Solutions to Exercises

Chapter 4

54. Assume that the set a is transitive. Show:

1. $a \in \mathbf{G}$ iff $\in$ is well-founded on a,
2. $a \subset \mathbf{TR}$ iff $\in$ is transitive on a.

Thus, an ordinal is the same as a transitive set on which $\in$ is a transitive and well-founded relation. (This is the standard definition of the notion.)
Solution.
1. If $a \notin \mathbf{G}$, then there would exist a $b \ni a$ such that $\forall y \in b (y \cap b \neq \emptyset)$, and then $a \cap b$ would be a nonempty subset of a with no $\in$-minimal element (since for any $y \in a \cap b$, $y \subset a$ by transititive of a, and so $y \cap (a \cap b) = y \cap b \neq \emptyset$). Conversely, if $c \subset a$ were a nonempty set without an $\in$-minimal element, then $b = c \cup \{a\}$ would satisfy $\forall y \in b : y \cap b \neq \emptyset$, and $a \notin \mathbf{G}$. It follows that $a \in \mathbf{G}$ iff $\in$ is well-founded on a.
2. $a \subset \mathbf{TR}$ iff for all x, y, z with $z \in a$, $(x \in y \in z \;\Rightarrow\; x \in z)$.
$\in$ is transitive on a iff for all x, y, z with $x, y, z \in a$, $(x \in y \in z \;\Rightarrow\; x \in z)$.
By transitivity of a, the two are equivalent. □

58. Show:

1. $\alpha \leq \beta \;\Leftrightarrow\; \alpha \subset \beta$,
2. if K is a non-empty class of ordinals, then $\bigcap K$ is the least element of K,
3. if A is a *set* of ordinals, then $\bigcup A$ is an ordinal that is the *sup* of A (the least ordinal $\geq$ every $\alpha \in A$).

Solution.
1. $(\alpha \in \beta \vee \alpha = \beta) \;\Rightarrow\; \alpha \subset \beta \;\Rightarrow\; \beta \notin \alpha \;\Rightarrow\; (\alpha \in \beta \vee \alpha = \beta)$
2. A reformulation of transfinite induction yields: if $K \subset OR$ is nonempty, then $\exists \alpha \in OR : (\forall \beta < \alpha (\beta \notin K)) \wedge \alpha \in K$, i.e. K has a least element. Let $\alpha \in K$ be the least element. Then $\bigcap K \leq \alpha$, and $\alpha \leq \bigcap K$ is trivial.
3. $\bigcup K$ is the union of transitive sets of ordinals, therefore itself a transitive set of ordinals, therefore an ordinal. It is also the smallest set containing as subsets all the sets in K, so by (1) it is the *sup* of K. □

61. Assume that $(A, \prec)$ is a well-ordering and $B \subset A$.
Show that $\text{type}(B, \prec) \leqslant \text{type}(A, \prec)$.
Solution. Suppose that α and β are the order types of A and B. If $\beta \nleqslant \alpha$, then $\alpha < \beta$. Let $f : \beta \to B$ and $g : A \to \alpha$ be order-isomorphisms. Then $gf : \beta \to \alpha$ is an order-preserving injection such that $gf(\alpha) < \alpha$. This contradicts Lemma 4.8 (p. 29). □

64. Prove Theorem 4.13: suppose that ε is a well-founded relation on the class $\mathbf{U}$ such that for all $a \in \mathbf{U}$, $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}$ is a set, then for every operation $H : \mathbf{V} \to \mathbf{V}$ there is a unique operation $F : \mathbf{U} \to \mathbf{V}$ such that for all $a \in \mathbf{U}$:

$$F(a) = H(F|\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}).$$

Solution
Let us call a function or operation f *good* if $\mathrm{Dom}(F) \subset \mathbf{U}$ and for all $a \in \mathrm{Dom}(F)$, $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\} \subset \mathrm{Dom}(F)$ and $f(a) = H(f|\{b \in \mathbf{U} \mid b \,\varepsilon\, a\})$.
If f and g are both good, then they agree on their common domain: otherwise, if a is the ε-minimal element on which they disagree, then $f(a) = H(f|\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}) = H(g|\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}) = g(a)$, a contradiction. It follows that the union F of all good functions is an operation that satisfies the recursion equation on its domain.
Furthermore, $\mathrm{Dom}(F) = \mathbf{U}$. For otherwise, let a be the ε-minimal element of $\mathbf{U} - \mathrm{Dom}(F)$. Then $\{b \in \mathbf{U} \mid b \,\varepsilon^*a\}$ would be a set, and setting $f = F|\{b \in \mathbf{U} \mid b \,\varepsilon^*a\})$ and $f' = f \cup \{a, H(f)\}$ we would obtain a good function f' with $f' \not\subset F$, a contradiction.
Since good operations on $\mathbf{U}$ have to agree on their domain, F is unique. □

65. Let $a_0 \in \mathbf{V}$ be a set and $G : \mathbf{V} \to \mathbf{V}$ an operation. Show: there exists a unique operation $F : \mathrm{OR} \to \mathbf{V}$ on OR such that

- $F(0) = a_0$,
- $F(\alpha + 1) = G(F(\alpha))$,
- for limits γ: $F(\gamma) = \bigcup_{\xi<\gamma} F(\xi)$.

Solution
Applying the Recursion Theorem on OR to the operation H defined by

- $H(f) = G(f(\alpha))$ if $\mathrm{Dom}(f) = \alpha + 1$, $\alpha \in \mathrm{OR}$
- $H(f) = \bigcup_{\xi<\gamma} f(\xi)$ if $\mathrm{Dom}(f) = \gamma$ and γ is a limit ordinal $\neq 0$.
- $H(x) = a_0$ otherwise

yields an operation satisfying the requirements. □

70. Show that the *single* recursion equation $H{\upharpoonleft}\alpha = \bigcup_{\xi<\alpha} H(H{\upharpoonleft}\xi)$ defines the same operation as the one defined in Definition 4.14 by *three* equations. (And, of course, $H{\downharpoonleft}\alpha = \bigcap_{\xi<\alpha} H(H{\downharpoonleft}\xi)$ is a single equation defining the greatest fixed point hierarchy — cf. Exercise 72.)
Solution.
Claim: $H{\upharpoonleft}\alpha \subset H{\upharpoonleft}(\alpha + 1)$.
Induction. Obviously, $H{\upharpoonleft}0 \subset H{\upharpoonleft}1$. And if $H{\upharpoonleft}\alpha \subset H{\upharpoonleft}(\alpha + 1)$, then, by monotonicity, $H{\upharpoonleft}(\alpha + 1) \subset H{\upharpoonleft}(\alpha + 2)$. Finally, if γ is a limit, then, if $\xi < \gamma$, $H{\upharpoonleft}\xi \subset \bigcup_{\xi<\gamma} H{\upharpoonleft}\xi$, hence, $H(H{\upharpoonleft}\xi) \subset H(\bigcup_{\xi<\gamma} H{\upharpoonleft}\xi)$; and so $H{\upharpoonleft}\gamma = \bigcup_{\xi<\gamma} H{\upharpoonleft}\xi \subset \bigcup_{\xi<\gamma} H(H{\upharpoonleft}\xi) \subset H(\bigcup_{\xi<\gamma} H{\upharpoonleft}\xi) = H{\upharpoonleft}(\gamma + 1)$.
Now:
$H{\upharpoonleft}0 = \emptyset = \bigcup_{\xi<0} H(H{\upharpoonleft}\xi)$;
$H{\upharpoonleft}(\alpha + 1) = H(H{\upharpoonleft}\alpha) = H(H{\upharpoonleft}\alpha) \cup H{\upharpoonleft}\alpha$ (since $H{\upharpoonleft}\alpha \subset H(H{\upharpoonleft}\alpha)) = H(H{\upharpoonleft}\alpha) \cup \bigcup_{\xi<\alpha} H(H{\upharpoonleft}\xi)$ (by IH) $= \bigcup_{\xi<\alpha+1} H(H{\upharpoonleft}\xi)$;
$H{\upharpoonleft}\gamma = \bigcup_{\xi<\gamma} H{\upharpoonleft}\xi = \bigcup_{\xi<\gamma} H(H{\upharpoonleft}\xi)$ (since $H{\upharpoonleft}\xi \subset H{\upharpoonleft}(\xi + 1) \subset H{\upharpoonleft}\gamma$). □

72 Let H be a monotone operator over a *set* $\mathbf{U}$. The *greatest fixed point hierarchy* is the sequence $\{H{\downharpoonleft}\alpha\}_\alpha$ recursively defined by

- $H{\downharpoonleft}0 = \mathbf{U}$,
- $H{\downharpoonleft}(\alpha + 1) = H(H{\downharpoonleft}\alpha)$,
- $H{\downharpoonleft}\gamma = \bigcap_{\xi<\gamma} H{\downharpoonleft}\xi$ (for limits γ).

Show that:

1. the hierarchy is *descending*, i.e., that $\alpha < \beta \Rightarrow H{\downharpoonleft}\beta \subset H{\downharpoonleft}\alpha$.
2. some stage $H{\downharpoonleft}\alpha_0$ is a fixed point of H.
3. $H{\downharpoonleft}\alpha_0 = \bigcap_\alpha H{\downharpoonleft}\alpha$ is the greatest fixed point of H.

Try to generalize for the case where $\mathbf{U}$ may be a proper class.
Solution.
Consider the dual operator of $H^d : \wp(\mathbf{U}_{\rightarrow}\wp(\mathbf{U})$ of H defined by $H^d(X) =_{\text{def}} \mathbf{U} - H(\mathbf{U} - X)$. By exercise 52, the least fixed point of H^d corresponds to the greatest fixed point of H. It is easy to see that the least fixed point hierarchy of H^d also corresponds exactly to the greatest fixed point hierarchy of H, i.e. $H{\downarrow}\alpha = \mathbf{U} - H^d{\uparrow}\alpha$ for $\alpha \in OR$. The desired properties follow.
If $\mathbf{U}$ is a class, then the first property will still hold, but the second and third properties may fail. For instance, consider the operator $H : \mathbf{V} \rightarrow \mathbf{V}$ given by $H(X) = X - \{\alpha\}$ if α is the least ordinal in X, and $H(X) = \emptyset$ if X does not contain any ordinals. Then for all ordinals α, $H{\downarrow}\alpha = \mathbf{V} - \alpha$, and yet the only fixed point of H is $H{\downarrow} = H{\uparrow} = \emptyset$. $\square$