Hints for Exercises

Chapter 7

140 Show that the following are Π_1^{ZF} or Π_1^{ZFC}.
Hint.

1. $y = \wp(x)$
2. $x <_1 y$: use $x <_1 y \iff_{\mathrm{ZFC}} \neg(y \leqslant_1 x)$.
3. α is an initial number: note that this is a condition on all $\beta < \alpha$.
4. $\gamma < \mathrm{cf}(\alpha)$: compare γ to all cofinal subsets of α.
5. α is regular: note that this is a condition on all $\beta < \alpha$.

What about the following?

1. $x \leqslant_1 y$: describe a countermodel K such that $K \models y <_1 x$ but $x =_1 y$. The last is best achieved by taking K countable. For the actual construction of such a model, see Exercise 186.
2. $x =_1 y$: see (1)
3. $\alpha = \omega_1$: describe a countermodel K such that $\omega_1^K \neq \omega_1$. The actual construction of such a model is beyond the scope of these exercises.
4. $\beta = \mathrm{cf}(\alpha)$: see (1), possibly with minor modifications.
5. "α is weakly inaccessible": rewrite it as a combination of known Π_1^{ZF} properties.

142 Every L_α is transitive. $\mathbf{L}$ is transitive.
Hint.
Induction w.r.t. α. For the successor step, show that if $y \in A$ and $y \subset A$, then $y \in \mathrm{Def}(A)$.

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1. For $a \in \mathbf{L}$, if $\mathbf{L} \cap \wp(a) \subset \mathrm{L}_\alpha$, then $\mathbf{L} \cap \wp(a) \in \mathrm{L}_{\alpha+1}$.
2. If $a \in \mathbf{L}$, then $\mathbf{L} \cap \wp(a) \in \mathbf{L}$.
3. *Thus*, the Powerset Axiom holds in $\mathbf{L}$.

Hint.

2. Define the operation $h : \mathbf{L} \to \mathrm{OR}$ by $h(x) =$ "the least ξ such that $x \in \mathrm{L}_\xi$" to construct an α satisfying the conditions of (1).
3. Show $b \in \mathbf{L} \cap \wp(a) \iff (b \subset a)^{\mathbf{L}}$ for $a, b \in \mathbf{L}$.

152 Show that Collection holds in $\mathbf{L}$.
Hint.
Define an operation $h : a \to \mathrm{OR}$ as in the previous exercise, to construct an α such that L_α witnesses the Collection Axiom.

157 Prove Lemma 7.28:

1. The intersection of two clubs is a club,
2. if each C_x (for every element x of a set a) is club, then so is $\bigcap_{x\in a} C_x$,
3. if each C_ξ ($\xi \in \mathrm{OR}$) is club, then so is $\left\{\alpha \in \mathrm{OR} \mid \alpha \in \bigcap_{\xi<\alpha} C_\xi\right\}$.

Hint.

1. This is a special case of (ii).
2. To show that $\bigcap_{x\in a} C_x$ is unbounded, let $\alpha \in \mathrm{OR}$, construct a monotone nondecreasing sequence $\alpha_{i\in\omega}$ with $\alpha_0 = \alpha$, such that $\alpha' = \bigcup_{n\in\omega} \alpha_n \in C_x$ for all $x \in a$. Note that you will have to differentiate between the cases that α' is a limit ordinal and that for some $n \in \omega$, $\forall m \geq n(\alpha' = \alpha_m)$.
3. To show that $C = \left\{\alpha \in \mathrm{OR} \mid \alpha \in \bigcap_{\xi<\alpha} C_\xi\right\}$ is unbounded, let $\alpha \in \mathrm{OR}$, and use (2) to construct a monotone nondecreasing sequence $\alpha_{i\in\omega}$ with $\alpha_0 = \alpha$ such that $\alpha' = \bigcup_{n\in\omega} \alpha_n \in C_\xi$ for all $\xi < \alpha'$.

159. Show that, in the reflection principle, $\{\alpha \mid A_\alpha \prec_\Sigma A\}$ is closed.
Hint.
Assume that $C_\Sigma = \{\xi \in \mathrm{OR} \mid A_\xi \prec_\Sigma A\}$ is unbounded in the limit ordinal α. Show that, for $\Phi \in \Sigma$, the equivalence $\Phi^{A_\alpha} \leftrightarrow \Phi^A$ holds on parameters from A_α. Use induction w.r.t. the number of logical symbols in Φ, in conjunction with the observation that for any parameters $\vec{a} \in A_\alpha$, there exist $\xi < \alpha$ such that $\xi \in C_\Sigma$ and $\vec{a} \in A_\xi$.

160. Suppose that the initial λ is strongly inaccessible (Definition 6.24 p. 50). Show that $\alpha < \lambda$ exists such that $\mathrm{V}_\alpha \prec \mathrm{V}_\lambda$. Show that the smallest such α has $\mathrm{cf}(\alpha) = \omega$.
Hint.
Let α be the supremum of the (countably many) least ordinals α_Σ such that $\mathrm{V}_{\alpha_\Sigma} \prec_\Sigma \mathrm{V}_\lambda$, where Σ ranges over all finite subformula-closed sets of formulas (implicitly using the apparatus from Section 7.6). Show that for all Σ, $C_\Sigma = \{\xi \mid \mathrm{V}_\xi \prec_\Sigma \mathrm{V}_\lambda\}$ contains α, and hence $V_\alpha \prec V_\lambda$. Then show that for all Σ, $\mathrm{V}_{\alpha_\Sigma} \not\prec \mathrm{V}_\lambda$ and hence $\alpha_\Sigma < \alpha$.