' \in NNIL(\vec{p})$.

Proof. Let \mathfrak{M} and \mathfrak{N} be finite models with $\mathfrak{M} \equiv \mathfrak{N}$. Then it follows from the assumption that

$$\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{N} \models \varphi.$$

Thus, by Theorem 6.5.13, φ is provably equivalent to a formula $\varphi' \in NNIL(\vec{p})$. \square

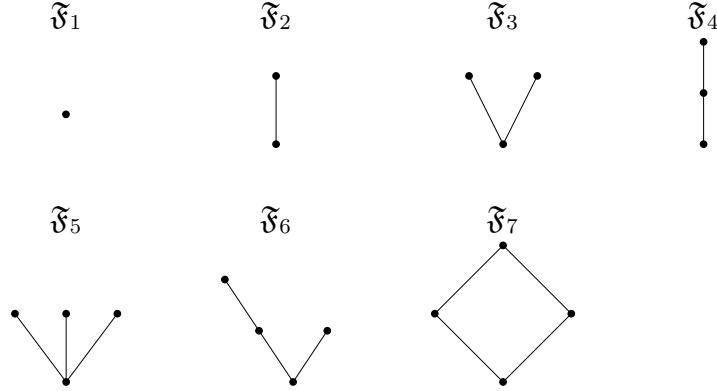
6.6 $\mathcal{U}(2)^{NNIL}$ and the subframe logics axiomatized by $NNIL(p, q)$ -formulas

The model $\mathcal{U}(n)^{NNIL}$ is special not only in the sense that it is a universal model for $NNIL(\vec{p})$ -formulas, but also in the sense that it suggests a method for future work on subframe logics. In this section, we exemplify this by obtaining the subframe logics defined by $NNIL(p, q)$ -formulas by observing the structure of $\mathcal{U}(2)^{NNIL}$. It will turn out that $NNIL$ -formulas are much more efficient than $[\wedge, \rightarrow]$ -formulas. We will find 8 subframe logics axiomatized by $NNIL(p, q)$ -formulas, whereas $[\wedge, \rightarrow]^2$ -formulas only axiomatize classical logic (by a Peirce formula).

We have seen in the previous section that each point $[\mathfrak{M}_0]$ in $\mathcal{U}(2)^{NNIL}$ corresponds to (is) a representative model \mathfrak{M}_0 . There are 7 different underlying frames of these representative models (see Figure 6.10). On the other hand, each upset X of $\mathcal{U}(2)^{NNIL}$ corresponds to a $NNIL(p, q)$ -formula $\theta^2(X)$. $\theta^2(X)$ refutes all the underlying frames of the representative models corresponding to the points of $\mathcal{U}(2)^{NNIL}$ that are not in X . This gives us some idea of axiomatizing subframe logics by two-variable $NNIL$ -formulas by some observation of the model $\mathcal{U}(2)^{NNIL}$.

To be precise, consider the upset X_1 generated by the points numbered 1, 2, 3, 4 in the model $\mathcal{U}(2)^{NNIL}$. The underlying frames of the representative

Figure 6.10: Underlying frames of the representative models in $\mathcal{U}(2)^{NNIL}$



models of these four points are the same, namely the frame \mathfrak{F}_1 in Figure 6.10. The $NNIL(p, q)$ -formula defining the set X_1 is

$$\theta^2(X_1) = (p \wedge \neg q) \vee (p \wedge q) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge q).$$

Consider the logic

$$L_1 = \mathbf{IPC} + \theta^2(X_1).$$

Observe that the axiom $\theta^2(X_1)$ can be simplified to

$$p \vee \neg p$$

and L_1 is the classical propositional logic **CPC**. **CPC** characterizes the frames consisting of a single reflexive element which are exactly the \mathfrak{F}_1 -like frames.

Consider the upset X_2 generated by the points numbered 5, 6, 7, 8, 9. The $NNIL(p, q)$ -formula defining the set X_2 is

$$\theta^2(X_2) = p \vee \neg q \vee (p \leftrightarrow q) \vee \neg p \vee q.$$

The logic

$$L_2 = \mathbf{IPC} + \theta^2(X_2)$$

is the 3-valued Gödel logic **G3** (also called *here-and-there logic* in [10]). The frames of here-and-there logic have depth less or equal to 2 and width 1. These frames are exactly those appearing in the corresponding set of frames

of the set X_2 , namely the \mathfrak{F}_1 -like or \mathfrak{F}_2 -like frames (see Figure 6.10). It is worthwhile to point out that in [10], many known axiomatizations of the here-and-there logic are given, however, the axiom $p \vee \neg q \vee (p \leftrightarrow q) \vee \neg p \vee q$ is not among those. As far as the author knows, this is a new axiomatization for the here-and-there logic.

The $NNIL(p, q)$ -formula defining the upset X_3 generated by the points numbered 5, 9, 10, 11, 12 is

$$\theta^2(X_3) = p \vee q \vee ((q \rightarrow p) \wedge (p \rightarrow q \vee \neg q)) \vee \neg(p \wedge q) \vee ((p \rightarrow q) \wedge (q \rightarrow p \vee \neg p)).$$

The underlying frames of the representative models of X_3 are \mathfrak{F}_1 , \mathfrak{F}_2 and \mathfrak{F}_3 in Figure 6.10. The logic

$$L_3 = \mathbf{IPC} + \theta^2(X_3)$$

defines the frames with depth less than or equal to 2 and width less than or equal to 2.

The $NNIL(p, q)$ -formula defining the upset X_4 generated by the points numbered 5, 9, 14 is

$$\theta^2(X_4) = p \vee q \vee ((p \rightarrow q \vee \neg q) \wedge (q \rightarrow p \vee \neg p)).$$

The underlying frames of the representative models of X_4 are $\mathfrak{F}_1 - \mathfrak{F}_3$ and \mathfrak{F}_5 in Figure 6.10. The axiom $\theta^2(X_4)$ can be simplified to

$$p \vee (p \rightarrow q \vee \neg q).$$

The logic

$$L_4 = \mathbf{IPC} + p \vee (p \rightarrow q \vee \neg q)$$

is the 3-Peirce logic and defines the frames with depth less than or equal to 2. 3-Peirce logic is commonly axiomatized by

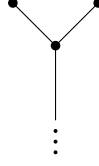
$$((p \rightarrow (((q \rightarrow r) \rightarrow r) \rightarrow r)) \rightarrow p) \rightarrow p.$$

As far as the author knows, the axiom $p \vee (p \rightarrow q \vee \neg q)$ is a new one. Moreover, this result can easily be generalized to the n -Peirce logic axiomatizing the frames of depth less or equal to n .

The $NNIL(p, q)$ -formula defining the upset X_5 generated by the points numbered 13, 15 is

$$\theta^2(X_5) = (p \rightarrow q) \vee (q \rightarrow p).$$

Figure 6.11: The frame for L_6



The logic

$$L_5 = \mathbf{IPC} + \theta^2(X_5)$$

is the Dummett's logic **LC** and defines the linear frames (frames with width 1). Note that the underlying frames of the representative models of X_5 are $\mathfrak{F}_1 - \mathfrak{F}_4$ in Figure 6.10. However, there exists a model on \mathfrak{F}_3 falsifying $\theta^2(X_5)$, namely the representative model of the point numbered 11. This means that \mathfrak{F}_3 is not an L_5 frame. The remaining frames \mathfrak{F}_1 , \mathfrak{F}_2 and \mathfrak{F}_4 are exactly the linear frames among the all 7 frames in Figure 6.10.

The $NNIL(p, q)$ -formula defining the upset X_6 generated by the points numbered 11, 13, 15 is

$$\theta^2(X_6) = \neg(p \wedge q) \vee (p \rightarrow q) \vee (q \rightarrow p).$$

The underlying frames of the representative models of X_6 are $\mathfrak{F}_1 - \mathfrak{F}_4$. The logic

$$L_6 = \mathbf{IPC} + \theta^2(X_6)$$

defines the frames with 2-split only right at the end of the frames (see Figure 6.11).

The $NNIL(p, q)$ -formula defining the upset X_7 generated by the points numbered 13, 14, 15 is

$$\theta^2(X_7) = (q \rightarrow p) \vee ((p \rightarrow q \vee \neg q) \wedge (q \rightarrow p \vee \neg p)) \vee (p \rightarrow q).$$

The underlying frames of the representative models of X_7 are $\mathfrak{F}_1 - \mathfrak{F}_5$. The logic

$$L_7 = \mathbf{IPC} + \theta^2(X_7)$$

defines the frames with any splits only right at the end of the frames (see Figure 6.12).

Figure 6.12: The frame for L_7

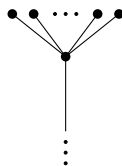
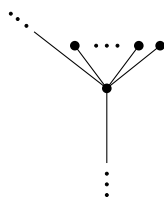


Figure 6.13: The frame for L_8



The $NNIL(p, q)$ -formula defining the upset X_8 generated by the points numbered 16, 17 is

$$\theta^2(X_8) = (p \rightarrow q \vee \neg q) \vee (q \rightarrow p \vee \neg p).$$

The underlying frames of the representative models of X_8 are $\mathfrak{F}_1 - \mathfrak{F}_6$. The logic

$$L_8 = \mathbf{IPC} + \theta^2(X_8)$$

defines the frames with any splits only right at the end except that one branch may continue linearly (see Figure 6.13).

All the above logics, their properties and connections with $\mathcal{U}(2)^{NNIL}$ are listed in Table 6.1.

Table 6.1: Subframe logics axiomatized by $NNIL(p, q)$ -formulas

| Name (if available) | Points | Axiomatization | Frame characterization property |
|----------------------|--------------|---|---|
| Classical logic | 1,2,3,4 | $p \vee \neg p$ | reflexive singleton |
| Here and there logic | 5,6,7,8,9 | $p \vee \neg q \vee (p \leftrightarrow q) \vee \neg p \vee q$ | depth 2 width 1 |
| - | 5,9,10,11,12 | $p \vee q \vee ((q \rightarrow p) \wedge (p \rightarrow q \vee \neg q)) \vee$ $\neg(p \wedge q) \vee ((p \rightarrow q) \wedge (q \rightarrow p \vee \neg p))$ | depth 2 width 2 |
| 3-Peirce logic | 5,9,14 | $p \vee (p \rightarrow q \vee \neg q)$ | depth 2 |
| Dummett's logic | 13,15 | $(p \rightarrow q) \vee (q \rightarrow p)$ | width 1 (linear) |
| - | 11,13,15 | $\neg(p \wedge q) \vee (p \rightarrow q) \vee (q \rightarrow p)$ | 2-split only at the end |
| - | 13,14,15 | $(q \rightarrow p) \vee ((p \rightarrow q \vee \neg q) \wedge (q \rightarrow p \vee \neg p)) \vee (p \rightarrow q)$ | any splits only at the end |
| - | 16,17 | $(p \rightarrow q \vee \neg q) \vee (q \rightarrow p \vee \neg p)$ | any splits only at the end except that one branch may continue linearly |

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