ULTRAFILTERS AS A TOOL IN MODAL LOGIC

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1. Introduction and motivation

The aim of these notes is to help the reader develop basic intuitions about the properties of ultrafilters over sets with an eye to applications in modal logic. Ultrafilters are a very powerful tool which is also used in other areas of mathematics and logic such as topology and model theory.

In sections 2.5 and 2.6 of [1], ultrafilters are introduced for two distinct purposes. In section 2.5 the goal is to find m-saturated models. What this amounts to technically is that a Kripke model $\langle W, R, V\rangle$ may have *too few points*. The solution is to take a new and bigger set of points, viz. the set $\mathrm{Uf}(W)$ of all ultrafilters over W. We will see that there is a natural injective function $f\colon W \hookrightarrow \mathrm{Uf}(W)$, which makes it a good starting point for an extension of $\langle W, R, V\rangle$. Note that here we are interested in using *all* ultrafilters over W.

In section 2.6 of [1], the objective is to construct a new big model $\mathfrak{N}$ out of an (infinite) bunch of 'small' ones, say $\mathfrak{M}_0, \mathfrak{M}_1, \dots$, such that $\mathfrak{N} \Vdash \phi$ iff there is a *majority* of $\mathfrak{M}_i$ such that $\mathfrak{M}_i \Vdash \phi$. Ultrafilters are a tool that allows us to define a precise mathematical notion of such a 'majority model', called an *ultraproduct* of $\mathfrak{M}_0, \mathfrak{M}_1, \dots$. The key here is that we can let an ultrafilter 'decide' what counts as a majority among the $\mathfrak{M}_i$. So when we are dealing with a specific ultraproduct of models $\prod_U \mathfrak{M}_i/\mathcal{F}$, we are concerned with *one* suitably chosen ultrafilter $\mathcal{F}$ over $\mathbb{N}$ (or some other index set I) which tells us what counts as a majority of indices.

2. Filters

Definition 1. A *filter over* W is a collection of subsets $\mathcal{F} \subseteq \mathcal{P}(W)$ such that:

(1) $W \in \mathcal{F}$,
(2) $X, Y \in \mathcal{F}$ implies $X \cap Y \in \mathcal{F}$,
(3) $X \in \mathcal{F}$, $Y \subseteq W$ and $X \subseteq Y$ implies $Y \in \mathcal{F}$.

We say that a filter $\mathcal{F}$ is *proper* if $\mathcal{F} \neq \mathcal{P}(W)$.

Exercise 2.1. Show that $\mathcal{F}$ is proper iff $\emptyset \notin \mathcal{F}$.

Example. We give three ubiquitous examples of filters.

(1) We call $\{W\}$ the *trivial filter* over W.
(2) Similarly, $\mathcal{P}(W)$ is the *improper filter* over W.
(3) If W is infinite, the set

$$\mathrm{Cof}(W) := \{X \subseteq W \mid W \setminus X \text{ is finite}\}$$

of *cofinite* subsets of W is a proper filter over W (sometimes called the Fréchet filter).

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Exercise 2.2. Show that the examples above are indeed filters. Why do we need W to be infinite in the third example?

Exercise 2.3. Show that the intersection of a non-empty collection of filters is again a filter.

Let $\mathcal{X} \subseteq \mathcal{P}(W)$ be a family of sets. Since $\mathcal{P}(W)$ is a filter, we know that $S = \{\mathcal{F} \mid \mathcal{F} \text{ is a filter and } \mathcal{X} \subseteq \mathcal{F}\} \neq \emptyset$. By the exercise above, we know that $\mathcal{G} = \bigcap S$ is a filter, so we may call $\mathcal{G}$ the filter *generated* by $\mathcal{X} \subseteq \mathcal{P}(W)$.

Definition 2. Let $\mathcal{X} \subseteq \mathcal{P}(W)$ be a non-empty family of sets.

$$\mathcal{X}\!\uparrow := \{Y \subseteq W \mid \exists n \in \mathbb{N}\, \exists X_0, \dots, X_n \in \mathcal{X} : X_0 \cap X_1 \cap \cdots \cap X_n \subseteq Y\}.$$

Using the definition above we can give a more concrete characterization of the filter generated by $\mathcal{X}$: a set $Y \subseteq W$ is in the filter generated by $\mathcal{X}$ iff there exists a finite collection of $X_0, \dots, X_n \in \mathcal{X}$ such that $X_0 \cap \cdots \cap X_n \subseteq Y$. To put it more precisely:

Proposition 1. *If we let $\mathcal{X} \subseteq \mathcal{P}(W)$ be a non-empty family of sets then $\mathcal{X}\!\uparrow$ is the filter generated by $\mathcal{X}$.*

Exercise 2.4. Prove Proposition 1.

We say that $\mathcal{X}$ has the *finite intersection property* if every finite intersection of elements from $\mathcal{X}$ is non-empty.

Proposition 2. *Let $\mathcal{X} \subseteq \mathcal{P}(W)$ be a non-empty family of sets. There exists a proper filter $\mathcal{F}$ extending $\mathcal{X}$ iff $\mathcal{X}$ has the finite intersection property.*

Exercise 2.5. Prove Proposition 2 above.

3. Ultrafilters

Definition 3. Let $\mathcal{F}$ be a proper filter over W. We call $\mathcal{F}$

(1) an *ultrafilter* if for every $X \subseteq W$, either $X \in \mathcal{F}$ or $W \setminus X \in \mathcal{F}$,
(2) a *maximal filter* if for every filter $\mathcal{F}' \supseteq \mathcal{F}$, we have either $\mathcal{F}' = \mathcal{F}$ or $\mathcal{F}' = \mathcal{P}(W)$,
(3) a *prime filter* if for all $X, Y \subseteq W$, if $X \cup Y \in \mathcal{F}$ then either $X \in \mathcal{F}$ or $Y \in \mathcal{F}$.

Example (Principal ultrafilters)**.** For every $w \in W$, $\pi_w := \{X \subseteq W \mid w \in X\}$ is an ultrafilter called the *principal ultrafilter generated by w*.

Exercise 3.1. Show that π_w is indeed an ultrafilter over W for any $w \in W$.

Thus the elements of W provide us with examples of ultrafilters. In fact, we often think of the principal ultrafilters over W as a copy of W inside the set $\mathrm{Uf}(W)$ of all ultrafilters over W: consider the function $f \colon W \to \mathrm{Uf}(W)$, defined by $w \mapsto \pi_w$.

Exercise 3.2. Show that $f \colon W \to \mathrm{Uf}(W)$ is injective.

But what about maximal and prime filters? It turns out that we may think of them as alternative characterizations of ultrafilters:

Proposition 3. *Let $\mathcal{F}$ be a proper filter over W. The following are equivalent:*

(1) *$\mathcal{F}$ is an ultrafilter,*

(2) *$\mathcal{F}$ is maximal,*
(3) *$\mathcal{F}$ is prime.*

Exercise 3.3. Prove Proposition 3 above.

4. The Ultrafilter Theorem

Above we showed that the set of all ultrafilters over W is non-empty because it has a 'copy' of W inside of it (the principal ultrafilters). Can we prove that there are any other ultrafilters?

Theorem 4. *Let $\mathcal{F}$ be a proper filter over W. Then there exists an ultrafilter $\mathcal{F}'$ extending $\mathcal{F}$.*

Proof sketch. Use Zorn's Lemma on $\{\mathcal{G} \mid \mathcal{G}$ is a proper filter over W extending $\mathcal{F}\}$. □

Exercise 4.1. Let $\mathcal{F}$ be an ultrafilter over W. Show that if there exists $X \in \mathcal{F}$ with X finite then $\mathcal{F}$ is principal. Conclude that $\mathcal{F}$ is non-principal iff every $X \in \mathcal{F}$ is infinite.

Exercise 4.2. Let $\mathcal{F}$ be an ultrafilter over an infinite set W. Show that $\mathcal{F}$ is non-principal iff $\mathrm{Cof}(W) \subseteq \mathcal{F}$. Conclude that at least one non-principal ultrafilter exists.

These *non-principal ultrafilters* are the 'new' points we add to W.

Exercise 4.3. Show that any ultrafilter $\mathcal{F}$ over an infinite set W has uncountably many elements.

Acknowledgements

These notes are based almost entirely on material scattered throughout [1].

References

[1] Patrick Blackburn, Maarten de Rijke and Yde Venema: *Modal Logic*, 2001, Cambridge University Press.