Solutions to Exercises

Chapter 4

75. Show:

1. *Every* V_α *is transitive,*
2. $x \subset y \in \mathrm{V}_\alpha \;\Rightarrow\; x \in \mathrm{V}_\alpha$,
3. $\alpha < \beta \;\Rightarrow\; \mathrm{V}_\alpha \in \mathrm{V}_\beta$; $\alpha \leq \beta \;\Rightarrow\; \mathrm{V}_\alpha \subset \mathrm{V}_\beta$,
4. $\alpha \subset \mathrm{V}_\alpha$; $\alpha \notin \mathrm{V}_\alpha$; $\alpha = \mathrm{OR} \cap \mathrm{V}_\alpha$,
5. $\mathrm{OR} \cap (\mathrm{V}_{\alpha+1} - \mathrm{V}_\alpha) = \{\alpha\}$.

Solution.

1. From Exercise 22 we know that for all A, if every $x \in A$ is transitive, then so is $\bigcup A$, and if A itself is transitive, then so is $\wp(A)$. Since $\emptyset$ is transitive, it follows by transfinite induction that V_α is transitive for all α.
2. By induction on α: if $x \subset y \in \mathrm{V}_{\alpha+1} = \wp(\mathrm{V}_\alpha)$, then $x \subset y \subset \mathrm{V}_\alpha$, so $x \in \mathrm{V}_{\alpha+1}$. If $x \subset y \in \mathrm{V}_\gamma$ for a limit γ, then $x \subset y \in \mathrm{V}_\xi$ for some $\xi < \gamma$, so $x \in \mathrm{V}_\xi \subset \mathrm{V}_\gamma$.
3. For all $\beta \geq \alpha$, $\mathrm{V}_\alpha \subset \mathrm{V}_\beta$. This holds by definition for $\beta = \alpha$ and for limits β, and for successor ordinals we have by induction on β that $\mathrm{V}_\alpha \subset \mathrm{V}_\beta \subset \wp(\mathrm{V}_\beta) = \mathrm{V}_{\beta+1}$ (since V_β is transitive). Consequentially, for all $\beta > \alpha$, $\mathrm{V}_\alpha \in \mathrm{V}_{\alpha+1} \subset \mathrm{V}_\beta$.
4. By induction, $\mathrm{OR} \cap \mathrm{V}_\alpha = \alpha$ for all α. For we have $\mathrm{OR} \cap \mathrm{V}_0 = 0$, $\mathrm{OR} \cap \mathrm{V}_{\beta+1} = \{\xi \in \mathrm{OR} \mid \xi \subset \mathrm{V}_\beta\} = \{\xi \in \mathrm{OR} \mid \xi \subset \beta\} = \beta + 1$. and $\mathrm{OR} \cap \mathrm{V}_\gamma = \bigcup_{\xi<\gamma}(\mathrm{OR} \cap \mathrm{V}_\xi) = \bigcup_{\xi<\gamma} \xi = \gamma$. Consequentially, for all α, $\alpha \subset V_\alpha$ and $\alpha \notin V_\alpha$.
5. $\mathrm{OR} \cap (\mathrm{V}_{\alpha+1} - \mathrm{V}_\alpha) = (\alpha + 1) - (\alpha) = \{\alpha\}$.

□

76 Show:

1. $\rho(\alpha) = \rho(\mathrm{V}_\alpha) = \alpha$,
2. $\mathrm{V}_\alpha = \{a \mid \rho(a) < \alpha\}$; $a \in b \;\Rightarrow\; \rho(a) < \rho(b)$,
3. $\rho(a) = \bigcup\{\rho(b) + 1 \mid b \in a\} = \{\rho(b) \mid b \in \mathrm{TC}(a)\}$
4. express $\rho(a \cup b)$, $\rho(\bigcup a)$, $\rho(\wp(a))$, $\rho(\{a\})$, $\rho((a, b))$ and $\rho(\mathrm{TC}(a))$ in terms of $\rho(a)$ and $\rho(b)$.

Solution.

1. From Lemma 4.17 it follows that $\alpha \subset \mathrm{V}_\beta$ and $\mathrm{V}_\alpha \subset \mathrm{V}_\beta$ are both equivalent to $\alpha \leq \beta$.
2. If $\rho(a) < \alpha$, then $a \subset \mathrm{V}_{\rho(a)} \in \mathrm{V}_\alpha$, so $a \in \mathrm{V}_\alpha$. Conversely, if $a \in \mathrm{V}_\alpha$, then $a \subset \mathrm{V}_\beta$ for some $\beta < \alpha$, so $\rho(a) < \alpha$. If $a \in b$, then $a \in b \subset \mathrm{V}_{\rho(b)}$, so $\rho(a) < \rho(b)$.

3. $a \subset \mathrm{V}_\alpha$ iff $\forall b \in a[b \in \mathrm{V}_\alpha]$, iff $\forall b \in a[\rho(b)+1 \leq \alpha]$, iff $\alpha \geq \bigcup\{\rho(b)+1 \mid b \in a\}$.
   The other follows by $\in$-induction: $\rho(a) = \bigcup\{\rho(b)+1 \mid b \in a\} = \bigcup\{\rho(b) \cup \{\rho(b)\} \mid b \in a\} = \bigcup\{\{\rho(c) \mid c \in TC(b) \lor c = b\} \mid b \in a\} = \{\rho(c) \mid c \in TC(a)\}$.
4. $\rho(a \cup b) = \rho(a) \cup \rho(b)$, $\rho(\bigcup a) = \bigcup(\rho(a))$, $\rho(\wp(a)) = \rho(a)+1$, $\rho(\{a\}) = \rho(a)+1$, $\rho((a,b)) = (\rho(a) \cup \rho(b)) + 2$, $\rho(\mathrm{TC}(a)) = \rho(a)$.

□

78 Assuming the Foundation Axiom, prove the Collection Principle:
$\forall x \in a\, \exists y\, \Phi(x,y) \;\Rightarrow\; \exists b\, \forall x \in a\, \exists y \in b\, \Phi(x,y)$ (b not free in Φ).
Solution.
Assume that $\forall x \in a\ \exists y\ \Phi(x,y)$. For all x, $\mathrm{Bottom}(\{y \mid \Phi(x,y)\})$ is a nonempty set. So $b = \bigcup\{\mathrm{Bottom}(\{y \mid \Phi(x,y)\}) \mid x \in a\}$ satisfies the given condition, as well as the condition for the Strong Collection Principle from Exercise 79. □

85 Show that the function $h : \mathrm{V}_\omega \to \mathbb{N}$ recursively defined by

$$h(x) = \sum_{y \in x} 2^{h(y)}$$

is a bijection.
Solution.
Define $i : \mathbb{N} \to \mathrm{V}_\omega$ recursively by setting, for all n,

$$i(n) = \{i(m) \mid \text{the } m\text{-th least significant bit of } n \text{ is } 1\}$$

Then for all n, $h(i(n)) = \sum\{2^{h(i(m))} \mid \text{the } m\text{-th least significant bit of } n \text{ is } 1\}$ so by induction on n, $\forall n[h(i(n)) = n]$. Conversely, if $x \in \mathrm{V}_\omega$ and $\forall y \in x[i(h(y)) = y]$, then h is injective on x, so $i(h(x)) = \{i(h(y)) \mid y \in x\} = x$, and by $\in$-induction it follows that for all $x \in \mathrm{V}_\omega$, $i(h(x)) = x$. □

91 Prove Lemma 4.30:

1. every ω_α is an initial,
2. every initial has the form ω_α,
3. $\alpha < \beta \;\Rightarrow\; \omega_\alpha < \omega_\beta$.

Solution.

1. From Lemma 4.28 it follows directly that ω_0 and $\omega_{\alpha+1}$ are initials. If γ is a limit ordinal and $\xi < \omega_\gamma$, then $\xi < \omega_\beta$ for some $\beta < \gamma$, so $\xi \leqslant_1 \omega_\beta <_1 \omega_{\beta+1} \leqslant_1 \omega_\gamma$.
2. Let β be an initial, and let α' be the least ordinal such that $\beta < \omega_{\alpha'}$. If α' were a limit ordinal, then for some $\xi < \alpha'$, $\beta < \omega_\xi$, contradicting our choice of α'. So $\alpha' = \alpha+1$ for some α, and $\omega_\alpha \leq \beta < \omega_{\alpha+1}$. Since β is an initial and $\omega_{\alpha+1}$ is the least initial $> \omega_\alpha$, it follows that $\beta = \omega_\alpha$.
3. By induction on β. First, $\omega_\alpha < \Gamma(\omega_\alpha) = \omega_{\alpha+1}$. Second, if $\omega_\alpha < \omega_\beta$, then $\omega_\alpha < \omega_\beta < \Gamma(\omega_\beta) = \omega_{\beta+1}$. Finally, if $\gamma > \alpha$ is a limit ordinal, then there exists a β with $\alpha < \beta < \gamma$, and then $\omega_\alpha < \omega_\beta \leq \bigcup_{\xi<\gamma} \omega_x i = \omega_\gamma$.

□

93 Let $\alpha \in \mathrm{OR}$ be arbitrary. Recursively define $\alpha_0 = \alpha$ and $\alpha_{n+1} = \omega_{\alpha_n}$. Put $\beta := \bigcup_n \alpha_n$. Show: β is the least ordinal $\gamma \geqslant \alpha$ for which $\omega_\gamma = \gamma$.
Solution.
If $\gamma \geq \alpha$ is such that $\omega_\gamma = \gamma$, then for all $\xi \leq \gamma$, $\omega_\xi \leq \gamma$ (by Lemma 4.30). By induction on n it follows that $\alpha_n \leq \gamma$ for all n, and hence $\beta \leq \gamma$.
For the converse, if $\alpha = \omega_\alpha$, then $\beta = \alpha$ and we are done, so assume $\alpha < \omega_\alpha$. Then by induction on n, $\alpha_n < \alpha_{n+1}$ for all n. For all $\xi < \beta$ there exists an n with $\xi < \alpha_n$, and therefore both $\xi + 1 < \alpha_{n+1} \leq \beta$ and $\omega_\xi < \alpha_{n+1} \leq \beta$. It follows that β is a limit, and $\omega_\beta = \bigcup_{\xi<\beta} \omega_\xi \leq \beta$. □

95 For $\alpha \geqslant \omega$, the following are equivalent:
1. α is critical for $+$; 2. $\beta < \alpha \Rightarrow \beta + \alpha = \alpha$; 3. $\exists \xi \, (\alpha = \omega^\xi)$.
Solution.
(1) $\Rightarrow$ (2): Let α be critical for $+$, and $\beta < \alpha$. Now, α is a limit ordinal (for if $\alpha = \alpha' + 1$, then α would not be critical for $+$). So $\beta + \alpha = \bigcup_{\xi < \alpha}(\beta + \xi) \leq \alpha$. On the other hand, it is straightforward to show by induction that for all ξ, ξ', $\xi + \xi' \geq \xi'$. It follows that $\beta + \alpha = \alpha$.
$\neg$(3) $\Rightarrow$ $\neg$(2): First, for all $\alpha > 0$ there exists a ξ such that $\omega^\xi \leq \alpha < \omega^{\xi+1}$. For let ξ' be the least ordinal such that $\alpha < \omega^{\xi'}$. Now, if ξ' were a limit, then by Definition 4.31 there would exist a $\xi'' < \xi'$ with $\alpha < \omega^{\xi''}$, contradicting our choice of ξ'. So $\xi' = \xi + 1$ for some ξ, and $\omega^\xi \leq \alpha < \omega^{\xi+1}$. Similarly, there exists an $n \in \omega$ such that $\omega^\xi \cdot n \leq \alpha < \omega^\xi \cdot (1 + n)$. This implies $\alpha < \omega^\xi + \omega^\xi \cdot n \leq \omega^\xi + \alpha$. Now if $\alpha \neq \omega^\xi$, then this contradicts (2).
(3) $\Rightarrow$ (1): Let $\alpha = \omega^\xi$, and let $\beta, \gamma < \alpha$. Then there exist $\xi' < \xi$ and $n \in \omega$ such that $\beta, \gamma < \omega^{\xi'} \cdot n$, and hence $\beta + \gamma < \omega^{\xi'} \cdot 2n < \omega^{\xi'} \cdot \omega = \omega^{\xi'+1} \leq \omega^\xi = \alpha$.

$\square$