# Take-Home Exam AST: Solutions

**67.** Prove the following generalization of Theorem 4.10:
If $\varepsilon$ is well-founded and extensional on the set $A$, then there is a unique transitive set $B$ such that $(B, \in) \cong (A, \varepsilon)$.
(This is called *Mostowski's Collapsing Lemma*, cf. Lemma 7.52 p. 79. Erasing the well-foundedness condition results in a statement —an example of an *Anti-Foundation Axiom*— that contradicts the Foundation Axiom.)
Generalize to the following theorem: If $\varepsilon$, next to satisfying the conditions from Theorem 4.13, is *extensional* on the class $\mathbf{U}$ (elements in $\mathbf{U}$ with the same $\varepsilon$-predecessors are the same), then there is a unique transitive class $T$ such that $(\mathbf{U}, \varepsilon) \cong (T, \in)$.
*Solution.*
Let $\varepsilon$ be a well-founded and extensional relation on a class $\mathbf{U}$ such that for all $a \in \mathbf{U}$, $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}$ is a set. Then by applying Theorem 4.13 to the operator $H$ with $H(f) = \mathrm{Ran}(f)$ for all functions $f$, we obtain a unique operator $F : \mathbf{U} \to \mathbf{V}$ such that for all $a \in \mathbf{U}$,

$$(*) \qquad F(a) = \{F(b) \mid b \in \mathbf{U}, b \,\varepsilon\, a\}$$

Set $T = F[\mathbf{U}]$. Then $T$ is transitive and $F$ is an isomorphism between $(\mathbf{U}, \varepsilon)$ and $(T. \in)$:

1. $F$ is surjective, by definition of $T$.
2. $T$ is transitive, for if $x \in T$, then for some $a \in \mathbf{U}$, $x = F(a) = \{F(b) \mid b \in \mathbf{U}, b \,\varepsilon\, a\} \subset T$.
3. $F$ is injective. For otherwise, let $a \in \mathbf{U}$ be a $\varepsilon$-minimal element such that $\exists b \in \mathbf{U} : a \neq b \wedge F(a) = F(b)$ (such an element exists, since $\varepsilon$ is well-founded on $\mathbf{U}$). Now for any $x \in \mathbf{U}$ with $x \,\varepsilon\, a$, if $F(x) = F(y)$ for some $y \in \mathbf{U}$, then $x = y$ (because of the $\varepsilon$-minimality of $a$). Since $\{F(x) \mid x \in \mathbf{U}, x \,\varepsilon\, a\} = F(a) = F(b) = \{F(y) \mid y \in \mathbf{U}, y \,\varepsilon\, b\}$. it follows that for any $x \,\varepsilon\, a$ or $y \,\varepsilon\, b$ we can pick an $y \,\varepsilon\, b$ or $x \,\varepsilon\, a$ with $F(x) = F(y)$ and therefore $x = y$. We conclude that for all $x \in \mathbf{U}$, $x \,\varepsilon\, a \iff x \,\varepsilon\, b$, contradicting the extensionality of $\varepsilon$ on $\mathbf{U}$.
4. $F$ is an isomorphy. For $x, y \in \mathbf{U}$, if $x \,\varepsilon\, y$, then by definition of $F$, $F(x) \in F(y)$. Conversely, if $F(x) \in F(y)$, then for some $z \in \mathbf{U}$ with $z \,\varepsilon\, y$, $F(z) = F(x)$, so by injectivity of $F$, $x = z \,\varepsilon\, y$.

For any transitive class $T'$ and any isomorphism $F' : (\mathbf{U}, \varepsilon) \to (T'. \in)$, $F'$ must satisfy (*), and by uniqueness of $F$ this implies $F' = F$ and $T' = F'[\mathbf{U}] = T$. So $T$ is unique.
Mostowski's Collapsing Lemma is merely the restriction of the above theorem to the case where $\mathbf{U}$ is a set. Note that in that case, the condition that that for all $a \in \mathbf{U}$, $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\} \subset \mathbf{U}$ is a set, is trivially satisfied. That $T$ is a set follows from Substitution.
If we erase the wellfoundedness condition, we could apply Mostowski's Collapsing Lemma to the set $A = \{0\}$ and the relation $\varepsilon = \{(0, 0)\}$, to obtain a transitive set $T = \{x\}$ with $x = \{x\}$, contradicting the Foundation Axiom. □

**104** (H. Rubin) Assume the Foundation Axiom. Show: AC is equivalent with the statement that powersets of ordinals have well-orderings.
*Solution.*
The one implication is trivial, Since AC implies all sets have well-orderings, one direction is trivial. So we will assume that powersets of ordinals have well-orderings, and try to show AC.

Let $a$ be an arbitrary set. From the Foundation axiom, we know that $a \subset V_\kappa$ for some ordinal $\kappa$. Let $\lambda = \Gamma(V_\kappa)$, and fix a well-ordering $\prec_{\wp(\lambda)}$ of $\wp(\lambda)$,. We will use this ordening to recursively define well-orderings $\prec_\alpha$ for all $\alpha \leq \kappa$. Then $\prec_\kappa$ restricted to $a$ will well-order $a$.

1. For $V_0$, let $\prec_0$ be the trivial ordering.
2. For $\alpha+1$, note that since $V_\alpha <_1 \lambda$, the well-ordering $\prec_\alpha$ induces an order-preserving injection $\phi_\alpha \colon (V_\alpha, \prec_\alpha) \to (\lambda, \in)$, which in turn induces an injection $\psi_{\alpha+1} \colon \wp(V_\alpha) \ni a \to \phi_\alpha[a] \in \wp(\lambda)$. If we define the ordering $\prec_{\alpha+1}$ of $V_{\alpha+1}$ by setting, for $a, b \in V_{\alpha+1}$,
$$a \prec_{\alpha+1} b \ \equiv_{def} \ \psi_{\alpha+1}(a) \prec_{\wp(\lambda)} \psi_{\alpha+1}(b)$$
then $(V_{\alpha+1}, \prec_{\alpha+1})$ is order-isomorphic to $(\psi_{\alpha+1}[V_{\alpha+1}], \prec_{\wp(\lambda)})$, and hence a well-ordering.
3. For limits $\gamma$, we can define a well-ordering $\prec_\gamma$ by setting, for $a, b \in V_\gamma$,
$$a \prec_\gamma b \ \equiv_{def} \ (\rho(a) < \rho(b)) \ \vee \ ((\rho(a) = \rho(b)) \ \wedge \ (a \prec_{\rho(a)+1} b))$$
It is easily seen that $\prec_\gamma$ is a linear ordering, and that for any subset $X \subset V_\gamma$, if $Y = \text{Bottom}(X)$, then the $\prec_{\rho(Y)}$-minimal element of $Y$ is also the $\prec_\gamma$ minimal element of $X$. Hence $\prec_\gamma$ is a well-ordering. □

**113**

1. Give a direct proof, using AC, but not using Lemma 6.19, that a countable union of countable sets is countable. In particular, $\omega_1$ is not a countable union of countable sets. (It is known that this is unprovable without AC).
2. Show *without using AC* that $\omega_2$ is not a countable union of countable sets.

*Solution.*

1. Let $(A_i)_{i\in\omega}$ be a countable collection of countable sets, and let $A = \bigcup_{i\in\omega} A_i$. Using AC, pick for each $i \in \omega$ an injection $\phi_i : A_i \to \omega$. Now we can define the injection $\phi : A \to \omega \times \omega$ by setting, for $a \in A$, $\phi(a) = (i_a, \phi_{i_a}(a))$, where $i_a$ is the least number such that $a \in A_{i_a}$. Since by section 4.8 there is an injection $\psi : \omega \times \omega \to \omega$, it follows that $\psi \circ \phi : A \to \omega$ is an injection, and hence $A$ is countable.
2. Let $(A_i)_{i\in\omega}$ be a countable collection of countable sets, and assume $\omega_2 = \bigcup_{i\in\omega} A_i$. For any $i \in \omega$, $A_i$ is a set of ordinals ordered by $\in$, of countable order-type $\alpha_i$. This means that *without using AC* we can define canonical bijections $\phi_i : A_i \to \alpha_i \subset \omega_1$. Now we can define the injection $\phi : \omega_2 \to \omega \times \omega_1$ by setting, for $\xi \in \omega_2$, $\phi(\xi) = (i_\xi, \phi_{i_\xi}(\xi))$, where $i_\xi$ is the least number such that $\xi \in A_{i_\xi}$. Since by section 4.8 there is an injection $\psi : \omega_1 \times \omega_1 \to \omega_1$, it follows that $\psi \circ \phi : \omega_2 \to \omega_1$ is an injection, a contradiction. □

**146** An "$\omega$-incompleteness phenomenon".

1. The properties of Def from Corollary 7.17 suffice to show, for every *specific* natural number $n$, that $\mathrm{L}_n = \mathbf{V}_n$.
2. Show: if ZF is consistent, then it stays consistent upon the addition of (i) the properties of Def from Corollary 7.17 (*not* the definition of Def!), and (ii) the statement $\exists n \in \omega(\mathrm{L}_n \neq \mathbf{V}_n)$.

*Solution.*

1. By Corollary 7.18 it is possible to show, for any specific natural number $k$, that if $B \subset A$ has cardinality $\leq k$, then $B \in \text{Def}(A)$. Now let $n$ be a specific natural number and let $k = |\mathbf{V}_n|$, then it follows that for all $i \leq n$ and all $B \subset V_i$, $B \in \text{Def}(V_i)$. Hence for all $i \leq n$, $\text{Def}(V_i) = \wp(V_i) = V_{i+1}$, and by induction on $i$, for all $i \leq n$ $\mathrm{L}_i = \mathbf{V}_i$. In particular, $\mathrm{L}_n = \mathbf{V}_n$.

2. By the Compactness Theorem, it suffices to show that any *finite* collection of instances of Corollary 7.17 is consistent with ZF + $\exists n \in \omega(\mathrm{L}_n \neq \mathbf{V}_n)$. So let $\mathbf{V}$ be a model of ZF, and let $\Phi_1, \Phi_2, \ldots, \Phi_k$ be a collection of formulas, using free variables $a_1, \ldots, a_m$. We will assign the language symbol Def an interpretation Def$^*$ such that for all sets $A$,

   (a) $\mathrm{Def}^*(A) \subset \wp(A)$

   (b) For $i \leq k$ and $a_1, \ldots, a_m \in A$, $\{a \in A \mid \Phi_i^A(a, a_1, \ldots, a_m)\} \in \mathrm{Def}^*(A)$

   (c) $\exists n \in \omega(\mathrm{L}_n \neq \wp(\mathbf{V}_n))$.

   The interpretation itself is straightforward: simply set

   $$\mathrm{Def}^*(A) = \big\{\{a \in A \mid \Phi_i^A(a, a_1, \ldots, a_m)\} \mid i \leq k, a_1, \ldots, a_m \in A\big\}$$

   This satisfies our first two conditions by definition. As for the third, note that $|\mathrm{Def}^*(A)| \leq k|A|^m$, so if we pick $n$ such that $k|\mathbf{V}_n|^m < 2^{|\mathbf{V}_n|}$, then $\mathrm{Def}^*(\mathbf{V}_n) \subsetneq \wp(\mathbf{V}_n) = V_{n+1}$, and as a consequence $\mathrm{L}_{n+1} = \mathrm{Def}^*(\mathrm{L}_n) \subset \mathrm{Def}^*(\mathbf{V}_n) \subsetneq V_{n+1}$. □

**155** Show that ZF (provided consistent) is not finitely axiomatizable over *Zermelo set theory* Z (axiomatized by all axioms except Substitution). That is: there is no sentence $\Phi$ consistent with Z such that Z+$\Phi$ proves (all instances of) the Substitution Axiom.

*Solution.*

Suppose that $\Phi$ is a sentence consistent with Z such that such that Z+$\Phi$ proves (all instances of) the Substitution Axiom. Then there exist models of Z+$\Phi$, and any such model is a model of ZF. ZF proves that if ($\Phi \wedge$ Power $\wedge$ Infinity) holds, then (by Reflection) there exists an ordinal $\alpha$ such that ($\Phi \wedge$ Power $\wedge$ Infinity) hold in $\mathbf{V}_\alpha$. Now if the Power and Infinity Axioms hold, then $\alpha$ must be a limit ordinal $> \omega$, and then $\mathbf{V}_\alpha$ must be a model of Z as well as of $\Phi$. So any model $\mathbf{V}$ of Z+$\Phi$ must contain an ordinal $\alpha$ such that $\mathbf{V}_\alpha$ is a model of Z+$\Phi$.

So let $\mathbf{V}$ be a model of Z+$\Phi$, and let $\alpha \in$ OR be the smallest ordinal such that $\mathbf{V}_\alpha$ is a model of Z+$\Phi$. Since $\mathbf{V}_\alpha$ is a model of Z+$\Phi$, it must contain an ordinal $\beta < \alpha$ such that $\mathbf{V}_\beta^{\mathbf{V}_\alpha}$ is also a model of Z+$\Phi$. It is easily seen that $\mathbf{V}_\beta^{\mathbf{V}_\alpha} = \mathbf{V}_\beta$, contradicting the minimality of $\alpha$. □

**189** Show:

1. Every $\Sigma_1$ statement provable in ZFC (or ZF+$\mathbf{V} = \mathbf{L}$) is also provable in ZF.

2. The same thing holds for statements of the form $\forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$ where $\Phi$ is $\Sigma_1$.

*Solution.*

The first statement is merely a special case of the second statement (where the ordinal $\alpha$ is not used in $\Phi$). So assume that $\Phi$ is $\Sigma_1$, and that $\forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$ is provable in ZFC or ZF+$\mathbf{V} = \mathbf{L}$.. Let $\mathbf{V}$ be a model of ZF. Then $\mathbf{L}^\mathbf{V}$ is a model of ZFC and of ZF+$\mathbf{V} = \mathbf{L}$, so $\mathbf{L}^\mathbf{V} \models \forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$. Since $\mathrm{OR}^{\mathbf{L}^\mathbf{V}} = \mathrm{OR}^\mathbf{V}$ and since by upward persistence we have $\mathbf{V} \models \forall\alpha\in\mathrm{OR}(\Phi^\mathbf{L}(\alpha) \to \Phi(\alpha))$, it follows that $\mathbf{V} \models \forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$. This holds for all models of ZF, so ZF proves $\forall\alpha\in\mathrm{OR}\,\Phi(\alpha)$. □