Polarized partitions on the second level of the projective hierarchy

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Joint work with Jörg Brendle (Kobe University, Japan)

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Finite polarized partitions 1

Question: Let $m_0, \ldots, m_{k-1}$ be fixed integers ≥ 2. Is it true that for every partition $\pi : \omega^k \to 2$ there is a sequence $H_0, \ldots, H_{k-1}$ of finite subsets of ω with $|H_i| = m_i$, which is *homogeneous* for π, i.e., π is constant on $\prod_{i<k} H_i$?

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Then $x \in \prod_i H_i \ \to \ \mathrm{ran}(x) \subseteq H$, hence it is easy to see that $\langle H_i \rangle_{i\in\omega}$ is homogeneous.

Finite polarized partitions 2

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- False if the n_i are too small, e.g.: $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \not\to \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.

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- False if the n_i are too small, e.g.: $\begin{pmatrix} 2 \\ 2 \end{pmatrix} \not\to \begin{pmatrix} 2 \\ 2 \end{pmatrix}$.
- However, by induction we can compute the n_i's from the m_i's so that the partition holds.

Finite polarized partitions 2

$$(?) \to (m_0)$$

Finite polarized partitions 2

$$(2m_0) \to (m_0)$$

Finite polarized partitions 2

$$\begin{pmatrix} 2m_0 \\ {\color{red}?} \end{pmatrix} \longrightarrow \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}$$

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If $\begin{pmatrix} n_0 \\ \cdot \\ \cdot \\ n_{k-1} \end{pmatrix} \to \begin{pmatrix} m_0 \\ \cdot \\ \cdot \\ m_{k-1} \end{pmatrix}$ holds and m_k is given, then by defining

$$n_k := 2 \cdot m_k \cdot \prod_{i<k} \binom{n_i}{m_i}$$

the partition $\begin{pmatrix} n_0 \\ \cdot \\ \cdot \\ n_k \end{pmatrix} \to \begin{pmatrix} m_0 \\ \cdot \\ \cdot \\ m_k \end{pmatrix}$ holds as well.

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- If π is analytic, then the partition holds by Silver's theorem (all analytic sets are Ramsey).

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Question: Let $\langle m_i \rangle_{i\in\omega}$ and $\langle n_i \rangle_{i\in\omega}$ be fixed integers ≥ 2. Is it true that for every partition $\pi : \prod_{i\in\omega} n_i \to 2$ there is a sequence $\langle H_i \rangle_{i\in\omega}$ with $|H_i| = m_i$ and $H_i \subseteq n_i$ which is homogeneous for π?

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This question came under attention only recently, in works of DiPrisco, Llopis, Todorčević and Zapletal.

Analytic partitions

- DiPrisco & Todorčević, 2004: $\begin{pmatrix} n_0 \\ n_1 \\ \cdot \\ \cdot \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \cdot \\ \cdot \end{pmatrix}$ holds for analytic partitions. Bounds $\langle n_i \rangle_{i\in\omega}$ are computed from $\langle m_i \rangle_{i\in\omega}$ in terms of recursive, but not primitive-recursive algorithm.

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Our goal is to see what happens at the next level of the projective hierarchy: $\boldsymbol{\Delta}^1_2$ and $\boldsymbol{\Sigma}^1_2$.

The second level

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E.g. if $V = L$ then

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On the other hand, if $\forall a \, (\aleph_1^{L[a]} = \aleph_0)$ then all $\mathbf{\Sigma}^1_2$ sets are Lebesgue-measurable, have the Baire property and the Ramsey property.

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In fact, regularity of $\mathbf{\Sigma}^1_2$ and $\mathbf{\Delta}^1_2$ sets indicates the level of *transcendence over L*.

Examples

- Judah & Shelah, 1989: the following are equivalent:
  1. all $\boldsymbol{\Delta}^1_2$ sets are Lebesgue-measurable,
  2. for all $a \in \omega^\omega$ there is a random real over $L[a]$.

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  1. all $\mathbf{\Delta}^1_2$ sets are Sacks-measurable (Marczewski-measurable)
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- Same for Miller-measurable (super-perfect trees) & unbounded reals, and Laver-measurable & dominating reals.

Examples 2

- Ikegami, 2008: for a wide class of forcings $\mathbb{P}$, one can canonically define $\mathbb{P}$-measurability and a notion of $\mathbb{P}$-transcendence, such that the following are equivalent:
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Advantage: we can control regularity on the second level by iterated forcing constructions over L.

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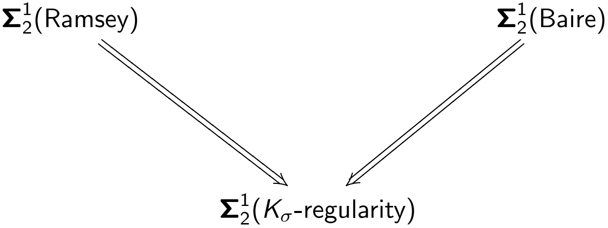
- Judah & Shelah, 1989: The following implications hold, and none other.



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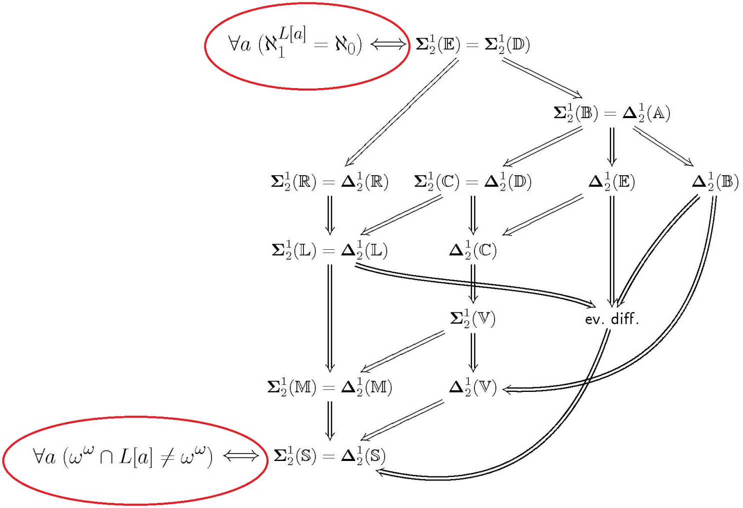

Diagram: Brendle & Löwe, *Eventually different functions and inaccessible cardinals*

Polarized partitions

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Polarized partitions

Back to the polarized partitions. What can we say about the statements

- "$\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ holds for $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$ partitions" and
- "$\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ holds for $\mathbf{\Sigma}^1_2/\mathbf{\Delta}^1_2$ partitions"?

Results

Easy to see:

1. If $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ holds for $\boldsymbol{\Sigma}^1_2 / \boldsymbol{\Delta}^1_2$ partitions, then $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ holds on the same level.

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If $\begin{pmatrix} \omega \\ \omega \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ holds for $\boldsymbol{\Delta}^1_2$ partitions, then for all $a \in \omega^\omega$ there is an eventually different real over $L[a]$.

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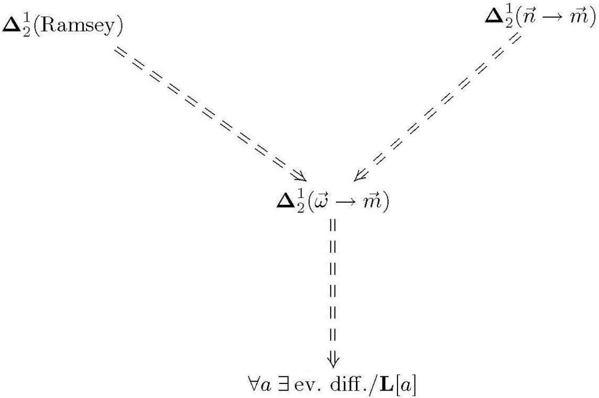
$\mathbf{\Delta}^1_2$ level: diagram of implications



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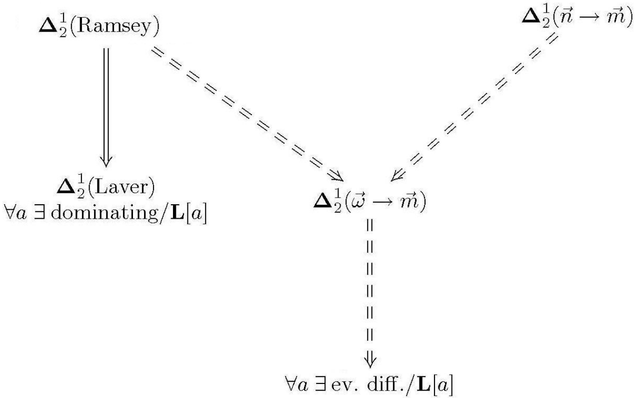

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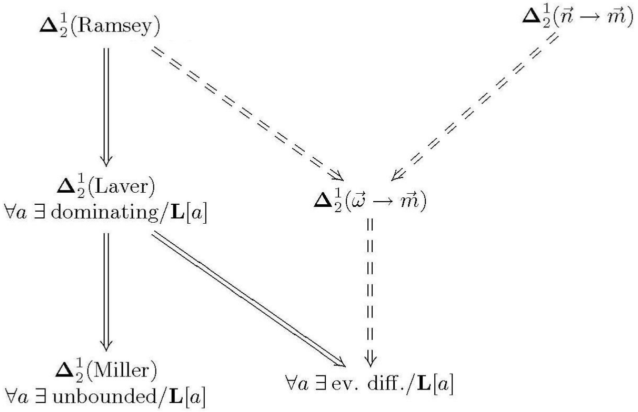

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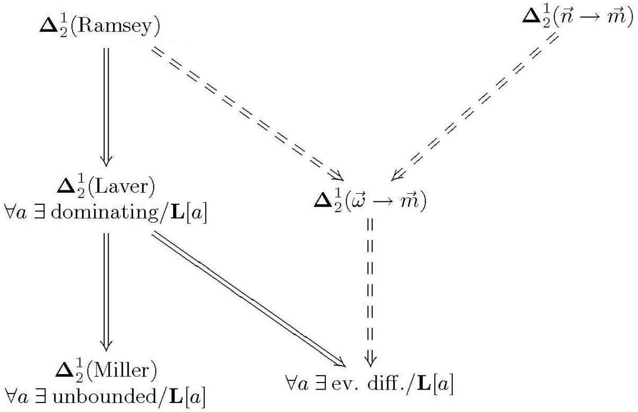

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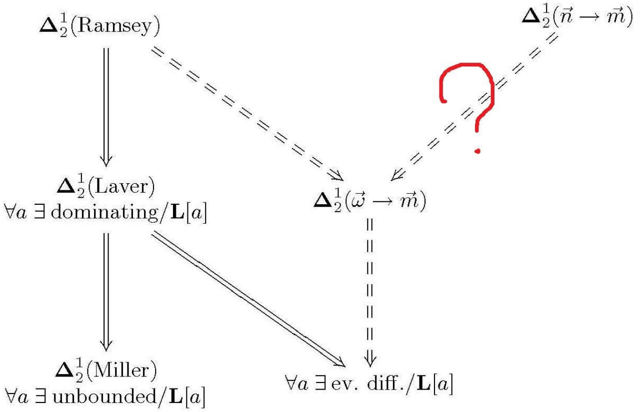

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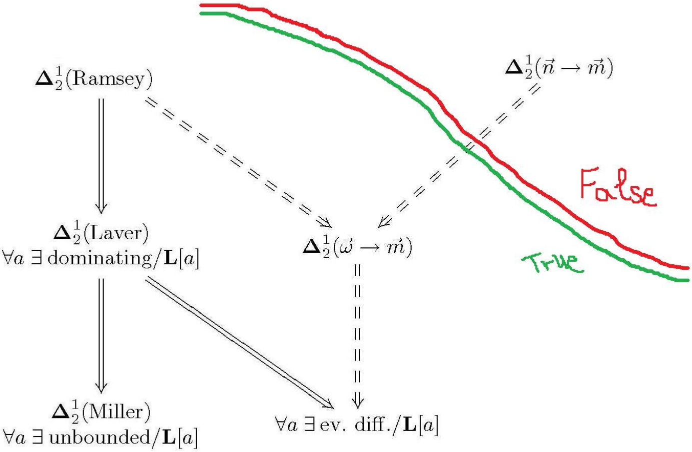
Mathias model

Theorem (Brendle-Kh)

In the ω_1-iteration of Mathias forcing starting from L, all $\mathbf{\Delta}^1_2$ sets are Ramsey but there is a $\mathbf{\Delta}^1_2$ partition which violates $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ (regardless of the values n_i, as long as they are computable from m_i).

$\boldsymbol{\Delta}^1_2$ level: diagram of implications



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In the proof, we use the fact that Mathias forcing satisfies the *Laver property*.

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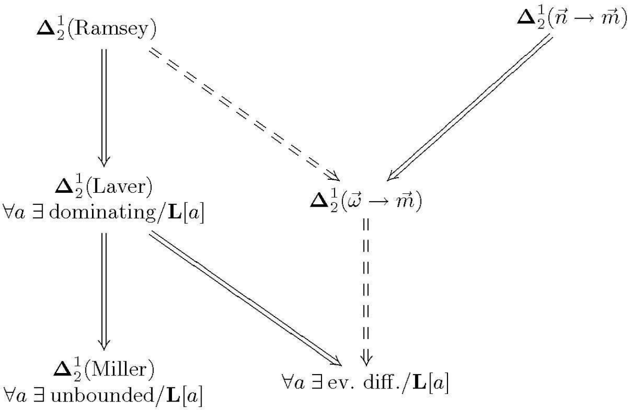

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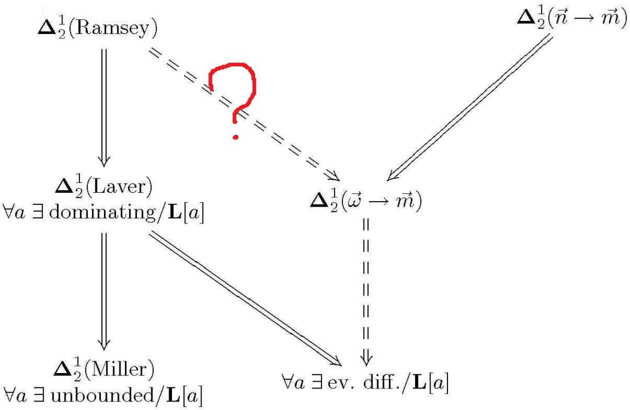

Creature forcing model

Theorem (Brendle-Kh)

There is a model in which $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ *holds for* $\boldsymbol{\Delta}^1_2$ *partitions, but not all* $\boldsymbol{\Delta}^1_2$ *sets are Miller-measurable (i.e., there are no unbounded reals).*

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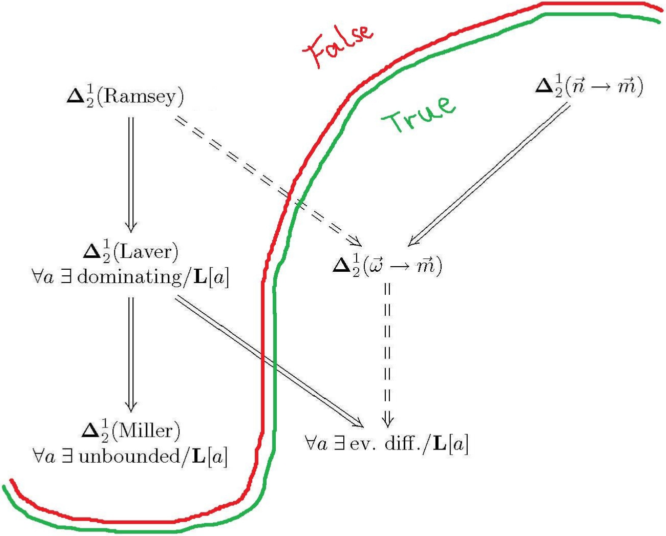

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Forcing conditions look like *uniform finitely branching trees* with a lower bound on the branching size. However, ordering is not simply inclusion.

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$\mathbb{P}_{\mathrm{KSZ}}$ adds a generic real $x_G := \bigcup\{\mathrm{stem}(p) \mid p \in G\}$, but the generic filter is not determined from the generic real in the usual fashion and $\mathbb{P}_{\mathrm{KSZ}}$ is not in general representable as $\mathcal{B}(\omega^\omega)/I$ for a σ-ideal I.

Bounds

Interesting fact: computations of bounds $\langle n_i \rangle_{i \in \omega}$ follows from purely forcing-theoretic considerations. Assuming all $m_i = 2$, we get:

$$n_i := 2^{\left(\left(2^{\prod_{j<i} n_j}\right)^i\right)}$$

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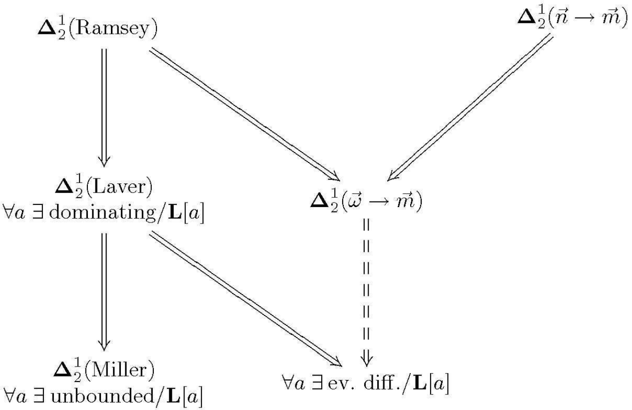

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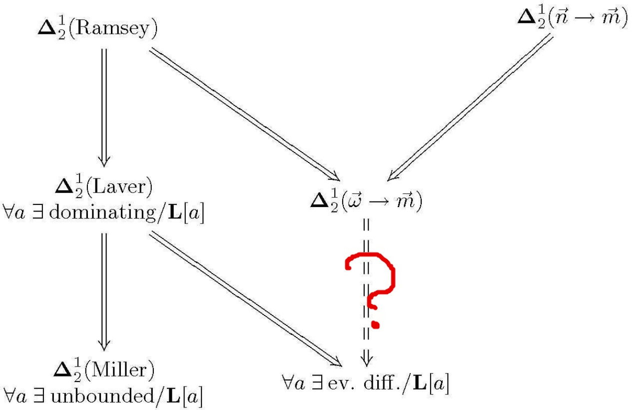

Open question

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Conjecture

"$\boldsymbol{\Delta}^1_2(\vec{\omega} \to \vec{m})$" fails in the Random model.

The $\mathbf{\Sigma}^1_2$ level

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Theorem (Brendle-Kh)

An ω_1-iteration of any (proper) forcing notion satisfying the clopification property yields a model where $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ *holds for* $\mathbf{\Sigma}^1_2$ *partitions.*

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Corollary

There is a model in which $\begin{pmatrix} n_0 \\ n_1 \\ \vdots \end{pmatrix} \to \begin{pmatrix} m_0 \\ m_1 \\ \vdots \end{pmatrix}$ *holds for* $\mathbf{\Sigma}^1_2$ *partitions but* $\mathbf{\Sigma}^1_2$*(Miller) fails.*

Computation of bounds

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3. For $\mathbf{\Delta}^1_2$ sets: same as above.
4. For $\mathbf{\Sigma}^1_2$ sets: currently much higher: non-primitive recursive.

Thank you!

Yurii Khomskii

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