Solutions for Exercises

Chapter 6

111. (AC) Suppose that p, q, r, s are cardinals such that $p < q$ and $r < s$. Show that $p + r < q + s$.
Solution.
If p and q are finite, then so is $p + q$, and the result is either trivial (if r or s are infinite) or follows from arithmetic (if r and s are both finite). Otherwise, since $p, r \leqslant \max\{q, s\}$ we have by Lemma 6.8 that $p + r = \max\{p, r\} < \max\{q, s\} = q + s$. □

114. Prove Lemma 6.22:

1. $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)} > \aleph_\alpha$,
2. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Solution.

1. $\aleph_\alpha = \sum_{\xi < cf(\omega_\alpha)} p_\xi < \prod_\xi \aleph_\alpha = \aleph^{cf(\aleph_\alpha)}$.
2. If not, then: $2^{\aleph_\alpha} = \sum_{\xi < \omega_\alpha} p_\xi < \prod_\xi 2^{\aleph_\alpha} = (2^{\aleph_\alpha})^{\aleph_\alpha} = 2^{\aleph_\alpha}$. □

116. (Hausdorff) Prove that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.
Solution.
If $\aleph_\beta \leq \aleph_\alpha$, then: $\aleph_{\alpha+1}^{\aleph_\beta} = |\bigcup_{\xi < \omega_{\alpha+1}} \xi^{\omega_\beta}| \leq \sum_{\xi < \omega_{\alpha+1}} \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \leq \aleph_{\alpha+1}^{\aleph_\beta}$.
If $\aleph_\beta \geq \aleph_{\alpha+1}$, then: $\aleph_{\alpha+1}^{\aleph_\beta} \leq (2^{\aleph_\alpha})^{\aleph_\beta} = 2^{\aleph_\beta} \leq \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \leq \aleph_{\alpha+1}^{\aleph_\beta}$. □

120. Show: if κ is a strongly inaccesible initial number, then

1. $\beta < \kappa \;\Rightarrow\; \mathrm{V}_\beta <_1 \kappa$,
2. $\mathrm{V}_\kappa =_1 \kappa$,
3. $(\mathrm{V}_\kappa, \in)$ satisfies all ZFC Axioms,
4. if κ is the *least* strong inaccessible, then $(\mathrm{V}_\kappa, \in) \models$"there is no strong inaccessible".

Solution.

1. By induction on β: $V_0 <_1 \kappa$ (since κ is uncountable), if $V_\beta <_1 \kappa$ then $V_{\beta+1} = \wp(V_\beta) <_1 \kappa$ (since κ is a strong limit), and if $\gamma < \kappa$ is a limit and $\forall \xi < \gamma [V_\xi <_1 \kappa]$, then $\gamma <_1 \mathrm{cf}(\kappa)$ (since κ is regular) and hence $V_\gamma = \bigcup_{\xi < \gamma} V_\xi <_1 \kappa$.
2. $\kappa \subset V_\kappa$, and conversely $|V_\kappa| = |\bigcup_{\beta < \kappa} V_\beta| \leq \sum_{\beta < \kappa} |V_\beta| \leq \sum_{\beta < \kappa} |\kappa| = |\kappa| \cdot |\kappa| = |\kappa|$.
3. By Lemma 4.17 and Theorem 4.18, for any limit ordinal α, $(V_\alpha, \in)$ is a model of all the axioms of ZFC (assuming AC) except Substitution. So all we need to do is to show that V_κ satisfies Substitution. Now let $a \in V_\kappa$, and let F be an operation such that $(V_\kappa, \in, a) \models \forall x \in a \exists y [y = F(x)]$. We can define the operation $F' : a \to V_\kappa$ by setting $F'(x) = y \;\Leftrightarrow\; (V_\kappa, \in, x, y) \models F(x) = y$. We need to show that $F'[a] \in V_\kappa$. Now first, for all $x \in a$, $\rho(F'(x)) <_1 \kappa$. Second, we have $a \subset V_\beta$ for some $\beta < \kappa$, and hence $a <_1 \kappa$. Since κ is regular, this implies $\rho(F'[a]) = \bigcup_{x \in a} \{\rho(F'(x)) + 1\} <_1 \kappa$. We conclude that $F'[a] \in V_\kappa$.

4. Let $\alpha < \kappa$. If α is not an uncountable strong limit initial, there exists an injection witnessing $\omega \geqslant_1 \alpha$ or $\beta \geqslant_1 \alpha$ or $\wp(\beta) \geqslant_1 \alpha$ (for some $\beta < \alpha$). This injection is an element of $V_{\alpha+3} \subset V_\kappa$. Similarly, if α is singular, then there exists a partition P of α and an injection from P into $\beta < \alpha$, and P and the injection are both elements of $V_{\alpha+5} \subset V_\kappa$.

   By induction on formulas it can be shown that for any 'bound' formula ϕ (where all quantifiers are of the form $\exists x \in y$ or $\forall x \in y$) and any $\vec{x} \in V_\kappa$, $(V_\kappa \models \phi(\vec{x}) \Leftrightarrow \phi(\vec{x})$. Since all the above mentioned conditions depend on a bound formula and the existence of a witness, it follows that if α is not strongly inaccessible, then $(V_\kappa, \in) \models \alpha$ is not strongly inaccessible. $\square$

122.

1. Suppose that $X \subset \alpha$ is cofinal in α. Show that α has a cofinal subset Y of type $\mathrm{cf}(\alpha)$ such that $Y \subset X$.
2. Show: if α and β have cofinal subsets of the same type, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.
3. Show: if α is a limit, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.

Solution.

1. Suppose that $f : \mathrm{cf}(\alpha) \to \alpha$ has $\mathrm{Ran}(f)$ cofinal in α. Define $g : \mathrm{cf}(\alpha) \to X$ by $g(\xi) = \bigcap\{\delta \in X \mid f(\xi) \leq \delta\}$. Clearly, $\mathrm{Ran}(g)$ is cofinal in α. Applying Lemma 6.26 (p. 51) yields an order-preserving function $h \subset g$ with $Y := \mathrm{Ran}(h) \subset X$ cofinal in α. Now $\mathrm{cf}(\alpha) \leq \mathrm{type}(Y)$ (since Y is cofinal in α) and $\mathrm{type}(Y) \leq \mathrm{cf}(\alpha)$ (since $\mathrm{type}(Y) = \mathrm{type}(\mathrm{Dom}(h))$ and $\mathrm{Dom}(h) \subset \mathrm{cf}(\alpha)$).
2. Suppose that $X \subset \alpha$ and $Y \subset \beta$ have the same type. By 1., let $X' \subset X$ be cofinal in α and have type $\mathrm{cf}(\alpha)$. By the order-isomorphism between X and Y, X' corresponds to some $Y' \subset Y$ that has the same type as X'. This shows $\mathrm{cf}(\beta) \leq \mathrm{type}(X') = \mathrm{cf}(\alpha)$. By symmetry, $\mathrm{cf}(\alpha) \leq \mathrm{cf}(\beta)$.
3. ω_α and α have cofinal subsets of the same type: $\{\omega_\xi \mid \xi < \alpha\}$, resp., α itself.