Hints to Exercises

Chapter 4

54. Assume that the set a is transitive. Show:

1. $a \in \mathbf{G}$ iff $\in$ is well-founded on a,
2. $a \subset \mathbf{TR}$ iff $\in$ is transitive on a.

Thus, an ordinal is the same as a transitive set on which $\in$ is a transitive and well-founded relation. (This is the standard definition of the notion.)
Hint.
1. If $b \ni a$ witnesses $a \notin \mathbf{G}$, show that $a \cap b$ has no $\in$-minimal element. Find a way to also do this the other way around.
2. Rewrite both as properties of x, y, z with $x \in y \in z \in a$.

58. Show:

1. $\alpha \leq \beta \;\;\Leftrightarrow\;\; \alpha \subset \beta$,
2. if K is a non-empty class of ordinals, then $\bigcap K$ is the least element of K,
3. if A is a *set* of ordinals, then $\bigcup A$ is an ordinal that is the *sup* of A (the least ordinal $\geq$ every $\alpha \in A$).

Hint.
1. Use $\alpha \not> \beta \;\Leftrightarrow\; \alpha \leq \beta$ for one of the implications.
2. Note that transfinite induction implies that K has a least element.
3. Show that $\bigcup A$ is an ordinal, and then use (1).

61. Assume that $(A, \prec)$ is a well-ordering and $B \subset A$.
Show that $\mathrm{type}(B, \prec) \leqslant \mathrm{type}(A, \prec)$.
Hint. Assume otherwise, and construct an order-preserving injection from an ordinal into a lesser ordinal to derive a contradiction.

64. Prove Theorem 4.13: suppose that ε is a well-founded relation on the class $\mathbf{U}$ such that for all $a \in \mathbf{U}$, $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}$ is a set, then for every operation $H : \mathbf{V} \to \mathbf{V}$ there is a unique operation $F : \mathbf{U} \to \mathbf{V}$ such that for all $a \in \mathbf{U}$:

$$F(a) = H(F|\{b \in \mathbf{U} \mid b \,\varepsilon\, a\}).$$

Hint
Ignore the hint from the syllabus. Use a proof analogous to that of Theorem 4.10, but requiring that a good function f has a domain satisfying $\{b \in \mathbf{U} \mid b \,\varepsilon\, a\} \subset \mathrm{Dom}(f)$ for all $a \in \mathrm{Dom}(f)$, and using ε-wellfoundedness instead of transfinite induction to show that good functions agree on their domain. Note that in the given conditions, $\{b \in \mathbf{U}^* \mid b \,\varepsilon\, a\}$ is a set for all $a \in \mathbf{U}$.
65. Let $a_0 \in \mathbf{V}$ be a set and $G : \mathbf{V} \to \mathbf{V}$ an operation. Show: there exists a unique operation $F : \mathrm{OR} \to \mathbf{V}$ on OR such that

- $F(0) = a_0$,

- $F(\alpha + 1) = G(F(\alpha))$,
- for limits γ: $F(\gamma) = \bigcup_{\xi<\gamma} F(\xi)$.

Hint
Applying the Recursion Theorem on OR to a suitable operation H.
70. Show that the *single* recursion equation $H{\uparrow}\alpha = \bigcup_{\xi<\alpha} H(H{\uparrow}\xi)$ defines the same operation as the one defined in Definition 4.14 by *three* equations. (And, of course, $H{\downarrow}\alpha = \bigcap_{\xi<\alpha} H(H{\downarrow}\xi)$ is a single equation defining the greatest fixed point hierarchy — cf. Exercise 72.)
Hint.
Use that the least fixed point hierarchy is cumulative, i.e. $\alpha < \beta \Rightarrow H{\uparrow}\alpha \subset H{\uparrow}\beta$.

72 Let H be a monotone operator over a *set* $\mathbf{U}$. The *greatest fixed point hierarchy* is the sequence $\{H{\downarrow}\alpha\}_\alpha$ recursively defined by

- $H{\downarrow}0 = \mathbf{U}$,
- $H{\downarrow}(\alpha + 1) = H(H{\downarrow}\alpha)$,
- $H{\downarrow}\gamma = \bigcap_{\xi<\gamma} H{\downarrow}\xi$ (for limits γ).

Show that:

1. the hierarchy is *descending*, i.e., that $\alpha < \beta \Rightarrow H{\downarrow}\beta \subset H{\downarrow}\alpha$.
2. some stage $H{\downarrow}\alpha_0$ is a fixed point of H.
3. $H{\downarrow}\alpha_0 = \bigcap_\alpha H{\downarrow}\alpha$ is the greatest fixed point of H.

Try to generalize for the case where $\mathbf{U}$ may be a proper class.
Hint.
Consider the dual operator H^d from exercise 52.