Solutions for Exercises

Chapter 7

125. Recall (Exercise 16 p. 12 and Theorem 4.18 p. 34) that ($\mathbf{G}$ is the least class such that) $\wp(\mathbf{G}) = \mathbf{G}$. Show, in ZF *minus* Foundation:

1. If K is a class such that $\wp(K) = K$, then every ZF axiom —with the possible exception of Foundation— holds in K.
2. Foundation is true in $\mathbf{G}$.

Thus, $\mathbf{G}$ is an inner model for ZF *including* Foundation in ZF *minus* Foundation. Therefore, if the latter theory is consistent, then so is the former.

Solution.

1. Assume that $K = \wp(K)$, i.e. for all a, $a \in K \iff a \subset K$. We will prove that the relativization of all the axioms with respect to K holds. In most cases, it suffices to show that the set whose existence is postulated by the axiom is in K, since the defining formula is bounded or Π_1^{ZF}, and K is a transitive inner model.

   Extensionality For all $a, b \in K$, if $\forall x \in K : [x \in a \iff x \in b]$, then $a \cap K = b \cap K$. Now since also $a, b \subset K$, it follows that $a = b$.

   Separation Let $a \in K$, and let Φ be a formula that doesn't contain b freely. Set $b = \{x \in a \mid \Phi^K(x, a)\}$. Then $b \subset a \subset K$, so $b \in K$.

   Pairing For all $a, b \in K$, $\{a, b\} \subset K$, so $\{a, b\} \in K$.

   Sumset Let $a \in K$. For all $b \in a$, $b \subset K$, so $\bigcup a \subset K$, so $\bigcup a \in K$.

   Powerset Let $a \in K$. Then $a \subset K$, so $\wp(a) \subset \wp(K) = K$, so $\wp(a) \in K$.

   Substitution Let $a \in K$, and let Ψ be a formula, not containing b freely, such that $\forall x \in a \exists! y \in K \Psi^K$. Then by Substitution $b = \{y \mid y \in K \wedge \exists x \in a \Psi^K\}$ is a set. Since $b \subset K$, we have $b \in K$, and $\forall y \in K (y \in b \leftrightarrow \exists x \in a \Psi^K)$.

   Infinity Since $\mathbf{T}(K) \subset \wp(K) = K$, we have $\mathrm{OR} \subset K$, so $\omega \in K$.

2. Let $a \in \mathbf{G}$. Since $a \in a \cup \{a\}$, we can find an $x \in a \cup \{a\}$ such that $x \cap (a \cup \{a\}) = \emptyset$. It follows that either $x = a = \emptyset$, or $x \in a \subset \mathbf{G}$ and $x \cap a = \emptyset$. Thus, Foundation holds in $\mathbf{G}$. □

127 Show that the following ZF axioms cannot be deduced from the others (modulo a consistency assumption):

1. Infinity,
2. Powerset,
3. Substitution (e.g., existence of $\omega + \omega$ is unprovable),
4. Sumsets.

Solution.

1. The class $K_1 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\}[y \text{ is finite}]\}$ of hereditary finite sets satisfies all axioms of ZFC except Infinity. All the axioms follow easily from the property that $\forall x[x \in K_1 \leftrightarrow (x \subset K_1 \wedge x \text{ is finite})]$. Note that it can be shown that $K_1 = V_\omega$.

   Since $\omega \notin K_1$, Infinity does not hold in K_1.

2. For any cardinal $\aleph_\alpha$, the class $K_2 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\}[|y| \leqslant \aleph_\alpha]\}$ of hereditary $\aleph_\alpha$-cardinality sets satisfies all axioms of ZFC except Powerset. All the axioms follow easily from the property that $\forall x[x \in K_2 \leftrightarrow (x \subset K_2 \wedge |x| \leqslant_1 \aleph_\alpha)]$. Note that it can be shown that $K_2 \subset V_{\omega_{\alpha+1}}$.

   Since $\wp(\omega_\alpha) \subset K_2$ and $\wp(\omega_\alpha) \notin K_2$, Powerset does not hold in K_2.

3. For any limit ordinal $\alpha > \omega$ (and in particular, for $\alpha = \omega + \omega$), the class $K_3 = V_\alpha$ satisfies all axioms except Substitution. All the axioms follow easily from the property that for all $\beta < \alpha$ and $n \in \omega$, $\beta + n < \alpha$ (and for Infinity, that $\alpha > \omega$).

   For $\alpha = \omega + \omega$, Substitution does not hold in K_3: the set $\{\omega, \omega + 1, \omega + 2, \ldots\}$ is not in K_3, even though it is constructible from ω using Substitution with the operator $(\omega + \cdot)$.

4. For any strong limit cardinal $\aleph_\alpha$ (for instance, $\aleph_\omega$ under GCH) the class $K_4 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\} : [|y| < \aleph_\alpha]\}$ of hereditary cardinality-less-than-$\aleph_\alpha$ sets satisfies all axioms of ZFC except Sumset. All the axioms follow easily from the property that $\forall x[x \in K_4 \leftrightarrow (x \subset K_4 \wedge |x| <_1 \aleph_\alpha)]$ (and for Powerset, that α is a limit ordinal). Note that it can be shown that $K_4 \subset V_{\omega_{\alpha+1}}$.

   If $\aleph_\alpha$ is singular (as $\aleph_\omega$ is), then there exists a cofinal subset $B \subset \omega_\alpha$ with $|B| < \aleph_\alpha$ (and hence $B \in K_4$), and since $\bigcup B = \omega_\alpha \notin K_4$, Sumset does not hold in K_4. $\square$

134 Prove a few items of Lemma 7.12: give bounded formulas expressing the following.
Solution.

1. $x = \emptyset \;\Leftrightarrow\; \forall u \in x(u \neq u)$
2. $x \subset y \;\Leftrightarrow\; \forall u \in x(u \in y)$
3. $z = \{x\} \;\Leftrightarrow\; x \in z \wedge \forall u \in z(z = x)$
   $z = \{x, y\} \;\Leftrightarrow\; x \in z \wedge y \in z \wedge \forall u \in z(z = x \vee z = y)$
4. $z = x \cup y \;\Leftrightarrow\; x \subset z \wedge y \subset z \wedge \forall u \in z(u \in x \vee u \in y)$
   $z = x \cup \{y\} \;\Leftrightarrow\; x \subset z \wedge y \in z \wedge \forall u \in z(u \in x \vee u = y)$
5. $x = 0$, $x = 1$, $x = 2$, $x = 3, \ldots,$

   $x = n + 1 \;\Leftrightarrow\; \exists y \in x(x = y \cup \{y\} \wedge y = n)$.
6. $x = \mathrm{V}_0$, $x = \mathrm{V}_1$, $x = \mathrm{V}_2$, $x = \mathrm{V}_3, \ldots,$

   $x = \mathrm{V}_{n+1} \;\Leftrightarrow\; \exists y \in x(y = \mathrm{V}_n \wedge \forall z \in x(z \subset y) \wedge \emptyset \in x \wedge \forall z \in x \forall i \in y(z \cup \{i\} \in x)$.
7. x is 0, S-closed $\;\Leftrightarrow\; \emptyset \in x \wedge \forall y \in x \exists z \in x(z = y \cup \{y\}))$,
8. $x \in \mathrm{OR} \;\Leftrightarrow\;$ x is a transitive set of transitive sets $\;\Leftrightarrow\; \forall y \in x \forall z \in y(z \in x \wedge \forall u \in z(u \in y))$ (cf. Exercise 130),
9. α is a successor ordinal $\;\Leftrightarrow\; \alpha \in \mathrm{OR} \wedge \exists x \in \alpha(\alpha = x \cup \{x\})$

   α is a limit ordinal $\;\Leftrightarrow\; \alpha \in \mathrm{OR} \wedge \neg(\alpha = \emptyset) \wedge \neg(\alpha$ is a successor ordinal$)$
10. $x \in \omega \;\Leftrightarrow\; x \in \mathrm{OR} \wedge \neg\lim(x) \wedge \forall y \in x \neg(\lim(y))$

    $x = \omega \;\Leftrightarrow\; x \in \mathrm{OR} \wedge \lim(x) \wedge \forall y \in x \neg(\lim(y))$
11. $y = \bigcup x \;\Leftrightarrow\; \forall z \in x \forall u \in z(u \in y) \wedge \forall u \in y \exists z \in x(u \in z)$
12. $z = (x, y) \;\Leftrightarrow\; \exists u \in z \exists v \in z(u = \{x, x\} \wedge v = \{x, y\} \wedge z = \{u, v\})$

13. p is an ordered pair $\Leftrightarrow$ $\exists u \in p \exists x \in u \exists y \in u(p = (x, y))$
14. R is a relation $\Leftrightarrow$ $\forall p \in R(p$ is an ordered pair$)$
    xRy $\Leftrightarrow$ $\exists z \in R(z = (x, y))$
15. f is a function $\Leftrightarrow$ f is a relation $\wedge$ $\forall p \in f \forall u \in p \forall x \in u \forall y \in u \forall p' \in f \forall u' \in p' \forall x' \in u' \forall y' \in u'$ $((p = (x, y) \wedge p' = (x', y') \wedge x = x') \to y = y')$
    $f(x) = y$ $\Leftrightarrow$ $\exists z \in R(z = (x, y))$
    f is an injection $\Leftrightarrow$ f is a function $\wedge$ $\forall p \in f \forall u \in p \forall x \in u \forall y \in u \forall p' \in f \forall u' \in p' \forall x' \in u' \forall y' \in u'$ $((p = (x, y) \wedge p' = (x', y') \wedge y = y') \to x = x')$
    f is an surjection onto Y $\Leftrightarrow$ f is a function $\wedge$ $Y = \mathrm{Ran}(f)$,
    f is a bijection $X \to Y$ $\Leftrightarrow$ f is an injective function $\wedge$ $X = \mathrm{Dom}(f)$ $\wedge$ $Y = \mathrm{Ran}(f)$
16. $X = \mathrm{Dom}(f)$ $\Leftrightarrow$ $\forall x \in X \exists p \in f \exists u \in p \exists y \in u(p = (x, y))$ $\wedge$ $\forall p \in f \forall u \in p \forall x \in u \forall y \in u$ $(p = (x, y) \to x \in X)$
    $Y = \mathrm{Ran}(f)$ $\Leftrightarrow$ $\forall y \in Y \exists p \in f \exists u \in p \exists x \in u(p = (x, y))$ $\wedge$ $\forall p \in f \forall u \in p \forall x \in u \forall y \in u$ $(p = (x, y) \to y \in Y)$
    $g = f|A$ $\Leftrightarrow$ $g \subset f$ $\wedge$ $\forall p \in f \forall u \in p \forall x \in u \forall y \in u((p = (x, y) \wedge x \in A) \to p \in g)$ □

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1. Decide which ZF Axioms/Axiom schemas hold in V_ω, and which are false.
2. Same question for $\mathrm{V}_{\omega+\omega}$.
3. Same question for V_{ω_1}.
4. Obtain some relative consistency results from 1–3.
5. What about the truth of Theorem 4.10 (p. 29) (*every well-ordering has a type*) in the above models?
6. Suppose that Theorem 4.10 holds in V_α and $\alpha > \omega$. Can you give lower bounds for α? And if AC holds in V_α?

Solution.
See also exercise 127.

1. All ZF Axioms except Infinity hold in V_ω.
2. All ZF Axioms except Substitution hold in $\mathrm{V}_{\omega+\omega}$
3. All ZF Axioms except Substitution hold in V_{ω_1} (note that $\mathrm{V}_{\omega+4}$ contains a well-ordering of type ω_1, namely the well-ordering of all the well-orderings (modulo order-isomorphism) of ω, ordered by inclusion).
4. If ZF is consistent, then so are (ZF-Infinity)+¬Infinity and (ZF-Substitution)+¬Substitution.
5. In V_ω, every set is finite, so every well-ordering has a finite type, which is in V_ω. In $\mathrm{V}_{\omega+\omega}$ and V_{ω_1}, the aforementioned set of all well-orderings of ω has no type.
6. If V_β contains a well-ordering for a set x, then $V_{\beta+4}$ contains a well-ordering of type $\Gamma(x)$ (namely the well-ordering of all the well-orderings (modulo order-isomorphism) of x). By induction on β, this implies that for all β, $V_{\omega+4\cdot\beta+1}$ contains a well-ordering of type ω_β Thus, α must satisfy $\forall\beta(\omega + 4 \cdot \beta < \alpha \to \omega_\beta < \alpha$. It follows that α is an initial and $\omega_\alpha = \alpha$. The smallest α which satisfies this is $\alpha = \bigcup\{\omega, \omega_\omega, \omega_{\omega_\omega}, \ldots\}$. Note that α might need to satisfy other contraints as well, so this is just a lower bound.

   If we assume AC, then we also have that for all $\beta < \alpha$, V_β has a well-ordering with type in α, so $|V_\beta| < |alpha|$. It follows that $\alpha = |V_\alpha|$. The smallest $\alpha > \omega$ which satisfies this is $\alpha = \bigcup\{\omega + 1, |V_{\omega+1}|, |V_{|V_{\omega+1}|}|, \ldots\}$. For this α, if $x \in V_\alpha$, then $x \in V_\beta$ for some $\beta < \alpha$, and then $|x| \le |V_\beta| < |V_\alpha| = \alpha$, thus any well-orderings of x have type in V_α: it follows that this is not merely a lower bound but also a valid choice for α.

   Note that under GCH, these two choices for (the lower bound of) α are of equal value. □