Solutions To Exercises

Chapter 3

36 Prove Lemma 3.19.1–3. Prove Lemma 3.19.4, and do *not* use 3.18, but use 3.19.1–3.
Solution.
1. $a = \mathrm{TC}(a, 0) \subset \mathrm{TC}(a)$.
2. If $x \in y \in \mathrm{TC}(a)$, then, for some $n \in \omega$, $y \in \mathrm{TC}(a, n)$. Thus, $x \in \bigcup \mathrm{TC}(a, n) = \mathrm{TC}(a, \mathrm{S}(n)) \subset \mathrm{TC}(a)$.
3. Suppose $b \supset a$ is transitive. Then $a = \mathrm{TC}(a, 0) \subset b$, and if for some n, $\mathrm{TC}(a, n) \subset b$, then $\mathrm{TC}(a, Sn) = \bigcup \mathrm{TC}(a, n) \subset \bigcup b \subset b$. Thus, by induction on n, $\mathrm{TC}(a, n) \subset b$ for all n, and therefore $\mathrm{TC}(a) \subset b$.
4. $\supset$: First, $a \subset \mathrm{TC}(a)$. Next, if $b \in a$, then $b \in \mathrm{TC}(a)$ (since $a \subset \mathrm{TC}(a)$), $b \subset \mathrm{TC}(a)$ (since $\mathrm{TC}(a)$ is transitive), and $\mathrm{TC}(b) \subset \mathrm{TC}(a)$ (by property 3).
$\subset$: By property 3, it suffices to show that $a \cup \bigcup_{b \in a} \mathrm{TC}(b)$ is a transitive superset of a. Transitivity: if $x \in y \in a \cup \bigcup_{b \in a} \mathrm{TC}(b)$, then $y \in a$, or $b \in a$ exists such that $y \in \mathrm{TC}(b)$. In the first case, $x \in \mathrm{TC}(y) \subset a \cup \bigcup_{b \in a} \mathrm{TC}(b)$. In the second, $x \in \mathrm{TC}(b) \subset a \cup \bigcup_{b \in a} \mathrm{TC}(b)$. □

39 Show that $x \in \mathrm{TC}(a)$ iff $x \in^\star a$.
Solution 1.
Define $xRy \equiv_{\text{def}} x \in y$ and $R' \equiv_{\text{def}} x \in TC(y)$. Now R and R' satisfy parts 1-3 of Lemma 3.21:
1. if $a \in b$, then $a \in TC(b)$, since $b \subset TC(b)$.
2. if $a \in TC(b)$ and $b \in TC(c)$, then by transitivity of $TC(c)$ we have that $b \subset TC(c)$ and $TC(b) \subset TC(c)$, and therefore $a \in TC(c)$.
3. Assume $R \subset S$ and S is transitive. It suffices to show that for any b, $\{x \mid xR'b\} \subset \{x \mid xSb\}$. This follows from $\{x \mid xR'b\} = TC(b)$ and the observation that $\{x \mid xSb\}$ is a transitive set (since for all x, y, if ySb and $x \in y$, then xSy and xSb).
It follows that $R' = R^\star$. □

Solution 2.
Define R_n by

$$aR_nb :\equiv_{\text{def}} \exists f[\mathrm{Dom}(f) = n{+}2 \wedge f(0) = a \wedge f(n{+}1) = b \wedge \forall i < n{+}1(f(i)Rf(i{+}1))]$$

It can easily be seen that $R_0 = R$, that for all a, b and n, $aR_{n+1}b \Leftrightarrow \exists c[aRc \wedge cR_nb]$, and that for all a and b: $aR^*b \Leftrightarrow \exists n{\in}\omega : aR_nb$. Now we can show by induction on n that for all n, x and a, $x{\in_n}a \Leftrightarrow x{\in}TC(a, n)$. For $n = 0$ it is trivial:

$$x{\in_0}a \Leftrightarrow x{\in}a \Leftrightarrow x{\in}TC(a, 0)$$

If we assume that for a given n and a and for all y, $y{\in_n}a \Leftrightarrow y{\in}TC(a, n)$, then for all x,

$$x{\in_{n+1}}a \Leftrightarrow \exists y[x{\in}y{\in_n}a] \Leftrightarrow \exists y[x{\in}y{\in}TC(a, n)] \Leftrightarrow x{\in}\bigcup TC(a, n) = TC(a, n{+}1)$$

Therefore $x \in^* a \Leftrightarrow \exists n x \in_n a \Leftrightarrow \exists n x \in TC(a, n) \Leftrightarrow x \in TC(a)$. □

43 $\mathbb{Z}$ is the set of integers. Define $H : \wp(\mathbb{Z}) \to \wp(\mathbb{Z})$ by $H(X) =_{\text{def}} \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$. Identify the fixed points of H.

Solution.
H is a finite operator, so $H\!\uparrow = \mathbb{N}$ is the least fixed point of H. For any fixed point K of H, by induction on n, $-1 \in K$ iff for all $n \in \omega$, $-(1+n) \in K$. It follows that the only other fixed point of H is $\mathbb{Z}$. □

44 Prove Theorem 3.27.
Solution.
1. We prove the equivalent statement that that for all n, m, $H\!\upharpoonright\! n \subset H\!\upharpoonright\!(n{+}m)$, by induction w.r.t. n:
Basis $n = 0$: $H\!\upharpoonright\! n = \emptyset \subset H\!\upharpoonright\!(n{+}m)$ is obvious.
Induction step: if $H\!\upharpoonright\! n \subset H\!\upharpoonright\!(n{+}m)$. then $H\!\upharpoonright\!(n{+}1) = H(H\!\upharpoonright\! n) \subset H(H\!\upharpoonright\!(n{+}m)) = H\!\upharpoonright\!(n{+}1{+}m)$.
2. Suppose that $H(X) \subset X$. By induction on n, it follows that $H\!\upharpoonright\! n \subset X$:
Basis $n = 0$: $H\!\upharpoonright\! 0 = \emptyset \subset X$ is obvious.
Induction step: if $H\!\upharpoonright\! n \subset X$. then $H\!\upharpoonright\!(n{+}1) = H(H\!\upharpoonright\! n) \subset H(X) \subset X$.
3. If $Y \subset H\!\upharpoonright\!\omega = \bigcup_n H\!\upharpoonright\! n$ is finite, then n exists s.t. $Y \subset H\!\upharpoonright\! n$: induction w.r.t. nr of elements of Y.
Basis, $Y = \emptyset$. Then $Y \subset \emptyset = H\!\upharpoonright\! 0$.
Induction step. IH: for n-element Y, the statement holds. Now let $Y \subset \bigcup_n H\!\upharpoonright\! n$ have $n + 1$ elements. For instance, $Y = Y' \cup \{y\}$, where Y' has n elements. By IH, n_1 exists s.t. $Y \subset H\!\upharpoonright\! n_1$. Furthermore, n_2 exists s.t. $y \in H\!\upharpoonright\! n_2$. Let $m = \max(n_1, n_2)$. Then clearly (by 1), $Y \subset H\!\upharpoonright\! m$.
4. $H(H\!\upharpoonright\!\omega) \subset H\!\upharpoonright\!\omega$:
Assume that $a \in H(H\!\upharpoonright\!\omega)$. By finiteness, a finite $Y \subset H\!\upharpoonright\!\omega$ exists s.t. $a \in H(Y)$. By 3 we can assume that for some n, $Y \subset H\!\upharpoonright\! n$. Then $a \in H(Y) \subset H(H\!\upharpoonright\! n) = H\!\upharpoonright\! n + 1 \subset H\!\upharpoonright\!\omega$. □

45 Let $A = \omega \cup \{\omega\}$ and define $H : \wp(A) \to \wp(A)$ by $H(X) = \{0\} \cup \{\mathrm{S}(x) \mid x \in X\} \cap A$ if $\omega \not\subset X$, and $H(X) = A$ otherwise. Show: H is monotone, H is not finite, $H\!\uparrow = A$, $\forall n \in \omega\ H\!\upharpoonright\! n = n$. Thus, $H\!\uparrow \neq \bigcup_n H\!\upharpoonright\! n$.
Solution.
H is monotone: Let $X \subset Y \subset A$. If $\omega \subset Y$, then $H(X) \subset A = H(Y)$. Otherwise, $\omega \not\subset X, Y$, so $H(X) = \{0\} \cup \{\mathrm{S}(x) \mid x \in X\} \subset \{0\} \cup \{\mathrm{S}(x) \mid x \in X\} = H(Y)$.
H is not finite: Since $\omega \in H(\omega)$, and for all finite sets $X \subset \omega$, $\omega \notin H(X)$, we see that H is not finite.
For all n, $H\!\upharpoonright\! n = n$, by induction on n: For $n = 0$, $H\!\upharpoonright\! 0 = \emptyset = 0$. If $H\!\upharpoonright\! n = n$, then $H\!\upharpoonright\! n + 1 = H(H\!\upharpoonright\! n) = H(n) = \{0\} \cup \{suc(x) \mid x \in n\} = n + 1$.
$H\!\uparrow = A$: Studying the proof of Theorem 3.24 it is apparent that $\bigcup_n H\!\upharpoonright\! n$ is inductive even if H is not finite. So $\omega = \bigcup_n H\!\upharpoonright\! n \subset H\!\uparrow$. Therefore $A = H(\omega) \subset H(H\!\uparrow) = H\!\uparrow$. We conclude that $H\!\uparrow = A$.
□

51 (Simultaneous inductive definitions.) Suppose that $\Pi, \Delta : \wp(A) \times \wp(A) \to \wp(A)$ are monotone operators in the sense that if $X_1, Y_1, X_2, Y_2 \subset A$ are such that $X_1 \subset X_2$ and $Y_1 \subset Y_2$, then $\Pi(X_1, Y_1) \subset \Pi(X_2, Y_2)$ (and similarly for Δ). Show that K, L exist such that

1. $\Pi(K, L) \subset K$, $\Delta(K, L) \subset L$; in fact, $\Pi(K, L) = K$, $\Delta(K, L) = L$,
2. if $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$, then $K \subset X$ and $L \subset Y$.

Show that, similarly, *greatest* (post-) fixed points exist. Generalize to more operators.
Solution.
Consider the operator $H : \wp(A{\times}A) \to \wp(A{\times}A)$ defined by $H(Z) = \Pi(\pi_1[Z], \pi_2[Z]) \times \Delta(\pi_1[Z], \pi_2[Z])$ (where, as usual, π_1 and π_2 denote the projection onto the first and second coordinates).
H is monotone: assume that $Z \subset Z' \subset (A \times A)$. Then $\pi_1[Z] \subset \pi_1[Z']$ and $\pi_2[Z] \subset \pi_2[Z']$. So by our assumption for Π, $\Pi(\pi_1[Z], \pi_2[Z]) \subset \Pi(\pi_1[Z'], \pi_2[Z'])$, and analogous for Δ. It follows that $H(Z) \subset H(Z')$.
Since H is monotone, it has a least fixed point $H\!\uparrow$. Setting $K = \pi_1[H\!\uparrow]$, $L = \pi_2[H\!\uparrow]$, we have that $H\!\uparrow = H(H\!\uparrow) = \Pi(K, L) \times \Delta(K, L)$, so $K = \pi_1[H\!\uparrow] = \Pi(K, L)$ and $L = \pi_2[H\!\uparrow] = \Delta(K, L)$ (and $H\!\uparrow = K \times L$).
For the second part, assume that for $X, Y \subset A$, $\Pi(X, Y) \subset X$ and $\Delta(X, Y) \subset Y$. Then $H(X{\times}Y) = \Pi(X, Y) \times \Delta(X, Y) \subset X \times Y$, and hence $K \times L = H\!\uparrow \subset X \times Y$. Therefore $K \subset X$ and $L \subset Y$.