Solutions to Exercises

Chapter 4

98 Show: every initial is critical for addition, multiplication and exponentiation.
Solution.
If $\beta.\gamma < \omega$, then β, γ are finite, and it is easy to show by induction that then $\beta + \gamma$, $\beta \cdot \gamma$ and β^γ are also finite. Hence, ω is critical for addition, multiplication and exponentiation.
For higher initials we need Corollary 4.35: $\omega_\alpha \times \omega_\alpha =_1 \omega_\alpha$ for all initials ω_α.
First, for all initials ω_α and all $\beta, \gamma \leqslant_1 \omega_\alpha$, $\beta + \gamma \leqslant_1 \omega_\alpha$ This follows by induction on γ: for $\gamma = 0$ this holds trivially, for $\gamma + 1$ this follows from $\beta + (\gamma + 1) = (\beta + \gamma) \cup \{\beta + \gamma\} \leqslant_1 (\beta + \gamma) \times \omega_\alpha \leqslant_1 \omega_\alpha \times \omega_\alpha \leqslant_1 \omega_\alpha$, and for limits γ we have $\beta + \gamma = \bigcup_{\xi<\gamma}(\beta + \xi) \leqslant_1 \bigcup_{\xi<\gamma}(\beta + \xi) \times \{\xi\} \leqslant_1 \bigcup_{\xi<\gamma} \omega_\alpha \times \{\xi\} = \omega_\alpha \times \gamma \subset \omega_\alpha \times \omega_\alpha \leqslant_1 \omega_\alpha$. In the last part we simultaneously select injections $\beta+\xi \to \omega_\alpha$ for all $\xi < \gamma$; the use of AC for this can be avoided by defining the injection recursively using this same induction.
Similarly, we have that for all $\beta, \gamma \leqslant_1 \omega_\alpha$, $\beta \cdot \gamma \leqslant_1 \omega_\alpha$: the proof is analogous, except that for $\gamma + 1$ we have that if $\beta \cdot \gamma \leqslant_1 \omega_\alpha$, then applying the result for $+$ yields $\beta \cdot (\gamma + 1) = (\beta \cdot \gamma) + \beta \leqslant_1 \omega_\alpha$.
Finally, we also have that for all β, γ, $\beta^\gamma \leqslant_1 \beta \cup \gamma \cup \omega$: the proof is again analogous, except that for $\gamma + 1$ we have that if $\beta \cdot \gamma \leqslant_1 \omega_\alpha$, then applying the result for $\cdot$ yields $\beta^{\gamma+1} = (\beta^\gamma) \cdot \beta \leqslant_1 \omega_\alpha$.
Now, if $\omega_{\alpha'}$ is an initial $> \omega$, and $\beta, \gamma < \omega_{\alpha'}$, then for some initial $\omega_\alpha <_1 \omega_{\alpha'}$, $\beta, \gamma \leqslant_1 \omega_\alpha$, so then $\beta + \gamma, \beta \cdot \gamma, \beta^\gamma \leqslant_1 \omega_\alpha <_1 \omega_{\alpha'}$, and hence $\beta + \gamma, \beta \cdot \gamma, \beta^\gamma < \omega_{\alpha'}$. We conclude that $\omega_{\alpha'}$ is critical for addition, multiplication and exponentiation. □

99 Show:

1. $<$ well-orders OR × OR,
2. every product $\gamma \times \gamma$ is an initial segment
   (if $(\alpha, \beta) < (\alpha', \beta') \in \gamma \times \gamma$, then $(\alpha, \beta) \in \gamma \times \gamma$),
3. the product $\omega \times \omega$ is well-ordered in type ω,
4. every product $\omega_\alpha \times \omega_\alpha$ $(\alpha > 0)$ is well-ordered in type ω_α.

Solution.

1. Let $K \subset \mathrm{OR} \times \mathrm{OR}$ be a class.
   Set γ to be the smallest ordinal satisfying $\exists\alpha\exists\beta[(\alpha, \beta) \in K \ \wedge\ \max(\alpha, \alpha) = \gamma]$.
   Set α to be the smallest ordinal satisfying $\exists\beta[(\alpha, \beta) \in K \ \wedge\ \max(\alpha, \beta) = \gamma]$.
   Finally, set β to be the smallest ordinal satisfying $(\alpha, \beta) \in K \wedge \max(\alpha, \beta) = \gamma$. Then $(\alpha, \beta) \in K$. Furthermore, for all $(\alpha', \beta') \in K$, either $\max(\alpha', \beta') > \gamma$, or $\max(\alpha', \beta') = \gamma \ \wedge\ \alpha' > \alpha$, or $\max(\alpha', \beta') = \gamma \ \wedge\ \alpha' > \alpha \ \wedge\ \beta' \geq \beta$. It follows that (α, β) is a $<$-minimal element of K.
2. If $(\alpha, \beta) < (\alpha', \beta') \in \gamma \times \gamma$, then $\max(\alpha, \beta) \leq \max(\alpha', \beta') < \gamma$, so $(\alpha, \beta) \in \gamma \times \gamma$.
3. By the previous point, for any $(\alpha, \beta) \in \mathrm{OR} \times \mathrm{OR}$, $\{(\alpha', \beta') \mid (\alpha', \beta') < (\alpha, \beta)\} \subset \gamma \times \gamma$ is a set (where $\gamma = \max(\alpha, \beta)$). Then we can use Theorem 4.13 to construct a unique order-preserving map $\Gamma : \mathrm{OR} \times \mathrm{OR} \Rightarrow \mathrm{OR}$.

   Now suppose that $\Gamma(\omega, \omega) > \omega$. Then there exist $(n, m) \in \omega \times \omega$ such that $\Gamma(n, m) = \omega$. But then $\omega \leqslant_1 \max(n, m) \times \max(n, m) = (\max(n, m))^2$ would be finite, a contradiction.

4. For any α, $\Gamma(\omega_\alpha, \omega_\alpha) = \omega_\alpha$, by induction on α. For if $\Gamma(\omega_\alpha, \omega_\alpha) > \omega_\alpha$, then there exist $(\beta, \gamma) \in \omega_\alpha \times \omega_\alpha$ such that $\Gamma(\omega_\alpha, \omega_\alpha) = \omega_\alpha$, Since $\beta, \gamma < \omega_\alpha$, there must exist a $\xi < \alpha$ such that $\max(\beta, \gamma) \leqslant_1 \omega_\xi$. Then $\omega_\xi <_1 \omega_\alpha \leqslant_1 \max(\beta, \gamma) \times \max(\beta, \gamma) \leqslant_1 \omega_\xi \times \omega_\xi$, contradicting the induction hypothesis.

□

Chapter 5

101

1. Assume AC. Prove DC: if the set A is non-empty and the relation $R \subset A^2$ is such that $\forall a \in A \exists b \in A (aRb)$, then a function $f : \omega \to A$ exists such that for all $n \in \omega$, $f(n)Rf(n+1)$.
2. Show the version of DC where A can be a proper class and $R \subset A^2$ is also provable from AC. (Use Foundation.)
3. Show that a relation $\prec$ is well-founded (every non-empty set has a $\prec$-minimal element) iff there is no function f on ω such that for all $n \in \omega$, $f(n+1) \prec f(n)$.

Solution.

1. Let j be a choice function for $\wp(A)$. Recursively define $f : \omega \to A$ by $f(0) = f(A)$ and $f(n+1) = j(\{a \in A \mid f(n)Ra\})$ (by assumption, $\{a \in A \mid bRa\} \neq \emptyset$ for all $b \in A$).
2. If we have Foundation, then we can use the Bottom operator of Definition 4.21 to define the operator $H(X) = \bigcup_{x \in X} \text{Bottom}(\{y \in A \mid xRy\})$. Then H is a finite operator, so $H{\uparrow} = H{\uparrow}\omega$ is a set, and $\forall a \in H{\uparrow} \exists b \in H{\uparrow} [aRb]$. Now we can apply DC on sets to $H{\uparrow}$.
3. If there exists a function f with the given property, then $\{f(n) \mid n \in \omega\}$ is a set with no $\prec$-minimal element. Conversely, if A is a set with no $\prec$-minimal element, then we can use *DC* to find a function $f : \omega \to A$ with the desired property.

□

103 (AC) Show: if A is infinite, then $\omega \leqslant_1 A$.
Show *without* AC that: if A is infinite, then $\omega \leqslant_1 \wp(\wp(A))$.
Solution.
(i) Let j be a choice function for $\wp(A)$. Recursively, define $f : \omega \to A$ by $f(n) = j(A - \{f(m) \mid m < n\})$ as long as $A - \{f(m) \mid m < n\}) \neq \emptyset$. Obviously, since A is infinite, if $f|n$ is an injection then $A - \{f(m) \mid m < n\}) \neq \emptyset$. By induction on n it follows that for all n, $f|n$ is defined and an injection. Thus, f is an injection as well.
(ii) If $B \subset A$ satisfies $|B| = n$, then $A - B \neq \emptyset$ and for all $a \in A - B$, $|B \cup \{a\}| = n + 1$. It follows by induction that for all n, $\{B \subset A \mid |B| = n\}$ is nonempty. Since all these subsets of $\wp(A)$ are disjoint, this is the required injection. □

105 Show that the following are equivalent for every two sets A and B:

1. $A <_1 B$, i.e.: there is no bijection : $A \to B$ and $A \leqslant_1 B$,
2. there is no surjection : $A \to B$ and $A \leqslant_1 B$,
3. there is no surjection : $A \to B$ and $B \neq \emptyset$.

For which of the six implications do you need AC?
Solution.

$2 \Rightarrow 1$ if there is no surjection $A \to B$, then there certainly is no bijection.

$2 \Rightarrow 3$ if $A \leqslant_1 B$ and $B = \emptyset$, then $A = \emptyset$ and there would exist a (trivial) surjection $A \to B$. Conversely, if 2) holds, then $B \neq \emptyset$.

$1 \Rightarrow 2$ if there exists a surjection : $A \to B$ and an injection : $A \to B$, then by Theorem 6.6 there exists a bijection : $A \to B$. Conversely, if 1) holds then there exists no surjection : $A \to B$.

$\neg 1 \Rightarrow \neg 3$ (AC) If $A \not\leqslant_1 B$, then $B \leqslant_1 A$, i.e. there exists an injection $f : B \to A$. Now either $B = \emptyset$, or we can define a surjection $g : A \to B$ by setting, for some $b_0 \in B$, $g(y) = x$ if $f(x) = y$, and $g(y) = b_0$ otherwise.

$3 \Rightarrow 1$ and $3 \Rightarrow 2$ are the only implications that seem to need AC. □

108 The *Teichmüller-Tukey Lemma* is the following statement.

> *Suppose that $\emptyset \neq A \subset \wp(X)$, and for all $Y \subset X$, Y is in A iff every finite subset of Y is in A. Then A has a ($\subset$-) maximal element.*

Show that this is equivalent with Zorn's Lemma.
Solution.
Zorn $\Rightarrow$ TT:
Suppose that A is as in the TT Lemma. For A to have a maximal element, by Zorn, it suffices to show that it is closed under unions of chains. Thus, suppose that $K \subset A$ is a chain. In order that $\bigcup K \in A$, it suffices to show that every finite subset is in A. Thus, suppose that $C \subset \bigcup K$ is finite. Then for some $Y \in K$, we have that $C \subset Y$. Therefore, $C \in A$.
TT $\Rightarrow$ Zorn:
Let P be a non-empty poset in which chains have upper bounds. Let A be the set of all chains of P. Then A satisfies the TT condition: (i) a finite subset of a chain is a chain, and (ii) if every finite subset of $K \subset A$ is a chain, then K is a chain (if $a, b \in K$, then $\{a, b\}$ is a finite subset, hence $a \leqslant b$ or $b \leqslant a$). By TT, A has a maximal element, which is a maximal chain of P. An upper bound of this chain is maximal in P. □