

Polarized Partition Properties on the Second Level of the Projective Hierarchy.

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Regularity Properties

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(Lebesgue measurability, Baire property, Ramsey property, Marczewski measurability)

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“More regularity on Δ_2^1/Σ_2^1 -level \propto L gets smaller”

Examples

1. $\Delta_2^1(\text{Lebesgue}) \iff \forall a \exists \text{ random-generic}/L[a]$

2. $\Delta_2^1(\text{Baire Property}) \iff \forall a \exists \text{ Cohen-generic}/L[a]$

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3. $\Delta_2^1(\text{Ramsey}) \iff \forall a \exists \text{ Ramsey real } /L[a]$
4. $\Delta_2^1(\text{Laver}) \iff \forall a \exists \text{ dominating real } /L[a]$
5. $\Delta_2^1(\text{Miller}) \iff \forall a \exists \text{ unbounded real } /L[a]$
6. $\Delta_2^1(\text{Sacks}) \iff \forall a \exists \text{ real } \notin L[a]$

Where

- $x \in [\omega]^\omega$ is *Ramsey* over $L[a]$ if for all $A \subseteq [\omega]^2 \cap L[a] \exists n$ s.t. $[x \setminus n]^2 \subseteq A$ or $[x \setminus n]^2 \subseteq ([\omega]^2 \setminus A)$
- $x \in \omega^\omega$ is *dominating* over $L[a]$ if $\forall y \in \omega^\omega \cap L[a] \forall^\infty n (y(n) < x(n))$
- $x \in \omega^\omega$ is *unbounded* over $L[a]$ if $\forall y \in \omega^\omega \cap L[a] \exists^\infty n (y(n) < x(n))$

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(Non-)implications

Given two regularity properties: Reg_1 and Reg_2 , we are interested in:

$$\Gamma_1(\text{Reg}_1) \implies \Gamma_2(\text{Reg}_2)?$$

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What has been established so far?

Diagram of implications

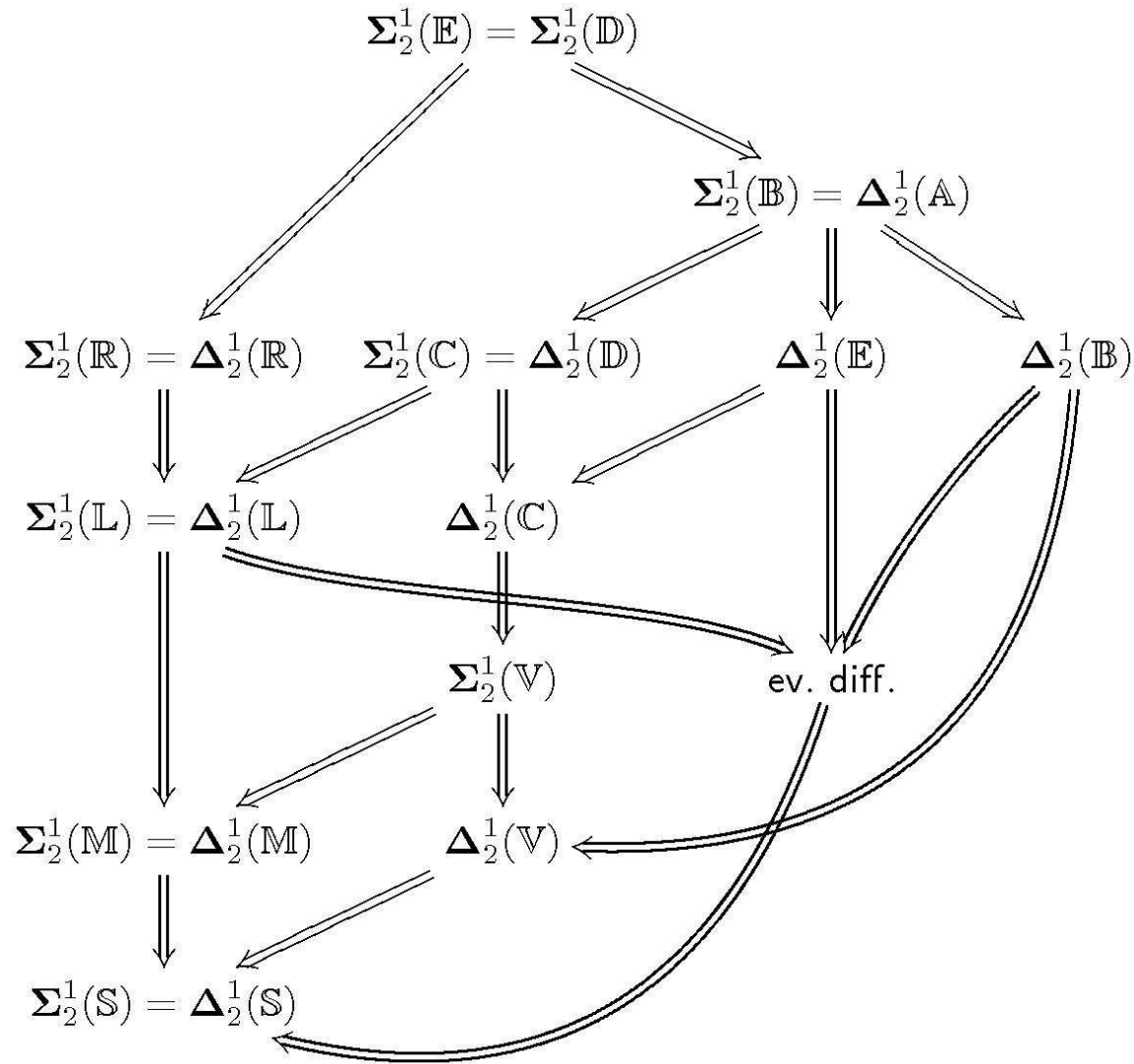


Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*

Some general theorems

Theorem (Ikegami, 2008) Let \mathbb{P} be a proper, tree-like forcing on ω^ω , and $I_{\mathbb{P}}$ a canonical σ -ideal such that $\mathbb{P} \dot{\dashv} \text{BOREL}(\omega^\omega)/I_{\mathbb{P}}$. Moreover suppose that the membership of Borel sets in $I_{\mathbb{P}}$ is a Σ_2^1 property. Call a set A \mathbb{P} -measurable if

$$\forall p \exists q \leq p ([q] \subseteq^* A \vee [q] \subseteq^* \omega^\omega \setminus A)$$

Then T.F.A.E.

1. Δ_2^1 (\mathbb{P} -measurability)
2. Σ_3^1 - \mathbb{P} -absoluteness
3. $\forall a \exists x$ quasi- $I_{\mathbb{P}}$ -generic over $L[a]$

where x is *quasi- $I_{\mathbb{P}}$ -generic* over M if $x \notin B$ for all Borel sets $B \in I_{\mathbb{P}}$, coded in M .

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Theorem (Ikegami, 2008) With additional (technical) assumptions on the ideal $I_{\mathbb{P}}$, T.F.A.E.

1. Σ_2^1 (\mathbb{P} -measurability)
2. $\forall a \exists$ co- $I_{\mathbb{P}}$ set of quasi- $I_{\mathbb{P}}$ -generics over $L[a]$

Polarized Partitions

Definition. Letters H, J etc. will denote infinite sequences of finite subsets of ω , i.e. $H : \omega \longrightarrow [\omega]^{<\omega}$. Use abbreviation: $[H] = \prod_{i \in \omega} H(i)$.

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(unbounded polarized partition) if

$$\exists H \text{ s.t. } \forall i |H(i)| = m_i \text{ and } [H] \subseteq A \text{ or } [H] \cap A = \emptyset$$

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and n_1, n_2, \dots are recursive in m_1, m_2, \dots

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From now on, use generic notations $(\vec{\omega} \rightarrow \vec{m})$ and $(\vec{n} \rightarrow \vec{m})$.

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In [DiPrisco & Todorčević, 2003]:

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On the other hand, easy to find counterexample using AC (i.e. well-ordering of ω^ω).

So, what about $\Delta_2^1/\Sigma_2^1(\vec{\omega} \rightarrow \vec{m})$ and $\Delta_2^1/\Sigma_2^1(\vec{n} \rightarrow \vec{m})$?

Upper bound

Fact. $\Gamma(\text{Ramsey}) \implies \Gamma(\vec{\omega} \rightarrow \vec{m})$.

Proof. Given A , let $X \in \omega^{\uparrow\omega}$ be homogeneous for $A \cap \omega^{\uparrow\omega}$. Then divide $\text{ran}(X)$ into X_0, X_1, \dots such that $|X_i| = m_i$. Now $H := \langle X_0, X_1, \dots \rangle$ witnesses that A satisfies $(\vec{\omega} \rightarrow \vec{m})$.

Eventually different reals

Theorem. (Brendle) If $\Delta_2^1(\vec{\omega} \rightarrow \vec{m})$ then $\forall a$ there is an eventually different real over $L[a]$.

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- Let H be homogeneous for A , w.l.o.g. $[H] \subseteq A$. But if $x \in [H]$ then let us change finitely many digits of x to produce a new real x' , such that the first n at which $x'(n) = y_x(n)$ is odd but still $x' \in [H]$. It is easy to see that $y_x = y_{x'}$, hence $x' \notin A$: contradiction. □

Diagram of implications

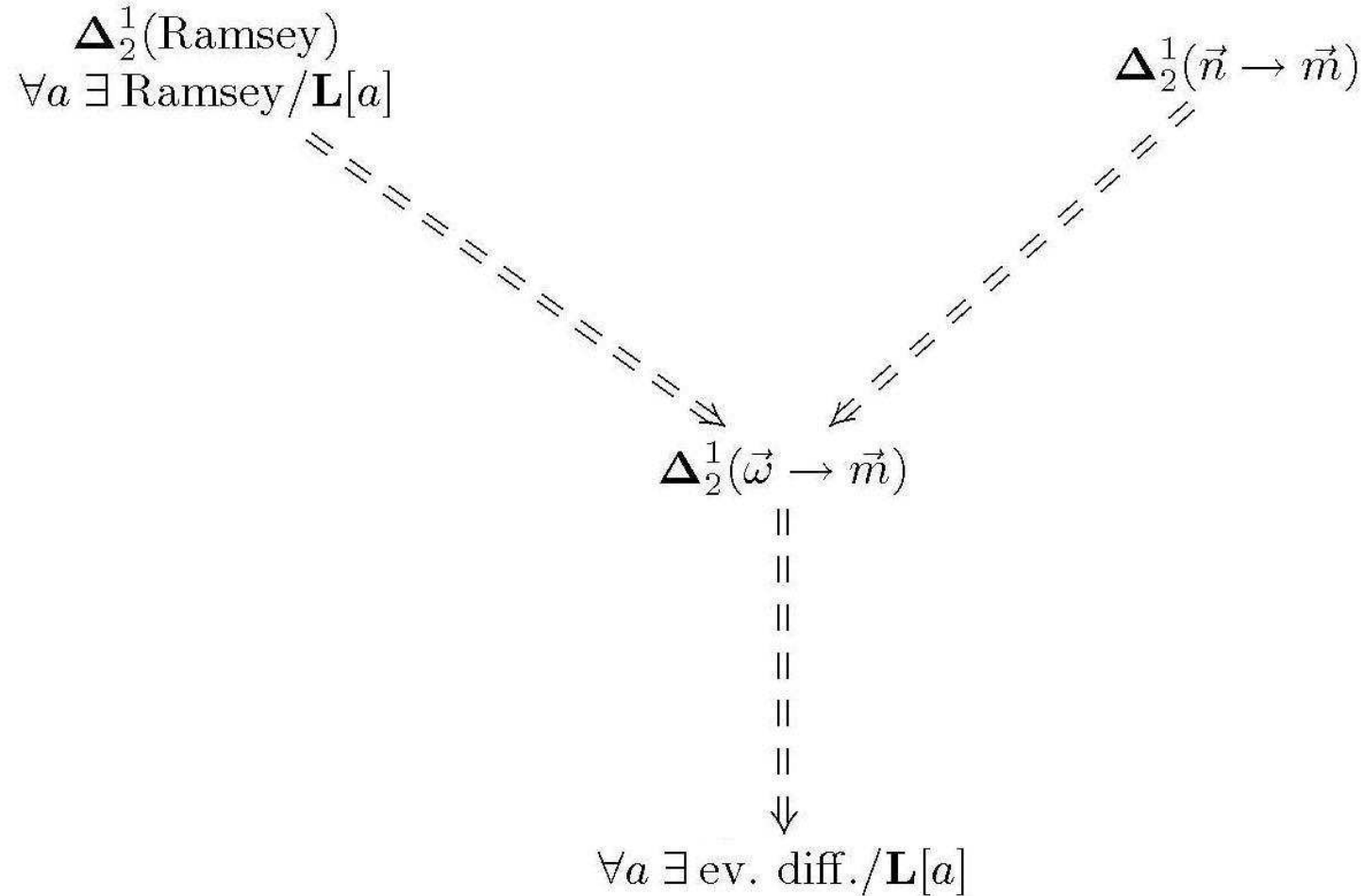


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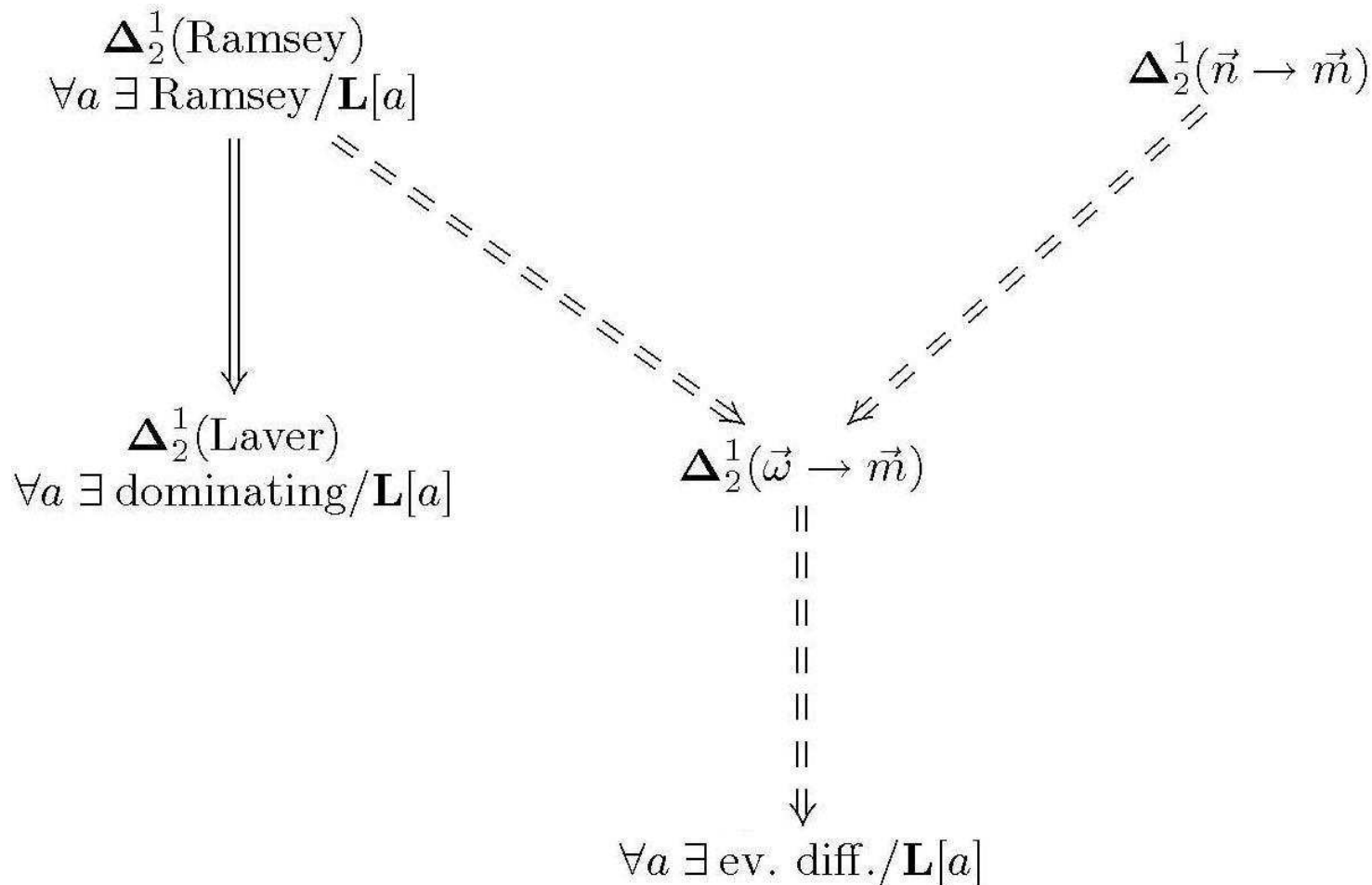


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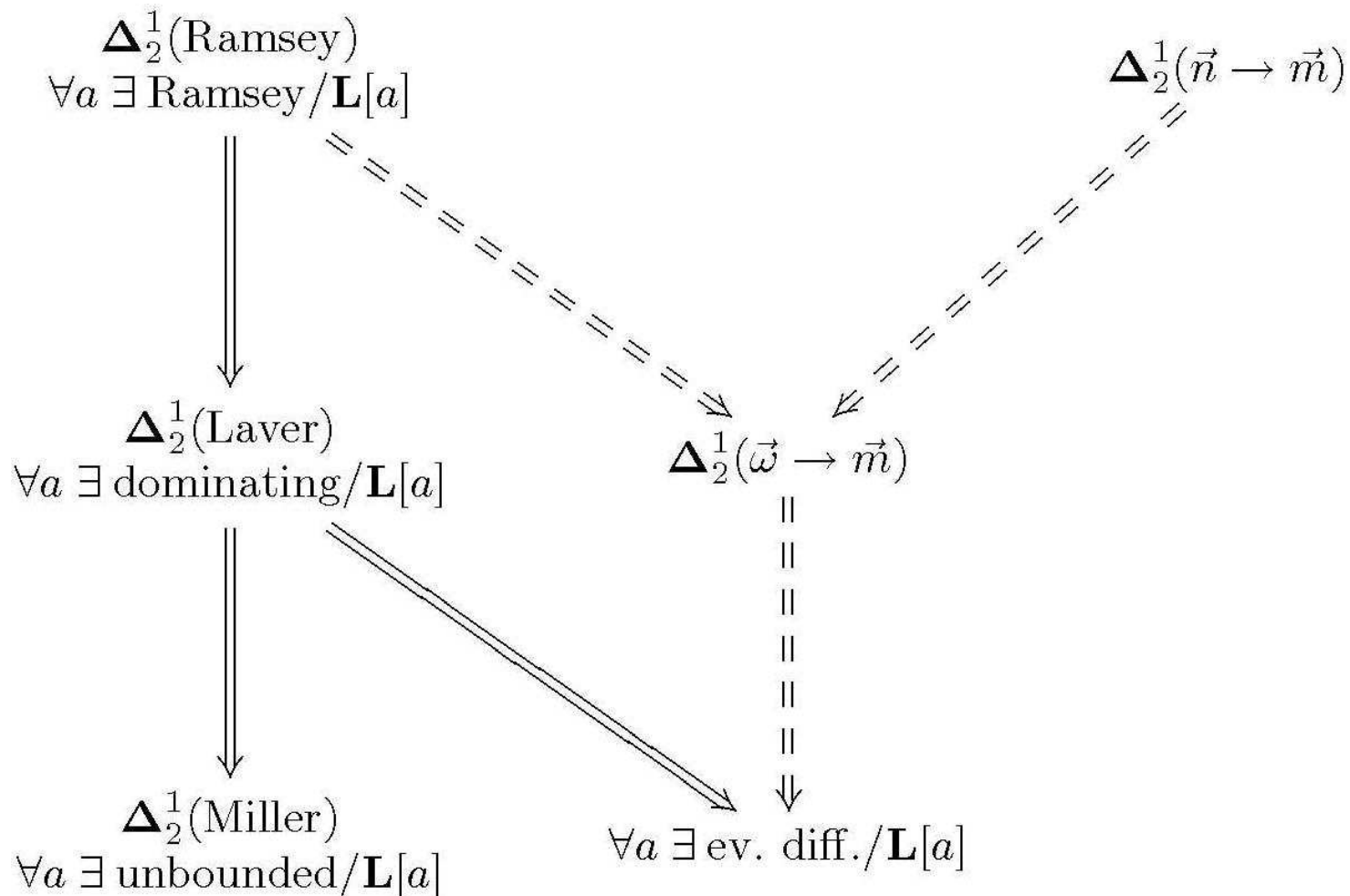
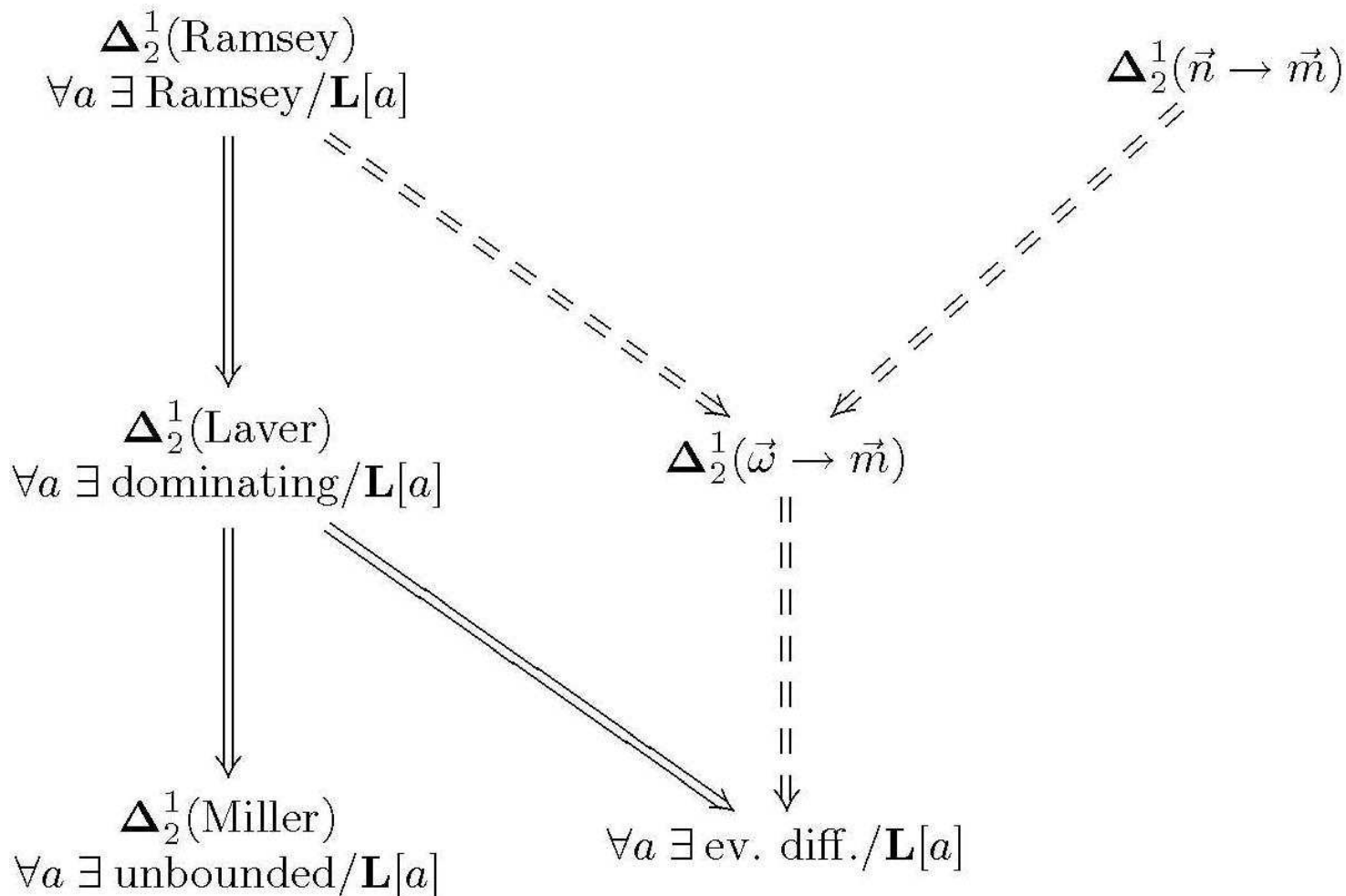
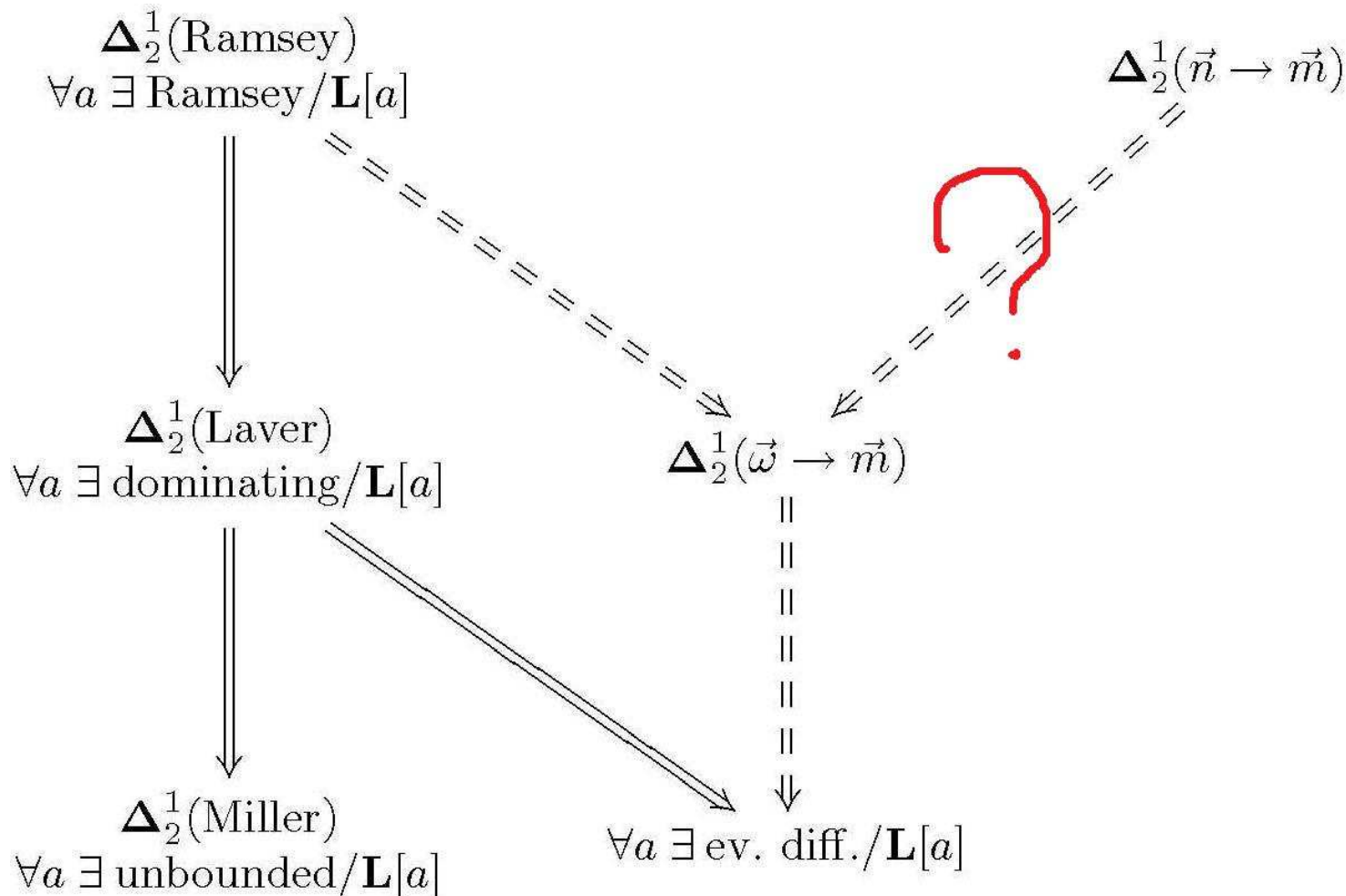


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Question: which implications cannot be reversed?

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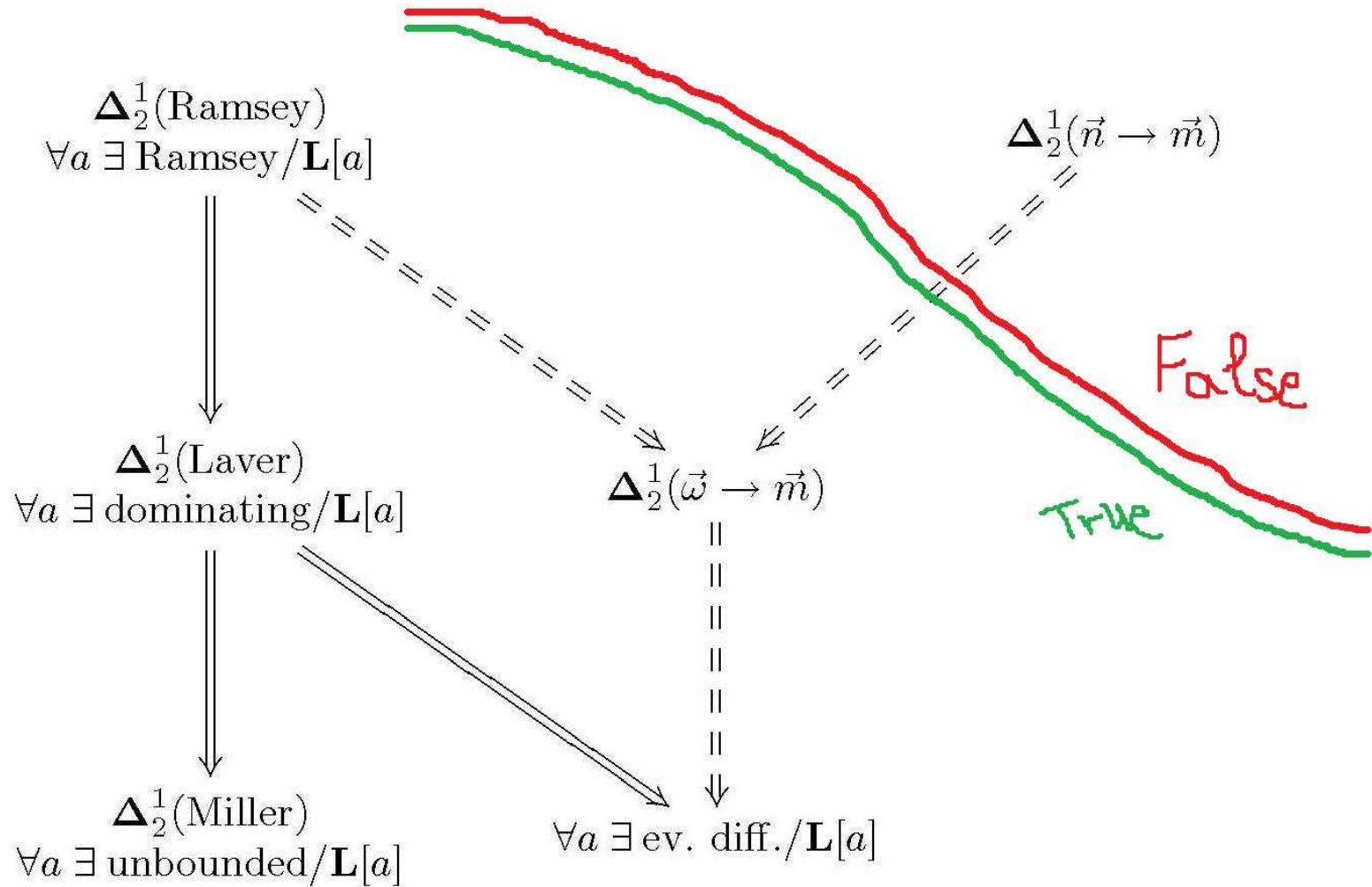


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- Let $\mathcal{C} := \{S : \omega \longrightarrow [\omega]^{<\omega} \mid \forall i |S(i)| \leq 2^i\}$. Mathias forcing satisfies the *Laver property*: For every $y \in M \cap \omega^\omega$ and \dot{x} s.t. $\Vdash \forall i \dot{x}(i) \leq y(i)$, there is an $S \in \mathcal{C} \cap M$ s.t. $\Vdash \forall i \dot{x}(i) \in S(i)$.

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- Use the Δ_2^1 -well-ordering of $L \cap \omega^\omega$ to define a Δ_2^1 -well-ordering of $L \cap \mathcal{C}$.
- Use that to define a Δ_2^1 set A which explicitly violates $(\vec{n} \rightarrow \vec{m})$, where the m_i grow faster than 2^i . This set is well-defined because of the Laver property. \square

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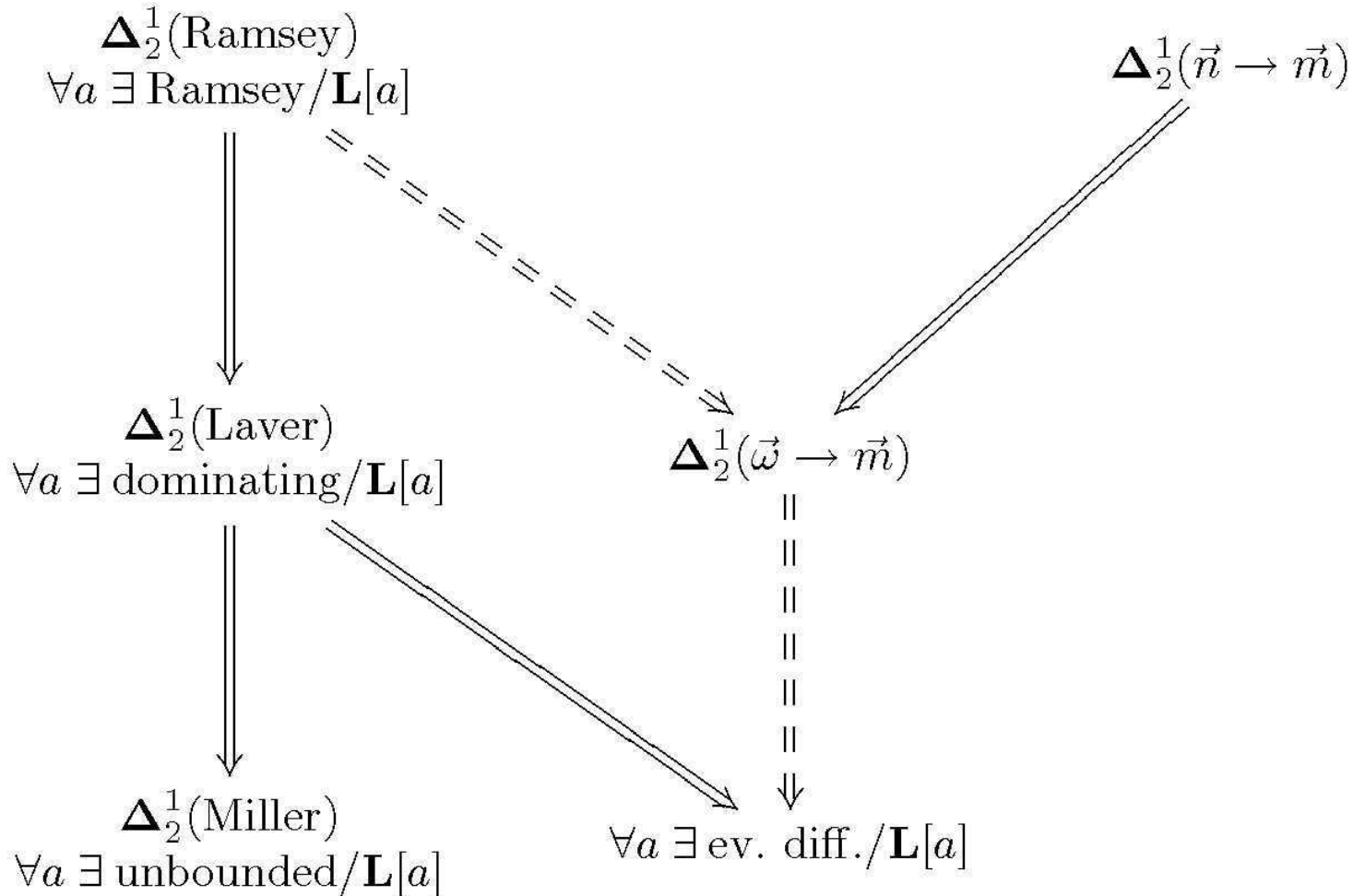
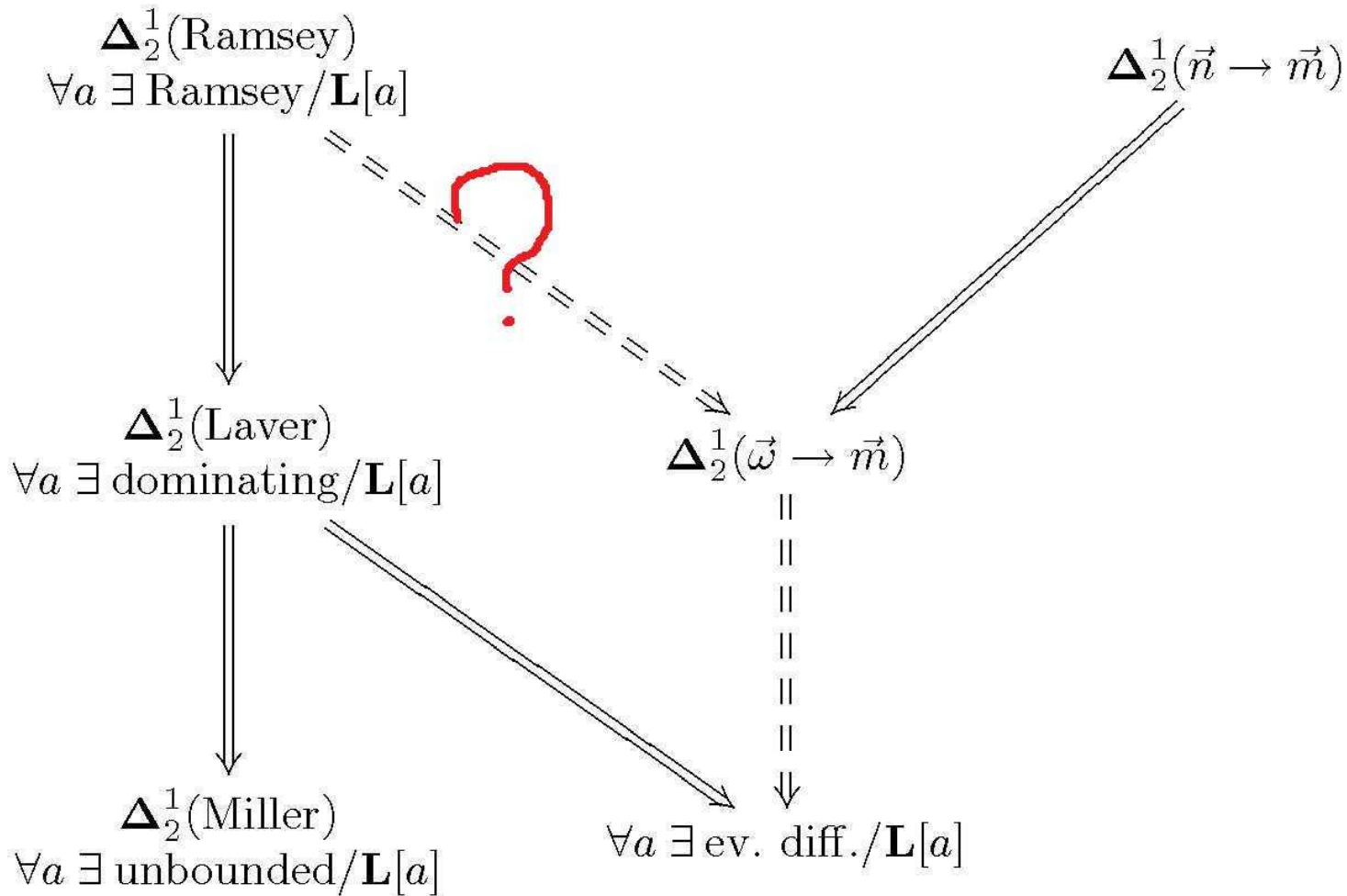


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A model for $\Delta_2^1(\vec{n} \rightarrow \vec{m})$

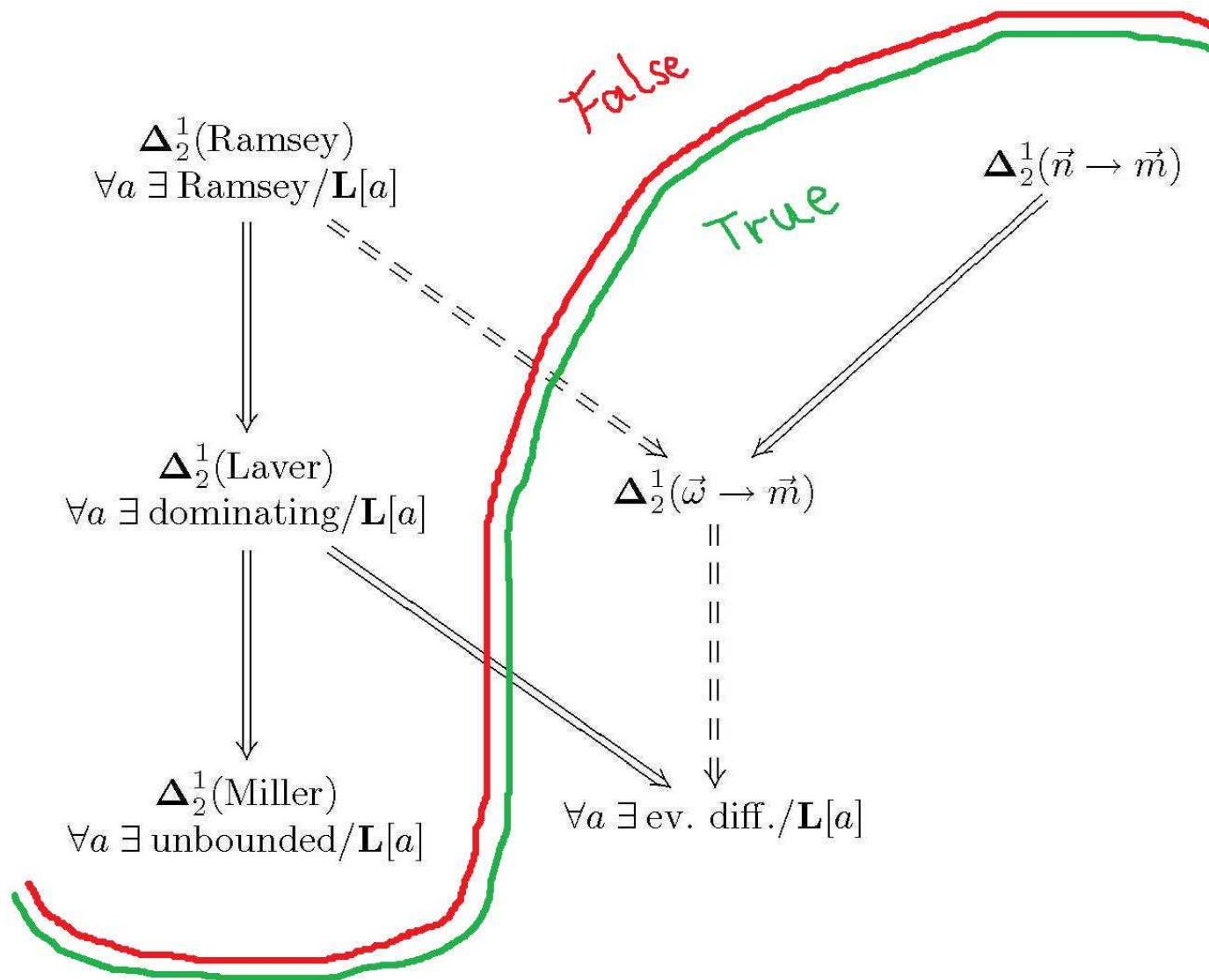
Goal. Force a model in which $\Delta_2^1(\vec{\omega} \rightarrow \vec{m})$ is true but $\Delta_2^1(\text{Ramsey})$ is false.

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Diagram of implications



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for all \dot{x} there is a y in the ground model and a p s.t. $p \Vdash \forall n \dot{x}(n) \leq y(n)$.

2. If $\forall a$ there is a generic over $L[a]$, then $\Delta_2^1(\vec{n} \rightarrow \vec{m})$ holds.

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- If $F(n)$ is large enough, then $\exists (c, k) \in \mathbb{P}_n$ s.t. $\text{norm}_n(c, k) \geq n$.

[To be precise: $F(n) \geq 2^{((2^{1/\epsilon_n})^n)}$]

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 - $\text{stem}(p') \supseteq \text{stem}(p)$
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Remark: \mathbb{P}_{KSZ} adds a generic real, but the generic filter is not determined from the generic real in the usual way, and \mathbb{P}_{KSZ} is not in general representable as $\text{BOREL}(\omega^\omega)/I$ for a σ -ideal I .

Proper and ω^ω -bounding

Theorem. (Kellner-Shelah, Shelah-Zapletal) If \mathbb{P}_{KSZ} is as above, and moreover

$$\forall n : \epsilon_n \leq \frac{1}{n \cdot \prod_{j < n} F(j)}$$

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The proof uses two properties from the general theory of creature forcings: for each n , \mathbb{P}_n satisfies “ ϵ_n -bigness” and “ ϵ_n -halving”.

Forcing $\Delta_2^1(\vec{n} \rightarrow \vec{m})$

Theorem. (Brendle-Kh) If for every a there is a \mathbb{P}_{KSZ} -generic over $L[a]$ then $\Delta_2^1(\vec{m} \rightarrow \vec{n})$ holds.

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- W.l.o.g. assume the former, and work in $L[a]$ from now on. Let $M \prec \mathcal{H}_\theta$ be countable and $q \leq p$ a $(M, \mathbb{P}_{\text{KSZ}})$ -Master condition. By pure decision, q has empty stem as well. Moreover, every $x \in [q]$ is M -generic and by standard absoluteness arguments $[q] \subseteq A$ follows.

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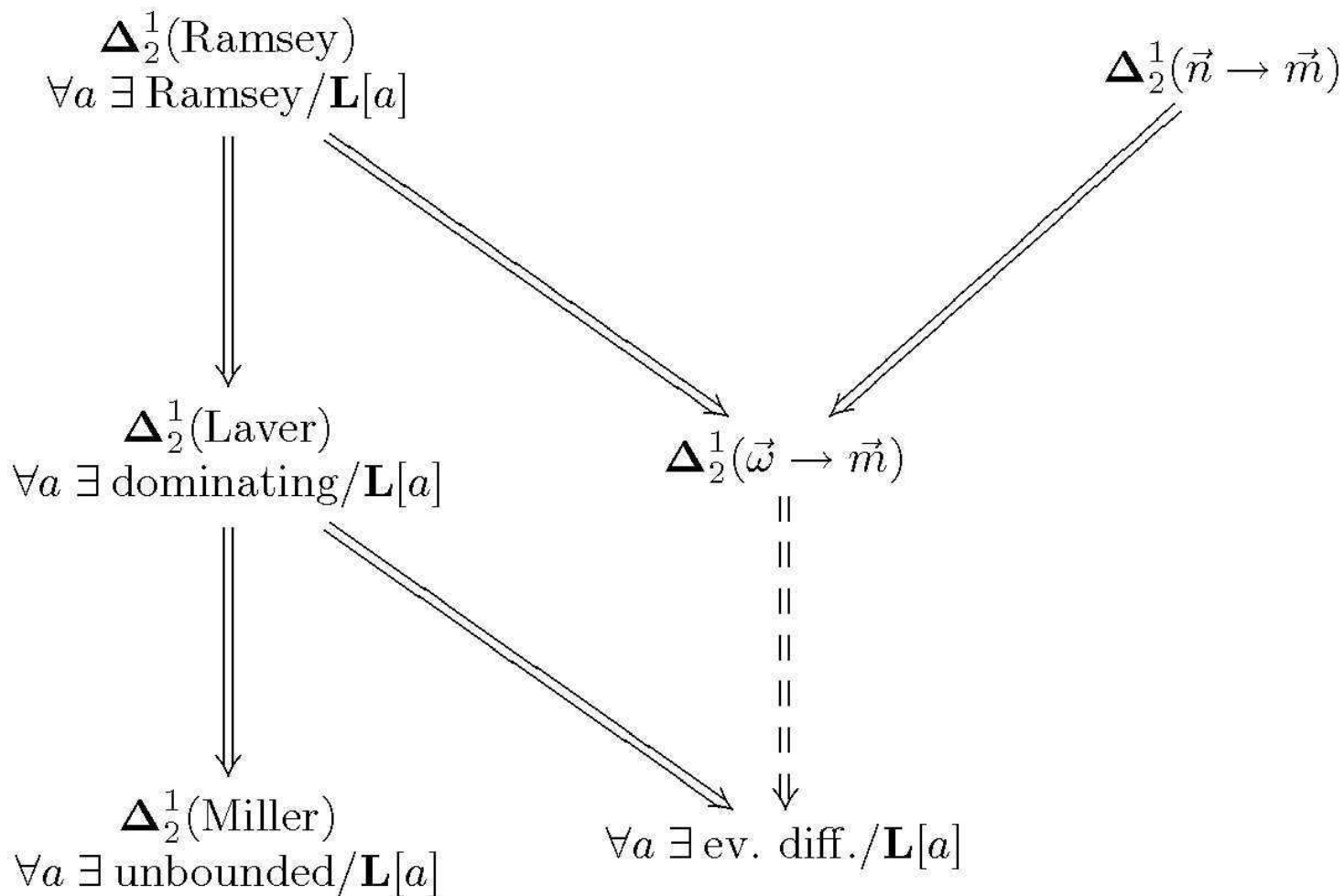
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- Since q has empty stem, it witnesses that A satisfies $(\vec{n} \rightarrow \vec{m})$.

Forcing $\Delta_2^1(\vec{n} \rightarrow \vec{m})$

Corollary. An ω_1 -iteration of \mathbb{P}_{KSZ} , starting from L , gives a model in which $\Delta_2^1(\vec{n} \rightarrow \vec{m})$ holds but $\Delta_2^1(\text{Miller})$ fails.

Notice that the bounds “ \vec{n} ” have been explicitly computed beforehand: they are the $F(n)$ ’s from the definition of \mathbb{P}_{KSZ} .

Diagram of implications



Other properties

Definition. A real $x \in [\omega]^\omega$ is *splitting* over M if for all $a \in [\omega]^\omega \cap M$, both $a \cap x$ and $a \setminus x$ are infinite.

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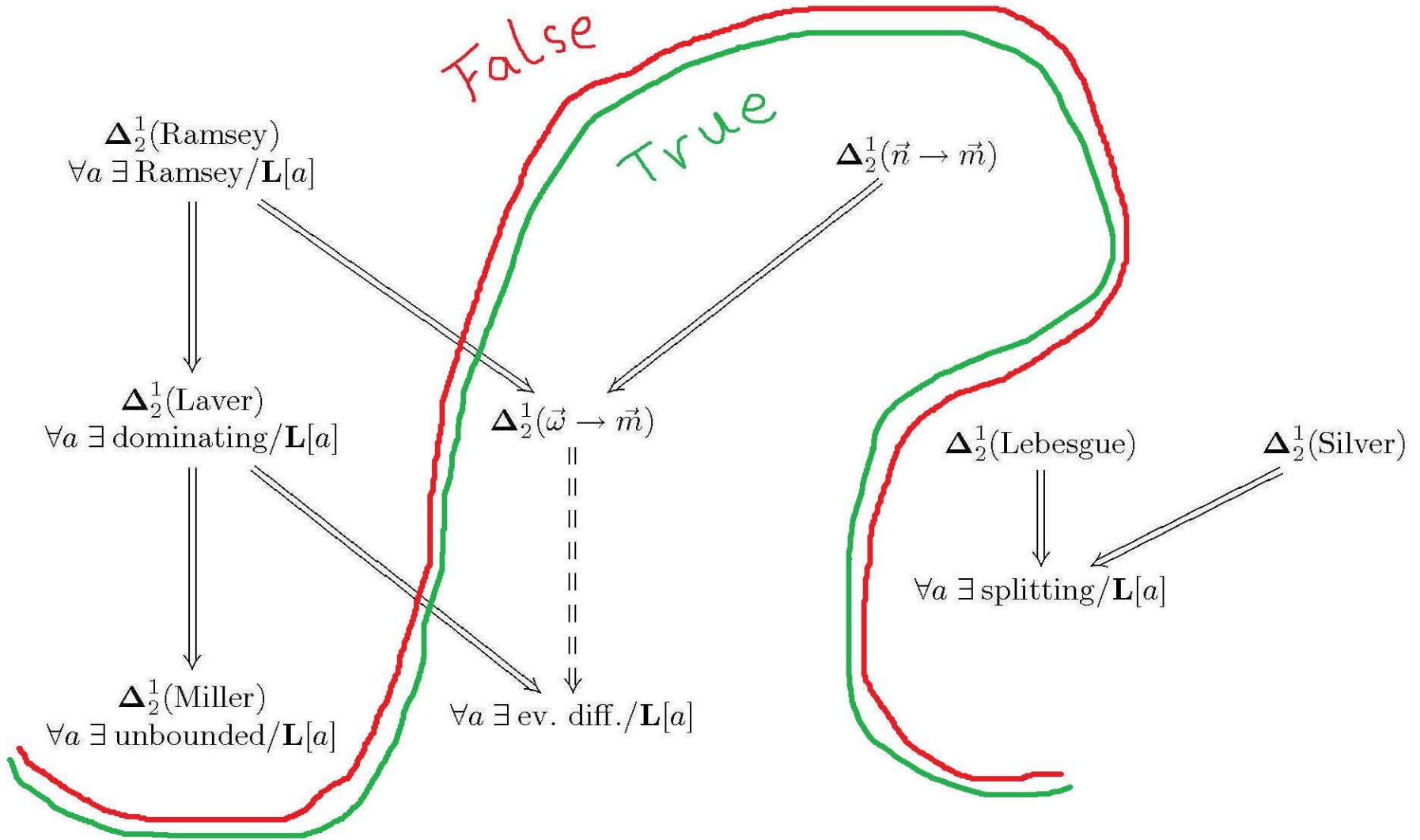
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By another result of Zapletal, the conjunction “ ω^ω -bounding and not adding splitting reals” is preserved in ω_1 -iterations, so:

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Open questions for Δ_2^1

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1. Is the implication $\Delta_2^1(\vec{\omega} \rightarrow \vec{m}) \Rightarrow \exists$ ev. diff. reals strict?

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2. Is there a characterization of $\Delta_2^1(\vec{\omega} \rightarrow \vec{m})$ and $\Delta_2^1(\vec{n} \rightarrow \vec{m})$ in terms of transcendence over \mathbb{L} ?

The property on the Σ_2^1 level

Recall that for Ramsey, Sacks, Miller and Laver measurability, Δ_2^1 and Σ_2^1 are equivalent.

Question: Are Δ_2^1 and Σ_2^1 equivalent for the polarized partition properties?

What we do know

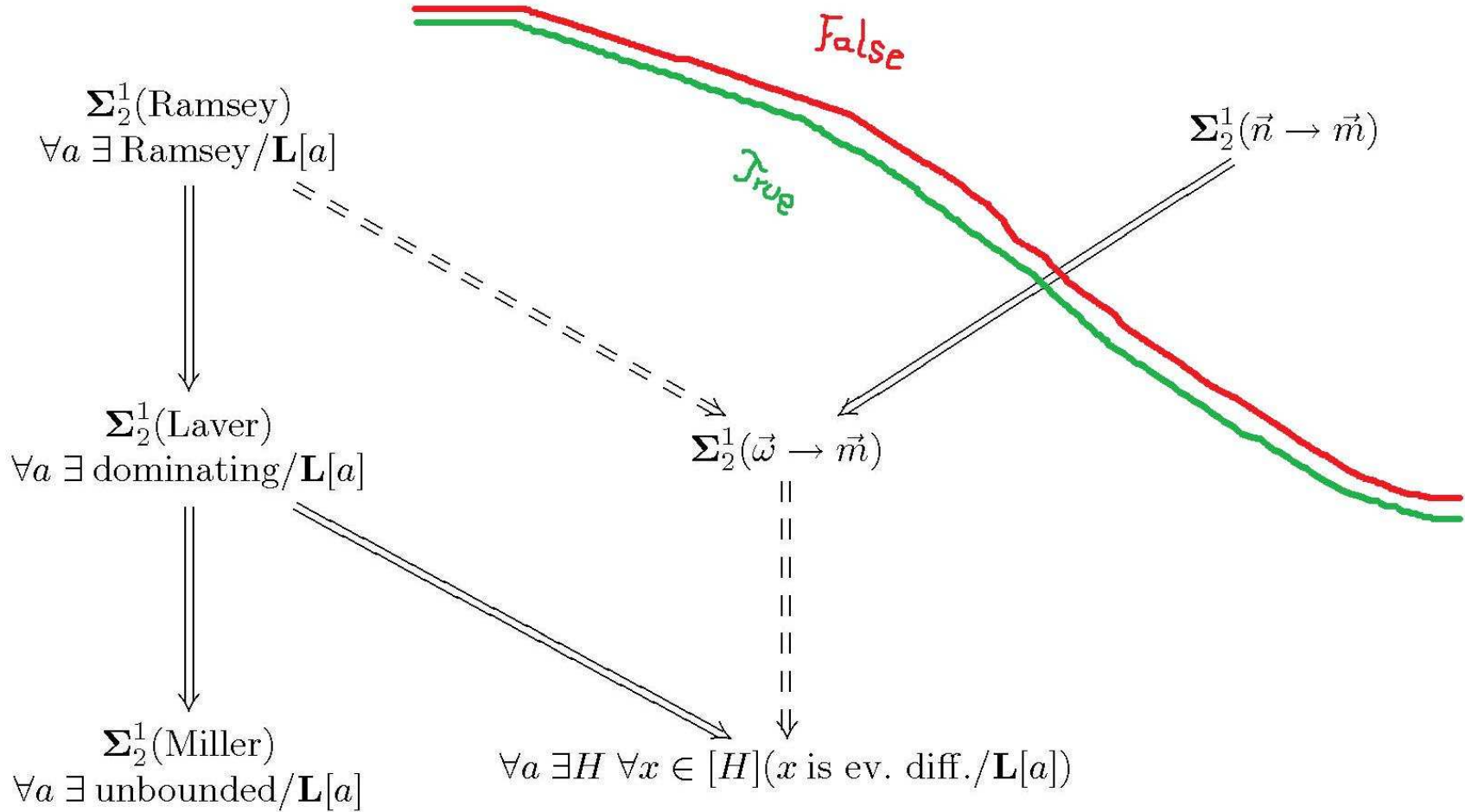
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Theorem. If $\Sigma_2^1(\vec{\omega} \rightarrow \vec{m})$ then $\forall a \exists H$ s.t. $\forall x \in [H] : x$ is eventually different over $L[a]$.

Theorem. In the Mathias model, $\Sigma_2^1(\text{Ramsey})$ holds while $\Sigma_2^1(\vec{n} \rightarrow \vec{m})$ fails.

Diagram of implications



Forcing $\Sigma_2^1(\vec{n} \rightarrow \vec{m})$

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[DiPrisco & Todorćević] use a forcing \mathbb{P}_{DPT} adding a whole *generic product* H_G with the following property:

For all Borel sets B in the ground model, (*)
 $B \cap [H_G]$ is relatively clopen in $[H_G]$.

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Corollary. There is a model where $\Sigma_2^1(\vec{n} \rightarrow \vec{m})$ holds but $\Sigma_2^1(\text{Miller})$ fails.

ありがとうございました