Hints for Exercises

Chapter 7

125. Recall (Exercise 16 p. 12 and Theorem 4.18 p. 34) that ($\mathbf{G}$ is the least class such that) $\wp(\mathbf{G}) = \mathbf{G}$. Show, in ZF *minus* Foundation:

1. If K is a class such that $\wp(K) = K$, then every ZF axiom —with the possible exception of Foundation— holds in K.
2. Foundation is true in $\mathbf{G}$.

Thus, $\mathbf{G}$ is an inner model for ZF *including* Foundation in ZF *minus* Foundation. Therefore, if the latter theory is consistent, then so is the former.

Hints.

1. Note that K is a transitive inner model, so for most axioms it suffices to show that the set whose existence is postulated by the axiom is in K. For Separation and Substitution you have to be careful to use Φ^K and Ψ^K rather than Φ and Ψ.
2. Assume $a \in \mathbf{G}$, and apply the defining condition of $\mathbf{G}$ to $a \cup \{a\}$.

127 Show that the following ZF axioms cannot be deduced from the others (modulo a consistency assumption):

1. Infinity,
2. Powerset,
3. Substitution (e.g., existence of $\omega + \omega$ is unprovable),
4. Sumsets.

Hints. We construct a transitive inner model in which all the axioms of ZF hold except one we single out.

1. Consider the class $K_1 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\}[y \text{ is finite}]\}$ of hereditary finite sets.
2. Consider, for any cardinal $\aleph_\alpha$, the class $K_2 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\}[|y| \leqslant \aleph_\alpha]\}$ of hereditary $\aleph_\alpha$-cardinality sets.
3. Consider, for any accessible limit ordinal $\alpha > \omega$ (and in particular, for $\alpha = \omega + \omega$), the class $K_3 = V_\alpha$.
4. For any strong limit cardinal $\aleph_\alpha$ (for instance, $\aleph_\omega$ under GCH) consider the class $K_4 = \{x \in \mathbf{G} \mid \forall y \in^* \{x\} : [|y| < \aleph_\alpha]\}$ of hereditary cardinality-less-than-$\aleph_\alpha$ sets.

134 Prove a few items of Lemma 7.12: give bounded formulas expressing the properties mentioned.
Hints.
You can use formulas you defined before. But be careful: if you gave a bounded formula for (for instance) $b = \wp(a)$, that does *not* mean you can simply use $\forall x \in \wp(a) \ldots$, as that evaluates to the unbounded formula $\exists y(y = \wp(a) \wedge \forall x \in y \ldots)$. Instead, try to find some other set z relating to what you want to express such that you can write $\forall x \in z(x \subset a \rightarrow \ldots)$.

5. $x = 0$, $x = 1$, $x = 2$, $x = 3$,...: use an inductive definition, i.e. for $x = n + 1$, give a formula which uses the (already defined) formula for $x = n$.
6. $x = \mathrm{V}_0$, $x = \mathrm{V}_1$, $x = \mathrm{V}_2$, $x = \mathrm{V}_3$,...: use an inductive definition, i.e. for $x = V_{n+1}$, give a formula which uses the (already defined) formula for $x = V_n$.
8. $x \in \mathrm{OR}$: see Exercise 130.
9. "α is a limit ordinal": rewrite it as "α is a non-zero, non-successor ordinal".
10. $x \in \omega$, $x = \omega$: use that ω is the lowest limit ordinal.
12. $z = (x, y)$: use that if $z = \{u, v\}$, then we can find these u, v by quantifying over z.
13. p is an ordered pair: use that if $p = (x, y)$, then we can find these x, y by quantifying over the elements of the elements of p.
15. f is an sur-/bijection: add "from X onto Y", otherwise the question is meaningless. Use the Domain and Range formula from the next item
16. $X = \mathrm{Dom}(f), Y = \mathrm{Ran}(f)$: express $\subset$ and $\supset$ separately.

    $g = f|A$: this is actually simpler if you do *not* use the Domain formula. Note that $g \subset f$.

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1. Decide which ZF Axioms/Axiom schemas hold in V_ω, and which are false.
2. Same question for $\mathrm{V}_{\omega+\omega}$.
3. Same question for V_{ω_1}.
4. Obtain some relative consistency results from 1–3.
5. What about the truth of Theorem 4.10 (p. 29) (*every well-ordering has a type*) in the above models?
6. Suppose that Theorem 4.10 holds in V_α and $\alpha > \omega$. Can you give lower bounds for α? And if AC holds in V_α?

Hints.
See also exercise 127.

3. Note that $\mathrm{V}_{\omega+4}$ contains the well-ordering of all well-orderings (modulo order-isomorphism) of ω.
4. If ZF is consistent, then so are ....
5. In V_ω, every set is finite. In $\mathrm{V}_{\omega+\omega}$ and V_{ω_1}, we have the aforementioned set of all well-orderings of ω.
6. Show by induction on β that for all β, $V_{\omega+4\cdot\beta+1}$ contains a well-ordering of type ω_β, and that this implies that $\omega_\alpha = \alpha$. Construct the lowest α satisfying this condition.

   With AC, show that $\alpha = |V_\alpha|$, and show that for any α satisfying this condition, V_α satisfies Theorem 4.10. Construct the lowest α satisfying this condition.