An Elementary Construction of an Ultrafilter on $\aleph_1$, using the Axiom of Determinateness

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Abstract

In this article we construct a free and σ-complete ultrafilter on the set ω_1, using the Axiom of Determinateness (AD). First we define for each $V \subseteq \omega_1$ a perfect information game $G(V)$. The axiom AD guarantees that in each game $G(V)$, either the first or the second player has a winning strategy. For several different constructions of V from other sets, we construct winning strategies in the game $G(V)$. We show that these constructions correspond to closure properties of the set $\{V \mid$ player I has a winning strategy in $G(V)\}$. Finally we show that this set is a free and σ-complete ultrafilter on ω_1.

1 Introduction

The set of ordinals $\omega_1 = \{\alpha \in ORD \mid \alpha \text{ is finite or countable}\}$ is a set of cardinality $\aleph_1$, the first uncountable cardinality. Let $V \subseteq \omega_1$. We will define auxiliary perfect information games $G(V)$ in which two players (I and II) independently construct countably many countable ordinals as subsets of the set of rational numbers $\mathbb{Q}$ and try to 'force' the supremum into or out of V. The game can be sketched as follows:

- The two players maintain separate (countable) collections of (initially empty) subsets of $\mathbb{Q}$.
- Each round, both players add a finite number of elements to finitely many of their own subsets.
- At the end of the game, 'after' playing countably infinitely many rounds, both players have generated countably many subsets of $\mathbb{Q}$, each one representing either a countable ordinal (if the set is well-ordered) or 0 (if it isn't).

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- If the supremum of these ordinals (which is itself a countable ordinal) is in V, player I wins, otherwise player II wins.

It is well-known that in any two-player finite perfect information game G, one of the two players always has a winning strategy. The *Axiom of Determinateness* (AD) holds that this is also true for any countably infinite game G, i.e. any game G of countably infinite maximum duration, with a countably infinite selection of moves each turn. Since in the games $G(V)$, each move can be described as a finite sequence of pairs of an index from some countable index-set and a rational number, there are only countably many possible moves at each turn, and AD applies. Hence under AD, in any game $G(V)$ either player I can force the supremum to be an ordinal in V, or player II can force the supremum to be an ordinal in $\omega_1 - V$.

A *σ-complete ultrafilter* on a set X is a collection of subsets of X, closed under countable intersection and taking supersets, and such that for each set $V \subseteq X$, exactly one of $V, X - V$ is in the ultrafilter. An ultrafilter is *free* if it is not of the form $\{V \subseteq X \mid x \in V\}$ for some $x \in X$. Free ultrafilters are used in the study of certain classes of large ordinals, notably the measurable ordinals.

An ultrafilter U can be thought of as a partitioning of the subsets of X into 'large' subsets (those in U) and 'small' subsets (those not in U). The property 'player I can force the supremum to be in V' intuitively seems likely to be a type of 'largeness' property. And indeed, we will show that the collection of all sets $V \subseteq \omega_1$, such that player I can win the game $G(V)$, is a free and σ-complete ultrafilter on ω_1.

2 Definitions

Definition 1 Let $\mathcal{A}$ and $\mathcal{B}$ be two countably infinite, disjoint sets. For any subset $V \subseteq \omega_1$, define the game $G_{\mathcal{A},\mathcal{B}}(V)$ as follows:

- In any round $i \geq 1$, first player I selects a finite set a_i of pairs $(a, q) \in \mathcal{A} \times \mathbb{Q}$, and then player II selects a finite set b_i of pairs $(b, r) \in \mathcal{B} \times \mathbb{Q}$.
- Set $\mathcal{I} := \mathcal{A} \cup \mathcal{B}$, and define the *result* z by $z := a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \subseteq \mathcal{I} \times Q$.
- Let the function $\pi : \mathcal{P}(\mathbb{Q}) \to \omega_1$ be defined by

  $$\pi(R) := \begin{cases} \text{the order type of } (R, <) & \text{if } (R, <) \text{ is well-ordered} \\ 0 & \text{otherwise} \end{cases}$$

  Set $\Pi_{\mathcal{A},\mathcal{B}}(z) := \sup_{i \in \mathcal{I}} \pi(\{q \in \mathbb{Q} \mid (i, q) \in z\})$.
  Player I wins if $\Pi_{\mathcal{A},\mathcal{B}}(z) \in V$, otherwise player II wins.

If we interpret $\Pi_{\mathcal{A},\mathcal{B}}$ as a function from $\mathcal{P}(\mathcal{I} \times \mathbb{Q})$ to ω_1, then we can set

$$\underline{V}_{\mathcal{A},\mathcal{B}} := \Pi^{-1}_{\mathcal{A},\mathcal{B}}[V] = \{z \subseteq \mathcal{I} \times \mathbb{Q} \mid \Pi_{\mathcal{A},\mathcal{B}}(z) \in V\}$$

and the winning condition of the game can be reformulated as 'Player I wins if $z \in \underline{V}_{\mathcal{A},\mathcal{B}}$, otherwise player II wins'.

Remark 2 For technical reasons, it is necessary at some places in the proof to be able to react to one's own moves as if they had been made by the opponent. This is done by temporarily considering some of one's own subsets-under-construction to belong to the other player, for the purpose of reacting to the moves made in them. The 'index-structure' $(\mathcal{A}, \mathcal{B})$ used in the definition above is used to facilitate this. When the distinction is not important, we write $G(V)$, $\Pi(z)$ and $\underline{V}$ for $G_{\mathcal{A},\mathcal{B}}(V)$, $\Pi_{\mathcal{A},\mathcal{B}}(z)$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$. Note that $\Pi_{\mathcal{A},\mathcal{B}}$ and $\underline{V}_{\mathcal{A},\mathcal{B}}$ depend on $\mathcal{A} \cup \mathcal{B}$ only.

Formally, a *strategy* for player I in a game $G(V)$ is a function f which takes an arbitrary finite even-length sequence of moves $\langle a_1, b_1, \ldots, a_{k-1}, b_{k-1}\rangle$ (the 'moves up to then') as an argument and produces a move a_k. Player I *plays according to* a strategy f if $a_k = f(\langle a_1, b_1, \ldots, a_{k-1}, b_{k-1}\rangle)$ for all k. A strategy f for player I in $G(V)$ is called a *winning strategy* if player I, playing according to f, will win the game no matter what moves player II plays, i.e. if for any sequence $\langle b_1, b_2, \ldots\rangle$ of moves for player II, if we set $a_k = f(\langle a_1, b_1, \ldots, a_{k-1}, b_{k-1}\rangle)$ for all k, then $a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \underline{V}$. Strategies and winning strategies for II are defined in a like manner.

Theorem 3 *Under AD, the set*

$$U := \{V \subseteq \omega_1 \mid \textit{player } I \textit{ has a winning strategy in } G(V)\}$$

is a free and σ-complete ultrafilter on ω_1.

The proof of this follows after some lemmas.

3 Lemmas

First we need an auxiliary lemma to justify writing $G(V)$ for $G_{\mathcal{A},\mathcal{B}}(V)$ when there are no other index-structures involved.

Lemma 4 *If player I [II] has a winning strategy in a game $G_{\mathcal{A},\mathcal{B}}(V)$, then I [II] has a winning strategy in $G_{\mathcal{A}',\mathcal{B}'}(V)$ for any index structure $(\mathcal{A}', \mathcal{B}')$.*

Proof:
Since $\mathcal{A}, \mathcal{B}, \mathcal{A}', \mathcal{B}'$ are all countably infinite, there exist bijective mappings $\mathcal{A} \leftrightarrow \mathcal{A}'$ and $\mathcal{B} \leftrightarrow \mathcal{B}'$. These in turn induce bijective mappings between the moves, games and (winning) strategies of $G_{\mathcal{A},\mathcal{B}}(V)$ and those of $G_{\mathcal{A}',\mathcal{B}'}(V)$.

$\square$

The following lemmas each correspond to a different property. of an ultrafilter.

Lemma 5 *If player I has a winning strategy in the game $G(V)$, and $V \subseteq W$, then I has a winning strategy in $G(W)$.*

Proof:
Let f be a winning strategy for I in $G(V)$, and suppose that player I plays according to f in the game $G(W)$. Then for any sequence of moves $b_1, b_2, \ldots$ for player II,

$$z = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \underline{V} \subseteq \underline{W}$$

So f is a winning strategy for I in $G(W)$ as well.

$\square$

Lemma 6 *If V is a singleton, then player II has a winning strategy in the game $G(V)$.*

Proof:
Suppose that $V = \{\alpha\}$ for some countable ordinal α. Then player II can force $\Pi(z)$ to be above α. For let $\mathcal{B}$ be player II's index-set, let $b \in \mathcal{B}$ and let $\{r_1, r_2, r_3, \ldots\} \subseteq \mathbb{Q}$ be a countable set of order-type $\alpha + 1$. If in each round i, player II plays (b, r_i), then for any sequence of moves $a_1, a_2, \ldots$ for player I,

$$\Pi(z) = \Pi(\,(\{b\} \times \{r_1, r_2, r_3 \ldots\}) \cup a_1 \cup a_2 \cup \ldots) \geq \pi(\{r_1, r_2, r_3, \ldots\}) = \alpha + 1$$

This is a winning strategy for player II.

$\square$

Lemma 7 *If player I has a winning strategy in the game $G(V)$, then player II has a winning strategy in $G(\omega_1 - V)$, and vica versa.*

Proof:
Suppose that player I has a winning strategy f in the game $G_{\mathcal{A},\mathcal{B}}(V)$. Then this is also a winning strategy for player II in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, except that since II does not have the first move, player II's response to any move is always 'delayed' by one round. Formally, we construct a strategy g for player II by setting

$$g(\langle b_1, a_1, b_2, a_2, \ldots, a_{k-1}, b_k \rangle) := f(\langle a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1} \rangle)$$

For any sequence of moves $b_1, b_2, \ldots$ for player I in $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, if player II plays according to g, then the resulting sequence of moves $\langle b_1, a_1, b_2, a_2, \ldots \rangle$ corresponds to a sequence of moves $\langle a_1, b_1, a_2, b_2, \ldots \rangle$ in the game $G_{\mathcal{A},\mathcal{B}}(V)$, such that player I's moves are according to the strategy f. Hence we have

$$z = b_1 \cup a_1 \cup b_2 \cup a_2 \cup \ldots = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \underline{V}$$

So g is a winning strategy for player II in $G_{\mathcal{B},\mathcal{A}}(V)$.

Now suppose that player II has a winning strategy g in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$. Then this is also a winning strategy for player I in the game $G_{\mathcal{A},\mathcal{B}}(V)$, except that player I has a first move in which she does nothing. Formally, we construct a strategy f for player I by setting

$$f(\langle\rangle) := \emptyset, f(\langle a_1, b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1} \rangle) := g(\langle b_1, a_2, b_2, \ldots, a_{k-1}, b_{k-1} \rangle)$$

For any sequence of moves $b_1, b_2, \ldots$ for player II in $G_{\mathcal{A},\mathcal{B}}(V)$, if player I plays according to f, then the resulting sequence of moves $\langle \emptyset, b_1, a_2, b_2, a_3, \ldots \rangle$ corresponds to a sequence of moves $\langle b_1, a_2, b_2, a_3, \ldots \rangle$ in the game $G_{\mathcal{B},\mathcal{A}}(\omega_1 - V)$, such that player II's moves are according to the strategy g. Hence we have

$$z = \emptyset \cup b_1 \cup a_2 \cup b_2 \cup a_3 \cup \ldots = b_1 \cup a_2 \cup b_2 \cup a_3 \cup \ldots \in \underline{V}$$

So f is a winning strategy for player I in $G_{\mathcal{A},\mathcal{B}}(V)$.

□

Lemma 8 *Let $(V^i)_{i \geq 0}$ be a countable sequence of subsets of ω_1. If for all $i \geq 0$, player II has a winning strategy in $G(V^i)$, then II has a winning strategy in $G(\bigcup_{i \geq 0} V^i)$.*

Proof:
Assume that for all $i \geq 0$, player II has a winning strategy in $G(V^i)$. Let $(\mathcal{A}, \mathcal{B})$ be an index-structure for players I and II, and let $(\mathcal{B}^i)_{i \geq 0}$ be a partitioning of $\mathcal{B}$ into a countably infinite number of disjoint countably infinite sets. Define $\mathcal{A}^i = (\mathcal{A} \cup \mathcal{B}) - \mathcal{B}^i$ for $i \geq 1$. Then for all $i \geq 1$, $(\mathcal{A}^i, \mathcal{B}^i)$ is a valid index-structure. By Lemma 4 we can find[1] winning strategies g^i for player II in each of the games $G_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$.

Now let $\langle a_1, b_1, a_2, b_2, \ldots \rangle$ be a sequence of moves, and suppose that all for $i \geq 0$ and $k \geq 1$, $b_{2^i(2k-1)} \in \mathcal{B}^i$. If we define for $i \geq 0$ and $k \geq 1$,

$$\begin{aligned}
a_1^i &= a_1 \cup b_1 \cup a_2 \cup \ldots \cup a_{2^i} \\
a_k^i &= a_{2^i(2k-3)+1} \cup b_{2^i(2k-3)+1} \cup a_{2^i(2k-3)+2} \cup \ldots \cup a_{2^i(2k-1)} \text{ if } k > 1 \\
b_k^i &= b_{2^i(2k-1)}
\end{aligned}$$

then for all $i \geq 0$ and $k \geq 1$, a_k^i and b_k^i are finite subsets of $\mathcal{A}^i \times \mathbb{Q}$ and $\mathcal{B}^i \times \mathbb{Q}$, so $\langle a_1^i, b_1^i, a_2^i, b_2^i \ldots \rangle$ is a valid sequence of moves in the game $G_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$. So construct the strategy g for player II in $G_{\mathcal{A},\mathcal{B}}(V)$ by setting, for $i \geq 0$ and $k \geq 1$,

$$g(\langle a_1, b_1, a_2, \ldots, a_{2^i(2k-1)} \rangle) = g^i(\langle a_1^i, b_1^i, a_2^i, \ldots, a_k^i \rangle)$$

It can easily be shown (inductively) that for all $i \geq 0$ and $k \geq 1$, $b_{2^i(2k-1)} \in \mathcal{B}^i$, and hence the strategy g is well-defined. Moreover, for all $i \geq 0$ and $k \geq 1$,

$$b_k^i = g^i(\langle a_1^i, b_1^i, a_2^i, \ldots, a_k^i \rangle)$$

so for all $i \geq 0$, $\langle a_1^i, b_1^i, a_2^i, b_2^i \ldots \rangle$ is a sequence of moves in the game $G_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$ in which player II's moves are according to the strategy g^i. It follows that for all $i \geq 0$,

$$z = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots = a_1^i \cup b_1^i \cup a_2^i \cup b_2^i \cup \ldots \in \underline{\omega_1 - V^i}$$

[1] This does not require the Axiom of Choice. Consider an auxiliary game where player I first selects $i \geq 0$, and the players then play $G_{\mathcal{A}^i,\mathcal{B}^i}(V^i)$. Obviously player I can't have a winning strategy, so by AD, player II has a winning strategy, from which the strategies g^i can be extracted.

and therefore

$$z = a_1 \cup b_1 \cup a_2 \cup b_2 \cup \ldots \in \bigcap_{i \geq 0} \underline{\omega_1 - V^i} = \underline{\omega_1 - V}$$

So g is a winning strategy for player II in the game $G_{\mathcal{A},\mathcal{B}}(V)$.

$\square$

4 Conclusions

Theorem 3 *Under AD, the set*

$$U := \{V \subseteq \omega_1 \mid \textit{player } I \textit{ has a winning strategy in } G(V)\}$$

is a free and σ-complete ultrafilter on ω_1.

Proof:
In any game $G_{\mathcal{A},\mathcal{B}}(V)$, there are only countably many possible moves each turn, since $\mathcal{A}$, $\mathcal{B}$ and $\mathbb{Q}$ are all countable, and therefore there are only countably many different finite collections of pairs $(a, q) \in \mathcal{A} \times \mathbb{Q}$ or $(b, q) \in \mathcal{B} \times \mathbb{Q}$. Hence the Axiom of Determinateness applies, and we have that for all $V \subseteq \omega_1$:

$$\begin{array}{lcl} V \in U & \Leftrightarrow & \text{player } I \text{ has a winning strategy in } G(V) \\ V \notin U & \Leftrightarrow & \text{player } II \text{ has a winning strategy in } G(V) \end{array}$$

The previous lemmas therefore yield the following properties of U:

1. For any $V, W \subseteq \omega_1$, if $V \in U$ and $V \subseteq W$, then $W \in U$.
2. For any $V \subseteq \omega_1$, if V is a singleton, then $V \notin U$.
3. For any $V \subseteq \omega_1$, $V \in U$ if and only if $\omega_1 - V \notin U$.
4. For any sequence $V_i \subseteq \omega_1$, if $V_i \notin U$ for all $i \geq 0$, then $\bigcup_{i \geq 0} V_i \notin U$.

and from the third and fourth property we can derive

5 For any sequence $V_i \subseteq \omega_1$, if $V_i \in U$ for all $i \geq 0$, then $\bigcap_{i \geq 0} V_i \in U$.

So U is a free and σ-complete ultrafilter on ω_1.

$\square$

Example 9 Some examples of $V \in U$ and the corresponding winning strategies for player I in $G(V)$ are:

- $V = \omega_1$: Trivial.
- V is co-countable: Player I constructs the ordinal $\sup(\omega_1 - V) + 1$.
- $V = \{\omega \cdot \alpha \mid \alpha \in \omega_1\}$: There exists an order-isomorphic bijection $h : \mathcal{B} \times \mathbb{Q} \to \mathcal{A} \times \mathbb{Q}_{<0}$. Each turn player I copies the moves player II makes using this bijection h, and then add the points $\{0, 1, 2, \ldots, \mathrm{k}\}$ to each one of her non-empty sets, where k is the number of the current round. In this manner, for each subset of $\mathbb{Q}$ produced by player II, player I produces a subset that is order-isomorphic, followed by a 'tail' of ω points.

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