Solutions to Exercises

Chapter 3

20 Show that ω is transitive.
Solution. We prove by induction on n that for all $n \in \omega$, $n \subset \omega$. For $n = 0$ this holds trivially. For $n \in \omega$, if $n \subset \omega$, then $S(n) = n \cup \{n\} \subset \omega$.

22. Are the following (always) true?

1. If every $x \in A$ is transitive, then so is $\bigcup A$.
2. If A is transitive, then so is $\bigcup A$.
3. If $\bigcup A$ is transitive, then so is A.
4. If A is transitive, then so is $\wp(A)$.
5. If $\wp(A)$ is transitive, then so is A.

Solution.

1. True. If $x \in y \in \bigcup A$, then some $z \in A$ exists s.t. $y \in z$. Since z is transitive, $x \in z$; and so $x \in \bigcup A$.
2. True. If $y \in \bigcup A$, then some $z \in A$ exists s.t. $y \in z$. Since A is transitive, $y \in A$, and so $y \subset \bigcup A$.
3. False. Consider $A := \{\{\emptyset\}\}$ (this is the simplest counter-example).
4. True. If $x \in y \in \wp(A)$, then $y \subset A$ and $x \in A$. Since A is transitive, $x \subset A$, and so $x \in \wp(A)$.
5. True. If $y \in A$, then since $A \in \wp(A)$ and $\wp(A)$ is transitive, $y \in \wp(A)$. and so $y \subset A$.

27 Show: $\exists n \in \omega \Psi(n) \;\Rightarrow\; \exists n \in \omega[\Psi(n) \;\wedge\; \forall m < n \neg\Psi(m)]$.
Solution. This statement is equivalent to $\forall n[\forall m < n \neg\Psi(m) \;\Rightarrow\; \neg\Psi(n)] \;\Rightarrow\; \forall n \neg\Psi(n)$, which is Strong Induction applied to $\neg\Psi(n)$.

28 Define the property Z by: $Z(x) \equiv_{\text{def}}$ there is no function f defined on ω such that (i) $f(0) = x$ and (ii) for all $n \in \omega$: $f(n+1) \in f(n)$. Show that the class $\mathcal{Z} =_{\text{def}} \{x \mid Z(x)\}$ is not a set, and that for every set A: $\{x \in A \mid Z(x)\} \notin A$.
Solution. Both statements follow from the proposition that for any class C, $\{x \in C \mid Z(x)\} \notin C$. So let C be a class, and assume that $X = \{x \in C \mid Z(x)\} \in C$. Note that the property Z is such that $\forall x[Z(x) \Leftrightarrow \forall y \in x Z(y)]$. X is a set since $X \in C$, and therefore $Z(X)$. It follows that $X \in X$. But then we can show that $\neg Z(X)$ by defining f on ω with $f(n) = X$ for all n, a contradiction.

33 Give an adequate definition of *natural number* that is not based on the Infinity Axiom.
Solution. The natural numbers are exactly the sets n such that for all X, if $0 \in X$ and $\forall m \in n(m \in X \;\Rightarrow\; \mathrm{S}(m) \in X)$, then $n \in X$. That the natural numbers satisfy this property follows from Exercise 32: that no other sets satisfy this property can be demonstrated by taking $X = \omega$.