Hints for Exercises

Chapter 6

111. (AC) Suppose that p, q, r, s are cardinals such that $p < q$ and $r < s$. Show that $p+r < q+s$.
Hint.
Use Lemma 6.8.

114. Prove Lemma 6.22:

1. $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)} > \aleph_\alpha$,
2. $\mathrm{cf}(2^{\aleph_\alpha}) > \aleph_\alpha$.

Hint. Use Theorem 6.15.

116. (Hausdorff) Prove that $\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}$.
Hint. Distinguish as to whether $\aleph_\beta$ is $<$ or $\geqslant$ than $\aleph_{\alpha+1}$.

120. Show: if κ is a strongly inaccesible initial number, then

1. $\beta < \kappa \;\Rightarrow\; \mathrm{V}_\beta <_1 \kappa$,
2. $\mathrm{V}_\kappa =_1 \kappa$,
3. $(\mathrm{V}_\kappa, \in)$ satisfies all ZFC Axioms,
4. if κ is the *least* strong inaccessible, then $(\mathrm{V}_\kappa, \in) \models$"there is no strong inaccessible".

Hint.

1. Straightforward by induction on β.
2. Use $|V_\kappa| = |\bigcup_{\beta<\kappa} V_\beta| \leq \sum_{\beta<\kappa} |V_\beta|$.
3. To show that V_κ satisfies Substitution, show that for any $a \in V_\kappa$ and any operator F with $F[a] \subset V_\kappa$, $\rho(F[a]) < \kappa$. The other axioms are straightforward for any limit ordinal.
4. You may assume that for any 'bound' formula ϕ (where all quantifiers are of the form $\exists x \in y$ or $\forall x \in y$) and any $\vec{x} \in V_\kappa$, $(V_\kappa \models \phi(\vec{x}) \;\Leftrightarrow\; \phi(\vec{x}))$. Show that if $\alpha < \kappa$ does not satisfy one of the conditions for being strongly inaccessible, then there exists a 'witness' for this in V_κ, and hence α fails this same condition in V_κ.

122.

1. Suppose that $X \subset \alpha$ is cofinal in α. Show that α has a cofinal subset Y of type $\mathrm{cf}(\alpha)$ such that $Y \subset X$.
2. Show: if α and β have cofinal subsets of the same type, then $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$.
3. Show: if α is a limit, then $\mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha)$.

Hint.

1. Suppose that $f : \mathrm{cf}(\alpha) \to \alpha$ has $\mathrm{Ran}(f)$ cofinal in α. Define $g : \mathrm{cf}(\alpha) \to X$ by $g(\xi) = \bigcap\{\delta \in X \mid f(\xi) \leq \delta\}$, and apply Lemma 6.26.

2. Use that a cofinal subset of a cofinal subset is a cofinal subset.
3. Show that ω_α and α have cofinal subsets of the same type.