Solutions for Exercises

Chapter 7

140 Show that the following are Π_1^{ZF} or Π_1^{ZFC}.
Solution.

1. $y = \wp(x) \quad \Leftrightarrow \quad \forall z(z \subset x \leftrightarrow z \in y)$
2. $x <_1 y \quad \Leftrightarrow_{\mathrm{ZFC}} \forall z \neg(z \text{ is an injection } y \to x)$
3. α is an initial number $\quad \Leftrightarrow \quad \alpha \in \mathrm{OR} \wedge \alpha \notin \omega \wedge \forall \beta \in \alpha(\beta <_1 \alpha)$
4. $\gamma < \mathrm{cf}(\alpha) \quad \Leftrightarrow \quad \forall x((x \subset \alpha \wedge \bigcup x = \alpha) \to \gamma <_1 x)$
5. α is regular $\quad \Leftrightarrow \quad \forall \gamma < \alpha(\gamma < \mathrm{cf}(\alpha))$

What about the following?

1. $x \leqslant_1 y$
2. $x =_1 y$
4. $\beta = \mathrm{cf}(\alpha)$
   These formulas are not Π_1^{ZFC}: if $\phi(x, y)$ is a formula equivalent in ZFC to $x \leqslant_1 y$ or $x =_1 y$ or $y = \mathrm{cf}(x)$, then ZFC proves $\exists x \in \mathrm{OR} \neg \phi(x, y)$. If K is a countable transitive inner model of ZFC (or of a sufficient fragment of ZFC to prove the above sentence), then for some ordinal $\alpha \in K$, $\neg \phi^K(\alpha, \omega)$. But since K is countable, α must be countable, so $\phi(\alpha, \omega)$ holds, and hence ϕ does not persist downward and is not Π_1.
3. $\alpha = \omega_1$ is not Π_1^{ZFC}: if we have an inner transitive model K of (a sufficient fragment of) ZFC containing ω_1 such that for some countable $\alpha \in K$, K does not contain any bijection between α and ω, then $\omega_1^K \leq \alpha < \omega_1$. Constructing such a model is beyond the scope of this exercise, though.
5. "α is weakly inaccessible" is Π_1^{ZF}: it can be written as
   "α is an initial $\wedge$ α is regular $\wedge$ $\forall \beta < \alpha \exists \gamma < \alpha(\beta <_1 \gamma)$".

□

142 Every L_α is transitive. **L** is transitive.
Solution.
Induction w.r.t. α. The only non-trivial case is the successor step. Suppose that $x \in \mathrm{L}_{\alpha+1}$ and $y \in x$. Then $x \subset \mathrm{L}_\alpha$ and hence $y \in \mathrm{L}_\alpha$. By IH, $y \subset \mathrm{L}_\alpha$. Thus, $y = \{u \in \mathrm{L}_\alpha \mid (u \in y)^{\mathrm{L}_\alpha}\} \in \mathrm{L}_{\alpha+1}$.
□

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1. For $a \in \mathbf{L}$, if $\mathbf{L} \cap \wp(a) \subset \mathrm{L}_\alpha$, then $\mathbf{L} \cap \wp(a) \in \mathrm{L}_{\alpha+1}$.
2. If $a \in \mathbf{L}$, then $\mathbf{L} \cap \wp(a) \in \mathbf{L}$.
3. *Thus*, the Powerset Axiom holds in **L**.

Solution.

1. If $a \in \mathbf{L}$, and $\mathbf{L} \cap \wp(a) \subset \mathrm{L}_\alpha$, then $a \in \mathrm{L}_\alpha$, and we can write $\mathbf{L} \cap \wp(a) = \{y \in \mathrm{L}_\alpha \mid (y \subset a)^{\mathrm{L}_\alpha}\} \in \mathrm{Def}(\mathrm{L}_\alpha) = \mathrm{L}_{\alpha+1}$.
2. Define the operation $h : \mathbf{L} \to \mathrm{OR}$ by $h(x) =$ "the least ξ such that $x \in \mathrm{L}_\xi$". If $a \in \mathbf{L}$, then since $\mathbf{L} \cap \wp(a) \subset \mathbf{L}$ we have $\mathbf{L} \cap \wp(a) \subset \mathrm{L}_\alpha$ and $\mathbf{L} \cap \wp(a) \in \mathrm{L}_{\alpha+1} \subset \mathbf{L}$ for $\alpha = \bigcup_{x \in \mathbf{L} \cap \wp(a)} h(x)$.
3. Now for any $a, b \in \mathbf{L}$, $b \in \mathbf{L} \cap \wp(a) \Leftrightarrow b \in \wp(a) \Leftrightarrow b \subset a \Leftrightarrow (b \subset a)^{\mathbf{L}}$. So the Powerset Axiom holds in $\mathbf{L}$, and $(\wp(a))^{\mathbf{L}} = \wp(a) \cap \mathbf{L}$. □

152 Show that Collection holds in $\mathbf{L}$.
Solution.
Let $a \in \mathbf{L}$, and suppose that $\forall x \in a \exists y \in \mathbf{L}\Psi^{\mathbf{L}}$. Define the operation $h : a \to \mathrm{OR}$ by $h(x) =$ the least ξ such that $\exists y \in \mathrm{L}_\xi \Psi^{\mathbf{L}}$. Now construct $\alpha = \bigcup\{h(x) \mid x \in a\}$. Then $\forall x \in a \exists y \in \mathrm{L}_\alpha \Psi^{\mathbf{L}}$. Since $\mathrm{L}_\alpha \in \mathrm{Def}(\mathrm{L}_\alpha) = \mathrm{L}_{\alpha+1} \subset \mathbf{L}$, Collection holds. □

157 Prove Lemma 7.28:

1. The intersection of two clubs is a club,
2. if each C_x (for every element x of a set a) is club, then so is $\bigcap_{x \in a} C_x$,
3. if each C_ξ ($\xi \in \mathrm{OR}$) is club, then so is $\left\{\alpha \in \mathrm{OR} \mid \alpha \in \bigcap_{\xi<\alpha} C_\xi\right\}$.

Solution.

1. This is a special case of (ii).
2. Let C_x be club for all $x \in a$. Now, if α is a limit ordinal in which $\bigcap_{x \in a} C_x$ is unbounded, then all the C_x are unbounded in α, so $\alpha \in \bigcap_{x \in a} C_x$. To show that $\bigcap_{x \in a} C_x$ is unbounded, let $\alpha \in \mathrm{OR}$, We can define a sequence $\alpha_{i \in \omega}$ by setting $\alpha_0 = \alpha$, $\alpha_{n+1,x} =$ "the smallest ordinal $\geq \alpha_n$ in C_x", and $\alpha_{n+1} = \bigcup_{x \in a} \alpha_{n+1,x}$. Now either $\alpha' = \bigcup_{n \in \omega} \alpha_n$ is equal to α_n for some $n \in \omega$, or it is a limit ordinal in which all the C_x are unbounded. In both cases, $\alpha' \in \bigcap_{x \in a} C_x$.
3. Let C_ξ be club for all $\xi \in \mathrm{OR}$, and let $C = \left\{\alpha \in \mathrm{OR} \mid \alpha \in \bigcap_{\xi<\alpha} C_\xi\right\}$. If α is a limit ordinal in which C is unbounded, then for all ξ, $C_\xi \cup (\xi + 1) \supset C$ is unbounded in α, and therefore for $\xi < \alpha$ so is C_ξ, and $\alpha \in C_\xi$. It follows that $\alpha \in C$. To show that C is unbounded, let $\alpha \in \mathrm{OR}$. We can define a sequence $\alpha_{i \in \omega}$ by setting $\alpha_0 = \alpha$ and $\alpha_{n+1} =$ "the smallest ordinal $\geq \alpha_n$ in $\bigcap_{\xi<\alpha_n} C_\xi$". Now either $\alpha' = \bigcup_{n \in \omega} \alpha_n$ is equal to α_n for some $n \in \omega$, or it is a limit ordinal in which all the C_ξ are unbounded for $\xi < \alpha'$. In both cases, $\alpha' \in C$. □

159. Show that, in the reflection principle, $\{\alpha \mid A_\alpha \prec_\Sigma A\}$ is closed.
Solution.
Assume that $C_\Sigma = \{\xi \in \mathrm{OR} \mid A_\xi \prec_\Sigma A\}$ is unbounded in the limit ordinal α. We need to show that, for $\Phi \in \Sigma$, the equivalence $\Phi^{A_\alpha} \leftrightarrow \Phi^A$ holds on parameters from A_α.
Induction w.r.t. the nr of logical symbols in Φ. The only problem arises when Φ is a quantification; say, $\Phi = \exists z \Psi(x, y, z)$. Assume that $a, b \in A_\alpha$.
($\Rightarrow$) Suppose that $[\exists z \Psi(a, b, z)]^{A_\alpha}$. Say, $c \in A_\alpha$ is such that $[\Psi(a, b, c)]^{A_\alpha}$. By IH on Ψ, it follows that $[\Psi(a, b, c)]^A$. Hence, $[\exists z \Psi(a, b, z)]^A$.
($\Leftarrow$) Conversely, suppose that $[\exists z \Psi(a, b, z)]^A$. Choose $\xi < \alpha$ in C_Σ such that $a, b \in A_\xi$. Then since $\xi \in C_\Sigma$, we also have that $[\exists z \Psi(a, b, z)]^{A_\xi}$. Thus, $c \in A_\xi$ exists such that $[\Psi(a, b, c)]^{A_\xi}$, and it follows that $[\Psi(a, b, c)]^A$. By IH on Ψ, $[\Psi(a, b, c)]^{A_\alpha}$. Therefore, $[\exists z \Psi(a, b, z)]^{A_\alpha}$.
Note that, by the Reflection Principle, $\{\alpha \mid A_\alpha \prec_\Sigma A\}$ is unbounded as well, and therefore club.
□

160. Suppose that the initial λ is strongly inaccessible (Definition 6.24 p. 50). Show that $\alpha < \lambda$ exists such that $\mathrm{V}_\alpha \prec \mathrm{V}_\lambda$. Show that the smallest such α has $\mathrm{cf}(\alpha) = \omega$.
Solution.
Since V_λ models ZF, for any finite subformula-closed set of formulas Σ, $C_\Sigma = \{\alpha \mid A_\alpha \prec_\Sigma A\}$

is closed and unbounded in λ. Let α_Σ be the smallest element of C_Σ. Technically we need the apparatus from Section 7.6 to be able to use the notion of satisfaction and range over sets of formulas. Assuming this apparatus, set $A = \{\alpha_\Sigma \mid \Sigma \text{ a finite subformula-closed set of formulas}\}$ and $\alpha = \bigcup A$.

Let Σ be an arbitrary finite subformula-closed set of formulas. Now for any $\beta < \alpha$ we can find Σ' such that $\beta < \alpha_{\Sigma'}$, and hence $\beta < \alpha_{\Sigma \cup \Sigma'} \in C_\Sigma$. It follows that either $\alpha = \alpha_{\Sigma \cup \Sigma'}$ for some Σ', or $A \cap C_\Sigma$ is unbounded in α, and in both cases $\alpha \in C_\Sigma$ and $V_\alpha \prec_\Sigma V_\lambda$. Since we chose Σ arbitrary, $V_\alpha \prec V_\lambda$. For any β with this property, we have $\alpha_\Sigma \leq \beta$ for all Σ, and therefore $\alpha \leq \beta$.

Now for any finite Σ, $\exists\beta(\mathrm{V}_\beta \prec_\Sigma \mathrm{V})$ can be expressed as a first-order formula, Obviously this formula holds in V_λ. If it were to hold in V_{α_Σ}, then for some $\beta < \alpha_\Sigma$ we would have $V_\beta \prec_\Sigma V_{\alpha_\Sigma} \prec_\Sigma V_\lambda$, contradicting the minimality of α_Σ. Thus $V_{\alpha_\Sigma} \not\prec V_\lambda$, and $\alpha_\Sigma < \alpha$. It follows that $\{\alpha_\Sigma\}$ is a countable cofinal subset of α. □