The Impossibility of a Paretian Liberal

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What Are We Going to See?

- Another impossibility result for preference aggregation
  - In Arrow's framework of social welfare functions (slightly generalised)
  - Impossibility caused by liberality (new) in connection with Pareto efficiency (as seen in Arrow)
- Liberality in the sense that there are "personal" decisions which should be taken by a single individual
  - Examples: having pink walls in ones apartment, sleeping on ones back or belly
  - Assumption: Preferences over social states, which are complete descriptions of society

Sen, A.: *The Impossibility of a Paretian Liberal*, The Journal of Political Economy, Vol. 78, No. 1, 1970, pp. 152-157.

Outline

1. The Author: AMARTYA SEN
2. SEN's Impossibility Result
   - Setting, Definitions and Conditions
   - Theorem
   - Proof
3. Critique and "Ways Out"
4. Discussion

SEN, A.: *Collective Choice and Social Welfare*, San Francisco: Holden-Day; and Edinburgh: Oliver & Boyd, 1970.

Amartya Sen

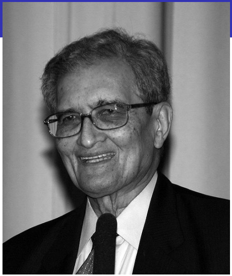

Figure: A. Sen in 2007

- born: 3 Nov 1933, India
- Professor of Economics and Philosophy
- Harvard University
- Publications:
  - 36 books
  - 375 articles in 19 fields
    - Focus on *Economic Development* (53), *Social, Political and Legal Philosophy* (38) and *Welfare Economics* (34)
    - In *Social Choice Theory* 23 articles published (+9 in Axiomatic Choice Theory)
- 143 professional elections and awards
  - including Nobel prize in economics in 1998

Setting, Notation and Basic Definitions

Notation

- A set of *social states* (or alternatives) S
- A finite set of *individuals* $I = \{1, \ldots, n\}$
- The set $\mathcal{B}$ of all binary relations on S
- The set of alternatives $C(X, R)$ that are "best"[1] of the set $X \subseteq S$ with respect to a relation $R \in \mathcal{B}$ (*choice set*)

$$x \in C(X, R) \iff (\forall y \in X)\, xRy$$

- The set $\mathcal{C}$ of all relations such that the choice set $C(X, R)$ is non-empty for any finite subset $X \subseteq S$ (*choice relations*)
  - Equivalently: reflexive, complete and acyclic relations (no transitivity)
- The set $\mathcal{R}$ of (non-strict) linear orders on S (*preference orderings* R)
- The set $\mathcal{P}$ of all strict linear orders on S (*strict preference orderings* P)
- Each individual has an *individual preference ordering* $R_i \in \mathcal{R}$, giving as the full picture a *preference profile* $\langle R_1, R_2, \ldots, R_n \rangle \in \mathcal{R}^n$
- Usually $R \in \mathcal{B}$ will denote the *social preference relation* to be determined

[1] stronger than the usual "maximal"

Types of Social Choice Functions

Most general:

Definition

A *collective choice rule* $f :\subseteq \mathcal{R}^n \rightarrow \mathcal{B}$ is a (potentially partial) function, which assigns a unique *social preference relation* $R \in \mathcal{B}$ to any preference profile $\boldsymbol{R} = \langle R_1, R_2, \ldots, R_n \rangle$.

ARROW:

Definition

A *social welfare function* $f :\subseteq \mathcal{R}^n \rightarrow \mathcal{R}$ is a collective choice rule, whose range is restricted to preference orderings, i.e. which assigns a unique *social preference ordering* $R \in \mathcal{R}$ to any preference profile $\boldsymbol{R} = \langle R_1, R_2, \ldots, R_n \rangle$.

SEN (here):

Definition

A *social decision function* $f :\subseteq \mathcal{R}^n \rightarrow \mathcal{C}$ is a collective choice rule, whose range is restricted to choice relations, i.e. which assigns a unique *choice relation* $R \in \mathcal{C}$ to any preference profile $\boldsymbol{R} = \langle R_1, R_2, \ldots, R_n \rangle$.

social welfare function $\Longrightarrow$ social decision function $\Longrightarrow$ collective choice rule

SEN's Three Conditions

Definition (Unrestricted domain ($\mathbf{U}$))

A social decision function has the property of unrestricted domain if it is total, i.e. if it is defined for any logically possible preference profile.

Definition (Weak Pareto efficiency ($\mathbf{P}$))

A social decision function has the weak Pareto property if for all alternatives $x, y \in S$, we have that xPy whenever xP_iy for all individuals $i \in I$.

$$((\forall i \in I)\, xP_iy) \rightarrow xPy$$

Definition (Liberalism ($\mathbf{L}$))

A social decision function is called liberal if for *each individual* $i \in I$ there is at least *one pair* of distinct alternatives, say (x, y), such that i is *decisive* over that pair of alternatives, i.e. if i prefers x to y, then society must do the same; and if i prefers y to x then society has to choose this preference.

$$(\forall i \in I)(\exists x, y \in S)[x \neq y \wedge (xP_iy \rightarrow xPy) \wedge (yP_ix \rightarrow yPx)]$$

Remark: Liberality implies non-dictatorship

Even Weaker Forms of Liberalism

Definition (Minimal liberalism ($\mathbf{L}^*$))

A social decision function is called *minimal liberal* if there are at least *two distinct individuals* $i, j \in I$ such that each of them is *decisive* over at least *one pair* of alternatives, say (x, y) and (z, w).

$$(\exists i, j \in I)(\exists x, y, z, w \in S) \quad [i \neq j \wedge x \neq y \wedge (xP_iy \rightarrow xPy) \wedge (yP_ix \rightarrow yPx) \wedge z \neq w \wedge (zP_jw \rightarrow zPw) \wedge (wP_jz \rightarrow wPz)]$$

Definition (Super-minimal liberalism ($\mathbf{L}^{***}$))

A social decision function is called *super-minimal liberal* if there are at least *two distinct individuals* $i, j \in I$ such that each of them is *semi-decisive* over at least *one pair* of alternatives, say (x, y) and (z, w), with $x \neq z$ and $y \neq w$.

$$(\exists i, j \in I)(\exists x, y, z, w \in S) \quad [i \neq j \wedge x \neq z \wedge y \neq w \wedge x \neq y \wedge (xP_iy \rightarrow xPy) \wedge z \neq w \wedge (zP_jw \rightarrow zPw)]$$

Remark: $\mathbf{L} \implies \mathbf{L}^* \implies \mathbf{L}^{***} \implies \mathbf{ND}$

SEN's Impossibility Theorem

Definition (Super-minimal liberalism ($\mathbf{L}^{***}$))

A social decision function is called *super-minimal liberal* if there are at least two distinct individuals $i, j \in I$ such that each of them is semi-decisive over at least one pair of alternatives, say (x, y) and (z, w), with $x \neq z$ and $y \neq w$. $(\exists i, j \in I)(\exists x, y, z, w \in S)[i \neq j \wedge x \neq z \wedge y \neq w \wedge x \neq y \wedge (xP_i y \rightarrow xPy) \wedge z \neq w \wedge (zP_j w \rightarrow zPw)]$

Theorem (Sen, 1970)

There is no social decision function that can simultaneously satisfy Conditions $\mathbf{U}$, $\mathbf{P}$ *and* $\mathbf{L}^{***}$.

Corollary (Sen, 1970)

There is no social decision function that can simultaneously satisfy Conditions $\mathbf{U}$, $\mathbf{P}$ *and* $\mathbf{L}^{*}$.

Proof of SEN's Impossibility Theorem

Definition (Super-minimal liberalism ($\mathbf{L}^{***}$))

A social decision function is called *super-minimal liberal* if there are at least two distinct individuals $i, j \in I$ such that each of them is semi-decisive over at least one pair of alternatives, say (x, y) and (z, w), with $x \neq z$ and $y \neq w$. $(\exists i, j \in I)(\exists x, y, z, w \in S)[i \neq j \wedge x \neq z \wedge y \neq w \wedge x \neq y \wedge (xP_i y \rightarrow xPy) \wedge z \neq w \wedge (zP_j w \rightarrow zPw)]$

Theorem (Sen, 1970)

There is no social decision function that can simultaneously satisfy Conditions $\mathbf{U}$, $\mathbf{P}$ *and* $\mathbf{L}^{***}$.

Proof (of the theorem).

- 1, 2 the two individuals of Condition $\mathbf{L}^{***}$; semi-decisive over pairs (x, y) and (z, w), respectively.
- Then, according to $\mathbf{L}^{***}$, $x \neq z$, $y \neq w$, $x \neq y$ and $z \neq w$.
- 3 cases:
  1. Two pairs contain same elements ($x = w$ and $y = z$). Consider $x >_1 y$ and $y = z >_2 w = x$. By $\mathbf{L}^{***}$, $x > y$ and $y > x$. Direct contradiction.
  2. Two pairs have one element in common (say $x = w$). Consider $x >_1 y >_1 z$ and $y >_2 z >_2 w = x$ (in domain by $\mathbf{U}$). By $\mathbf{L}^{***}$, $x > y$ and $z > x$ and by $\mathbf{P}$, $y > z$ yielding an empty choice set $C(\{w = x, y, z\}, \geq)$. Contradiction.
  3. Two pairs are distinct. Consider $w >_1 x >_1 y >_1 z$ and $y >_2 z >_2 w >_2 x$ (in domain by $\mathbf{U}$). By $\mathbf{L}^{***}$, $x > y$ and $z > w$ and by $\mathbf{P}$, $y > z$ and $w > x$. Hence, again no best alternative exists and the choice set $C(\{w, x, y, z\}, \geq)$ is empty for the considered alternatives. Contradiction.

Critique of Acyclicity

- Without acyclicity all three Conditions $\mathbf{U}$, $\mathbf{P}$ and $\mathbf{L}^*$ compatible and lead to pairwise choice function (as long as there are enough pairs to avoid an overlap of liberality [two individuals being decisive over the same pair of alternatives])
- Possible "irresistibility" of the three main conditions is the only argument SEN finds for dropping it
- Can lead to "irrationality":
  - Two individuals, three alternatives: $c >_1 a >_1 b$ and $a >_2 b >_2 c$
  - 1 (semi-)decisive over (a, c), 2 over (b, c)
  - Choice function $C(\{a, b\}, R) = \{a\}, C(\{b, c\}, R) = \{b\}, C(\{a, c\}, R) = \{c\}, C(\{a, b, c\}, R) = \{a\}$ (which can not be represented using a social preference relation)
- "Cheating": Conditions $\mathbf{P}$ and $\mathbf{L}^*$ defined for pairs of alternatives $\rightarrow$ no statement about choice functions
  - Redefinition brings back impossibility (choice set empty in some cases):
  - $\widehat{\mathbf{P}}$: $((\forall i \in I)\, xP_iy) \rightarrow y \notin C(S, R)$
  - $\widehat{\mathbf{L}^*}$: $(\exists i, j \in I)(\exists x, y, z, w \in S)[i \neq j$ $\wedge x \neq y \wedge (xP_iy \rightarrow y \notin C(S, R)) \wedge (yP_ix \rightarrow x \notin C(S, R))$ $\wedge z \neq w \wedge (zP_jw \rightarrow w \notin C(S, R)) \wedge (wP_jz \rightarrow z \notin C(S, R))]$
  - Now consider the above example ($\rightarrow C(\{a, b, c\}, R) = \emptyset$)

Critique of Unrestricted Domain

- Restricting the domain as much as to make Conditions $\mathbf{P}$ and $\mathbf{L}^*$ compatible appears hard to motivate, having the very small example from the paper in mind
- Formally, however, this might be a way out
- Potentially even with different domains for each of the two conditions
- Work in this direction?

Critique of the Pareto Principle

- Pareto principle a "sacred cow" in the literature on social welfare
- Possible attack: Reasons for preference must be considered; excessive nosiness not allowed
  - Mr. A (prude) PREFERS noone to read the book TO him reading the book TO Mr. B reading the book ($c >_1 a >_1 b$)
  - Second preference might be considered irrelevant
- But then whole concept of collective choice rule (and hence SDF and SWF) in doubt
- And evidence for relevance of preferences only indirect
  - Maybe based on social concern ("How will Mr. B behave after having read the 'dangerous' book?") rather than nosiness

Critique of Liberality

- $\mathbf{L}^{***}$ is stronger than non-dictatorship, but a very weak form of what can intuitively be associated with "liberality"
  - requires only two individuals to have liberality
  - and only over one pair each: does for instance not require decisiveness over the many pairs of alternatives that can be formed by varying the "other things" in the extension of "sleeping on ones back or belly" to a complete social state
- But idea of "personal" affairs as such could be considered insupportable (smoking marijuana, suppression of homosexuality, pornography, ...)
- Furthermore, potentially reduced space of alternatives, e.g. war against country1, war against country2, no war
- Not exercising ones right also an option (possibility theorem; Gibbard [1974])
  - Example (again): $c >_1 a >_1 b$ and $a >_2 b >_2 c$ yields cyclic preference $a > b > c > a$
  - If both individuals insist on their decisiveness, we get outcome b; but if 1 does not exercise his right, we get a
  - $\rightarrow$ 1 is better off waiving his liberal rights
- Strategic situation (Gaertner et al. [1992]): $ww >_1 bb >_1 bw >_1 wb$, $bw >_2 wb >_2 ww >_2 bb$
  - 1 decisive over (ww, bw) contradicts minimax behaviour (which would yield bw)
  - 1 decisive over only one of (ww, bw), (wb, bb)? If both then bw and wb out

Conclusion

- Concept of liberalism in ARROW's framework
  - New idea
- Generalization to social decision functions (SDF)
  - Every finite subset has at least one "best" element
  - For which ARROW's conditions are consistent
- Impossibility theorem under the assumption of unrestricted domain $\mathbf{U}$, Pareto principle $\mathbf{P}$ and liberalism $\mathbf{L}^{***}$
  - Relatively easy and straightforward proof
- Critique and possible relaxations of the conditions
  - Acyclicity:
    - Generally possible way out ($\longrightarrow$ pairwise choice; use?)
    - Redefinition of the conditions for choice functions brings back impossibility
  - Unrestricted domain:
    - Hard to motivate relaxation, but possible
    - Different domains for different conditions (work so far?)
  - Pareto principle:
    - Reasons for preferences to be considered
    - Attacks the concept of a collective choice rule (and hence SDF and SWF)
  - Liberality:
    - Already very weak form
    - Applicable to few alternatives?
    - Right-waiving as a way out
    - Problems in strategic domains