Hints for Exercises

Chapter 4

75. Show:

1. *Every* V_α *is transitive,*
2. $x \subset y \in \mathrm{V}_\alpha \;\Rightarrow\; x \in \mathrm{V}_\alpha$,
3. $\alpha < \beta \;\Rightarrow\; \mathrm{V}_\alpha \in \mathrm{V}_\beta$; $\alpha \leq \beta \;\Rightarrow\; \mathrm{V}_\alpha \subset \mathrm{V}_\beta$,
4. $\alpha \subset \mathrm{V}_\alpha$; $\alpha \notin \mathrm{V}_\alpha$; $\alpha = \mathrm{OR} \cap \mathrm{V}_\alpha$,
5. $\mathrm{OR} \cap (\mathrm{V}_{\alpha+1} - \mathrm{V}_\alpha) = \{\alpha\}$.

Hints

1. Use transfinite induction and Exercise 22.

3 Use transfinite induction on β and the transitivity of V_β to show that $\mathrm{V}_\alpha \subset \mathrm{V}_\beta$ for $\alpha \leq \beta$. The other statement follows as a consequence.

4 Use transfinite induction to prove the third statement: the other two follow as consequences.

76 Show:

1. $\rho(\alpha) = \rho(\mathrm{V}_\alpha) = \alpha$,
2. $\mathrm{V}_\alpha = \{a \mid \rho(a) < \alpha\}$; $a \in b \;\Rightarrow\; \rho(a) < \rho(b)$,
3. $\rho(a) = \bigcup\{\rho(b) + 1 \mid b \in a\} = \{\rho(b) \mid b \in \mathrm{TC}(a)\}$
4. express $\rho(a \cup b)$, $\rho(\bigcup a)$, $\rho(\wp(a))$, $\rho(\{a\})$, $\rho((a, b))$ and $\rho(\mathrm{TC}(a))$ in terms of $\rho(a)$ and $\rho(b)$.

Hints

1. Use Lemma 4.17.
2. Use Lemma 4.17, and the property that if $a \in \mathrm{V}_\alpha$, then $a \subset \mathrm{V}_\beta$ for some $\beta < \alpha$.
3. The first statement can be proved by direct rewriting of the condition $a \subset \mathrm{V}_\alpha$, the second follows from this by $\in$-induction.

78 Assuming the Foundation Axiom, prove the Collection Principle:
$\forall x \in a\, \exists y\, \Phi(x, y) \;\Rightarrow\; \exists b\, \forall x \in a\, \exists y \in b\, \Phi(x, y)$ (b not free in Φ).
Hint
Use the Bottom operator on $\{y \mid \Phi(x, y)\}$.

85 Show that the function $h : \mathrm{V}_\omega \to \mathbb{N}$ recursively defined by

$$h(x) = \sum_{y \in x} 2^{h(y)}$$

is a bijection.

Hint
Define $i : \mathbb{N} \to V_\omega$ recursively by setting, for all n,

$$i(n) = \{i(m) \mid \text{the } m\text{-th least significant bit of } n \text{ is } 1\}$$

and show that h and i are each other's inverse.

91 Prove Lemma 4.30:

1. every ω_α is an initial,
2. every initial has the form ω_α,
3. $\alpha < \beta \;\Rightarrow\; \omega_\alpha < \omega_\beta$.

Hints.

1 Straight from definition and Lemma 4.28.

2 Let β be an initial, and let α' be the least ordinal such that $\beta < \omega_{\alpha'}$. Show that $\alpha' = \alpha + 1$ and $\beta = \omega_\alpha$.

3 Transfinite induction on β.

□

93 Let $\alpha \in \mathrm{OR}$ be arbitrary. Recursively define $\alpha_0 = \alpha$ and $\alpha_{n+1} = \omega_{\alpha_n}$. Put $\beta := \bigcup_n \alpha_n$. Show: β is the least ordinal $\gamma \geqslant \alpha$ for which $\omega_\gamma = \gamma$.
Hint
If $\alpha < \omega_\alpha$, then $(\alpha_n)_n$ is strictly increasing. Using this show that β is a limit and rewrite ω_β to show that $\omega_\beta = \beta$. Then prove that β is the *least* such ordinal.

95 For $\alpha \geqslant \omega$, the following are equivalent:
1. α is critical for $+$; 2. $\beta < \alpha \;\Rightarrow\; \beta + \alpha = \alpha$; 3. $\exists \xi\ (\alpha = \omega^\xi)$.
Hint
$\neg(3) \;\Rightarrow\; \neg(2)$: First show that for all $\alpha > 0$ there exists a ξ such that $\omega^\xi \leq \alpha < \omega^{\xi+1}$. Then show that if $\alpha \neq \omega^\xi$, then $\omega^\xi + \alpha > \alpha$ (falsifying (2)).
$(3) \;\Rightarrow\; (1)$: Show that if $\alpha = \omega^\xi$ and $\beta, \gamma < \alpha$, then there exist $\xi' < \xi$ and $n \in \omega$ such that $\beta, \gamma < \omega^{\xi'} \cdot n$.