Solutions for Exercises

Chapter 7

163. Show that $\mathbf{L}$ is absolute w.r.t. every transitive collection that contains all ordinals and satisfies sufficiently many ZF axioms.
Solution.
Let Σ be the ZF-axioms needed to prove that $\forall\alpha \in \mathrm{OR}\exists y\mathcal{L}(\alpha, y)$ ($\mathcal{L}(\alpha, y)$ is a Σ_1-formula that, relative to ZF, amounts to $y = \mathrm{L}_\alpha$).
Claim: If K is transitive, $\mathrm{OR} \subset K$, and $(\forall\alpha \in \mathrm{OR}\exists y\mathcal{L}(\alpha, y))^K$, then $\mathbf{L}$ (which we take to be defined by the formula $\exists\alpha\exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$) is absolute w.r.t. K.
Proof:
$\mathbf{L}^K \subset \mathbf{L}$: Suppose that $x \in \mathbf{L}^K$. That is: $\exists\alpha\exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$ holds in K. Say, $\alpha \in \mathrm{OR}$, $y \in K$, $x \in y$, $\mathcal{L}^K(\alpha, y)$. By upward persistence, $\mathcal{L}(\alpha, y)$ holds as well. Thus, $y = \mathrm{L}_\alpha$, and $x \in \mathbf{L}$.
$\mathbf{L} \subset \mathbf{L}^K$: Suppose that $x \in \mathbf{L}$, say, $x \in \mathrm{L}_\alpha$. By assumption on K, $y \in K$ exists such that $\mathcal{L}^K(\alpha, y)$. By persistence, $\mathcal{L}(\alpha, y)$, i.e.: $x \in \mathrm{L}_\alpha = y$. Hence, $\exists\alpha\exists y[\mathcal{L}(\alpha, y) \wedge x \in y]$ holds in K. □

166. Define $A^{<\omega} = \{f \mid f$ is a finite function s.t. $\mathrm{Dom}(f) \subset \omega \wedge \mathrm{Ran}(f) \subset A\}$. Show that the formula $X = A^{<\omega}$ is Σ_1^{ZF}.

Solution.
$X = A^{<\omega}$ holds iff
$\emptyset \in X \wedge \forall g \in X \forall n \in (\omega - \mathrm{Dom}(g)) \forall a \in A[g \cup \{(n, a)\} \in X] \wedge$
$\wedge \forall g \in X \exists n \in \omega[g$ is a function $\wedge \mathrm{Dom}(g) \subset n \wedge \mathrm{Ran}(g) \subset A]$. □

174. Show: (if $A \neq \emptyset$, then) $\mathrm{Def}(A)$ contains all finite subsets of A.

Solution.
Define formulas* ϕ_n inductively by setting $\phi_0 = \ulcorner x_0 = x_0 \urcorner$ and $\phi_{n+1} = \phi_n \mathbin{\dot\wedge} \ulcorner x_0 \neq x_{n+1} \urcorner$. Then $\mathrm{FrV}(\phi_n) = \{\ulcorner 0 \urcorner, \dots, \ulcorner n \urcorner\}$ and $\mathrm{SAT}(A, \phi_n) = \{f \in A^{\{\ulcorner 0 \urcorner, \dots, \ulcorner n \urcorner\}} \mid f(\ulcorner 0 \urcorner) \neq f(\ulcorner 1 \urcorner), \dots, f(\ulcorner n \urcorner)\}$ for all n (by induction on n). It follows that for any function $f : \{\ulcorner 1 \urcorner, \dots, \ulcorner n \urcorner\} \to A$, $D(A, \dot\neg\, \phi_n, f) = \{f(\ulcorner 1 \urcorner), \dots, f(\ulcorner n \urcorner)\}$. If $B \subset A$ is finite, then for some $n \in \omega$ there exists a bijection $g : n \to B$, so if we set $f(\ulcorner i+1 \urcorner) = g(i)$, then $B = D(A, \dot\neg\, \phi_n, f) \in \mathrm{Def}(A)$. □

178. Suppose that $(A, <)$ is a wellordering and $f : A \to B$ a surjection. Define the relation $\prec$ on B by $x \prec y \equiv$ the $<$-first element of $f^{-1}(x)$ is $<$-smaller than the $<$-first element of $f^{-1}(y)$. Then $\prec$ wellorders B.
Solution.
The correspondence: $x \mapsto$ $<$-first element of $f^{-1}(x)$, embeds $(B, \prec)$ into $(A, <)$. □

186. Show that the formula $x =_1 y$ (which is Σ_1^{ZF}) is not Π_1^{ZF} (unless ZF is inconsistent).
Solution.
Assume that $x =_1 y$ is provably equivalent with the formula $\Phi(x, y)$.
Reason in ZF. Choose two infinite sets a and b such that $a \neq_1 b$. Hence, $\neg\Phi(a, b)$ is true.
Reflection: choose A satisfying Extensionality such that $a, b \in A$ and ($A \models \neg\Phi(a, b) \wedge a, b$ infinite)
Löwenheim-Skolem: choose a countable $B \subset A$ with $a, b \in B$ and ($B \models \neg\Phi(a, b) \wedge a, b$ infinite)
Mostowski's Collapsing Lemma: collapse B to a transitive C via an isomorphism h.
Then ($\neg\Phi(h(a), h(b)) \wedge h(a), h(b)$ infinite) is true in C. Since $h(a), h(b)$ are infinite and $\subset C$, $h(a)$ and $h(b)$ are both countably infinite, and hence $\Phi(h(a), h(b))$ holds in $\mathbf{V}$. Thus, Φ is not Π_1. □

197.

1. Assume that a set A exists such that $(A, \in)$ is a model of all ZF-axioms (considered as a certain subset of FORM). Show:
   (a) There is such a set A that is transitive.
   (b) There is such a set A that has the form L_α, where $\alpha < \omega_1$.
2. Assume that α is the least ordinal such that $(\mathrm{L}_\alpha, \in)$ is a ZF-model.
   Show that if A is a transitive set such that $(A, \in)$ is a ZF-model, then $\alpha \subset A$, and (hence) $\mathrm{L}_\alpha \subset A$.

Solution.

1. If $(A, \in)$ is a model of all ZF-axioms, then by the Downward Löwenstein-Skolem-Tarski Theorem, we can find a countable model $(A_2, \in)$ of ZF. Since $\in$ is well-ordered, and $(A_2, \in)$ satisfies Extensionality, by Mostowski's Collapsing Lemma (see Exercise 67) there exists a unique transitive set A_3 such that $(A_3, \in) \cong (A_2, \in)$. Now let $A_4 = \mathbf{L}^{A_3}$. Then A_4 is a countable transitive model of $\mathrm{ZF} + \mathbf{V} = \mathbf{L}$. So $\alpha = \mathrm{OR} \cup A_4$ is a countable limit ordinal, and by the Condensation Lemma (Corollary 7.32), $A_4 = \mathrm{L}_\alpha$.
2. Let α be the least ordinal such that $(\mathrm{L}_\alpha, \in)$ is a ZF-model, and let A be a transitive set such that $(A, \in)$ is a ZF-model. Then $(\mathbf{L}^A, \in)$ is a model of $\mathrm{ZF} + \mathbf{V} = \mathbf{L}$, so by the Condensation Lemma $\mathbf{L}^A = \mathrm{L}_\beta$ for $\beta = A \cap \mathrm{OR}$. By our choice of α, we now have $\alpha \leq \beta \subset A$, and hence $\mathrm{L}_\alpha \subset \mathrm{L}_\beta = \mathbf{L}^A \subset A$. □