Hints for Exercises

Chapter 7

163. Show that $\mathbf{L}$ is absolute w.r.t. every transitive collection that contains all ordinals and satisfies sufficiently many ZF axioms.
Hint.
Let Σ be the ZF-axioms needed to prove that $\forall\alpha \in \mathrm{OR}\exists y\mathcal{L}(\alpha, y)$ where $\mathcal{L}(\alpha, y)$ is a Σ_1-formula that, relative to ZF, amounts to $y = \mathrm{L}_\alpha$. Show that if K is transitive, $\mathrm{OR} \subset K$, and K satisfies Σ, then for all α, $\mathrm{L}^K_\alpha = \mathrm{L}_\alpha$.

166. Define $A^{<\omega} = \{f \mid f \text{ is a finite function s.t. } \mathrm{Dom}(f) \subset \omega \wedge \mathrm{Ran}(f) \subset A\}$. Show that the formula $X = A^{<\omega}$ is Σ^{ZF}_1.

Hint.
Note that $A^{<\omega}$ is the smallest collection of functions such that $\emptyset \in A^\omega$ and if $g \in A^{<\omega}$ then g extended with 1 entry is $\in A^\omega$. You need the existential quantifier to introduce ω.

174. Show: (if $A \neq \emptyset$, then) $\mathrm{Def}(A)$ contains all finite subsets of A.

Hint.
Define formulas* ϕ_n inductively such that $D(A, \dot{\neg}\, \phi_n, f) = \{f(\ulcorner 1 \urcorner), \dots, f(\ulcorner n \urcorner)\}$.

178. Suppose that $(A, <)$ is a wellordering and $f : A \to B$ a surjection. Define the relation $\prec$ on B by $x \prec y \equiv$ the $<$-first element of $f^{-1}(x)$ is $<$-smaller than the $<$-first element of $f^{-1}(y)$. Then $\prec$ wellorders B.
Hint.
Embed $(B, \prec)$ into $(A, <)$.

186. Show that the formula $x =_1 y$ (which is Σ^{ZF}_1) is not Π^{ZF}_1 (unless ZF is inconsistent).
Hint.
As before (Exercise 140), but now you can explicitly construct the countermodel.

197.

1. Assume that a set A exists such that $(A, \in)$ is a model of all ZF-axioms (considered as a certain subset of FORM). Show:
   (a) There is such a set A that is transitive.
   (b) There is such a set A that has the form L_α, where $\alpha < \omega_1$.
2. Assume that α is the least ordinal such that $(\mathrm{L}_\alpha, \in)$ is a ZF-model.
   Show that if A is a transitive set such that $(A, \in)$ is a ZF-model, then $\alpha \subset A$, and (hence) $\mathrm{L}_\alpha \subset A$.

Hint.

1. Use Downward Löwenstein-Skolem-Tarski Theorem, Mostowski's Collapsing Lemma (see Exercise 67, and note that $\in$ is well-founded on $(A, \in)$) and the Condensation Lemma (Corollary 7.32) to find L_α.
2. Show that $\mathbf{L}^A = \mathrm{L}_\beta$ for some $\beta \geq \alpha$.