Hints for Exercises

Chapter 3

36 Prove Lemma 3.19.1–3. Prove Lemma 3.19.4, and do *not* use 3.18, but use 3.19.1–3.
Hint. 3. Use induction on n for the stages $TC(a,n)$.
4. $\supset$: note that for all $b \subseteq TC(a)$, $TC(b) \subseteq TC(a)$.
$\subset$: show that $a \cup \bigcup_{b \in a} \mathrm{TC}(b)$ is transitive. □

39 Show that $x \in \mathrm{TC}(a)$ iff $x \in^\star a$.
Hint. Show that $xRy \equiv_{\mathrm{def}} x \in y$ and $R^\star \equiv_{\mathrm{def}} x \in TC(y)$ satisfy parts 1-3 of Lemma 3.21.
Alternatively, define R_n and use induction on n to show that $x \in TC(a,n)$ iff $x \in_n a$. □

43 $\mathbb{Z}$ is the set of integers. Define $H : \wp(\mathbb{Z}) \to \wp(\mathbb{Z})$ by $H(X) =_{\mathrm{def}} \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$. Identify the fixed points of H.
Hint. Show that for any fixed point K of H, $-1 \in K$ iff for all $n \in \omega$, $-(1+n) \in K$. Use this to show that H has exactly two fixed points. □

44 Prove Theorem 3.27.
Hint. Do *not* use Theorem 3.24. Show that $n < m \;\Rightarrow\; H{\upharpoonleft}n \subset H{\upharpoonleft}m$. Show: if $H(X) \subset X$, then, for all n, $H{\upharpoonleft}n \subset X$. Finally, show that $H(H{\upharpoonleft}\omega) \subset H{\upharpoonleft}\omega$. (For this, you will need the fact that if Y is finite and $Y \subset \bigcup_{n \in \omega} H{\upharpoonleft}n$, then for some $m \in \omega$, $Y \subset H{\upharpoonleft}m$. This is shown by induction w.r.t. the *number of elements* of Y, cf. Definition 3.15, p.17. □

45 Let $A = \omega \cup \{\omega\}$ and define $H : \wp(A) \to \wp(A)$ by $H(X) = \{0\} \cup \{\mathrm{S}(x) \mid x \in X\}$ if $\omega \not\subset X$, and $H(X) = A$ otherwise. Show: H is monotone, H is not finite, $H{\uparrow} = A$, $\forall n \in \omega\; H{\upharpoonleft}n = n$. Thus, $H{\uparrow} \neq \bigcup_n H{\upharpoonleft}n$.
Hint. To show that $H{\uparrow} = A$, first show that for all $n \in \omega$, $n \in A$. □

51 (Simultaneous inductive definitions.) Suppose that $\Pi, \Delta : \wp(A) \times \wp(A) \to \wp(A)$ are monotone operators in the sense that if $X_1, Y_1, X_2, Y_2 \subset A$ are such that $X_1 \subset X_2$ and $Y_1 \subset Y_2$, then $\Pi(X_1, Y_1) \subset \Pi(X_2, Y_2)$ (and similarly for Δ). Show that K, L exist such that

1. $\Pi(K,L) \subset K$, $\Delta(K,L) \subset L$; in fact, $\Pi(K,L) = K$, $\Delta(K,L) = L$,
2. if $\Pi(X,Y) \subset X$ and $\Delta(X,Y) \subset Y$, then $K \subset X$ and $L \subset Y$.

Show that, similarly, *greatest* (post-) fixed points exist. Generalize to more operators.
Hint. Consider the operator $H : \wp(A \times A) \to \wp(A \times A)$ defined by $H(Z) = \Pi(\pi_1[Z], \pi_2[Z]) \times \Delta(\pi_1[Z], \pi_2[Z])$ (where, as usual, π_1 and π_2 denote the projection onto the first and second coordinates). Show that H has a least fixed point of the form $K \times L$, and that K and L satisfy the given conditions. □