Hints for Exercises

Chapter 4

98 Show: every initial is critical for addition, multiplication and exponentiation.
Hint. For all $\beta, \gamma < \omega_\alpha$, if $\alpha > 0$, then for some $\alpha' < \alpha$, $\beta, \gamma \leqslant_1 \omega_{\alpha'}$. Therefore it suffices to show that for all initials $\omega_{\alpha'}$ and all $\beta, \gamma \leqslant_1 \omega_{\alpha'}$, $\beta + \gamma$, $\beta \cdot \gamma$ and $\beta^\gamma \leqslant_1 \omega_{\alpha'}$. This follows by induction on γ, using Corollary 4.35.

99 Show:

1. $<$ well-orders OR $\times$ OR,
2. every product $\gamma \times \gamma$ is an initial segment
   (if $(\alpha, \beta) < (\alpha', \beta') \in \gamma \times \gamma$, then $(\alpha, \beta) \in \gamma \times \gamma$),
3. the product $\omega \times \omega$ is well-ordered in type ω,
4. every product $\omega_\alpha \times \omega_\alpha$ $(\alpha > 0)$ is well-ordered in type ω_α.

Hint

1. Let $K \subset$ OR $\times$ OR be a class. In order, pick $\gamma = \max(\alpha, \beta)$, α and β using the wellfoundedness of OR, such that (α, β) is $<$-minimal in K.

3 Use Theorem 4.13 to show the existence of a unique order-preserving map $\Gamma :$ OR $\times$ OR $\Rightarrow$ OR, and use that to show that if $\Gamma(\omega, \omega) > \omega$, then $\Gamma(n, m) = \omega$ for some finite n, m. Derive a contradiction.

4 Show that if equality doesn't hold for ω_α, then $\Gamma(\beta, \gamma) = \omega_\alpha$ for some $\beta, \gamma \leqslant_1 \omega_{\alpha'} < \omega_\alpha$, and apply induction on α. $\square$

Chapter 5

101

1. Assume AC. Prove DC: if the set A is non-empty and the relation $R \subset A^2$ is such that $\forall a \in A \exists b \in A(aRb)$, then a function $f : \omega \to A$ exists such that for all $n \in \omega$, $f(n)Rf(n+1)$.
2. Show the version of DC where A can be a proper class and $R \subset A^2$ is also provable from AC. (Use Foundation.)
3. Show that a relation $\prec$ is well-founded (every non-empty set has a $\prec$-minimal element) iff there is no function f on ω such that for all $n \in \omega$, $f(n+1) \prec f(n)$.

Hint.

1. Given a choice function j for $\wp(A)$, define f recursively.
2. Using the Bottom operator of Definition 4.21, construct a set $A' \subset A$ satisfying $\forall a \in A' \exists b \in A'(aRb)$.

103 (AC) Show: if A is infinite, then $\omega \leqslant_1 A$.
Show *without* AC that: if A is infinite, then $\omega \leqslant_1 \wp(\wp(A))$.
Hint.
(i) Define $f : \omega \to A$ recursively in such a way that you can prove inductively that for all n, $f|n$ is an injection.
(ii) Show by induction on n that for all n, $\{B \subset A \mid |B| = n\}$ is nonempty.

105 Show that the following are equivalent for every two sets A and B:

1. $A <_1 B$, i.e.: there is no bijection : $A \to B$ and $A \leqslant_1 B$,
2. there is no surjection : $A \to B$ and $A \leqslant_1 B$,
3. there is no surjection : $A \to B$ and $B \neq \emptyset$.

For which of the six implications do you need AC?
Hint. $2 \Rightarrow 1$ and $2 \Rightarrow 3$ are trivial. For $1 \Rightarrow 2$ you can use Theorem 6.6. To prove $\neg 1 \Rightarrow \neg 3$, use AC to construct a surjection $A \to B$ if $A \not<_1 B$ and $B \neq \emptyset$.

108 The *Teichmüller-Tukey Lemma* is the following statement.

> *Suppose that $\emptyset \neq A \subset \wp(X)$, and for all $Y \subset X$, Y is in A iff every finite subset of Y is in A. Then A has a ($\subset$-) maximal element.*

Show that this is equivalent with Zorn's Lemma.
Hint.
Zorn $\Rightarrow$ TT:
Show that if A is as in the TT Lemma, then it is closed under unions of $\subset$-chains.
TT $\Rightarrow$ Zorn:
Assume that $(X, \preceq)$ is a partial ordering, let A be the set of (by $\preceq$) linearly ordered subsets of X, and show that A satisfies the conditions of the TT Lemma.