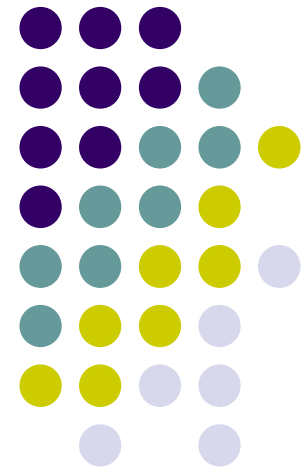# "A Difficulty in the Concept of Social Welfare" (1950)

The original statement of Kenneth J. Arrow's General Possibility Theorem

June 5, 2009

by Stefan Eichinger

Overview of the presentation

1. Preliminaries: surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. Blau (1957): sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

Overview of the presentation

1. Preliminaries: surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. Blau (1957): sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

# Utilitarianism and the origins of modern welfare economics

<u>Two views of ethics:</u>

- traditionally, "Platonism" [cf. p. 335]:

  the good, or ethically desirable, exists independently of people's actual desires and beliefs about what is morally good

  => philosophers' task: discover what is morally good in a genuine sense
  => people's desires and actions can be measured against an objective yardstick

- 19th century: Utilitarianism (Bentham, J. St. Mill, Sidgwick)

  <u>critique of objectivist notion:</u> the morally good is simply that which produces most collective pleasure (hedonist psychology) [cf. p. 335]

  => 'method': discover individual pleasures & calculate collective pleasure from it
  => pleasure can be measured and thus aggregated (unit: util)

  This approach profoundly influenced early welfare economics.

Neoclassical vs. new welfare economics
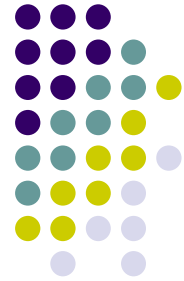
- until 1930s, neoclassical welfare economics (Marshall, Pigou):

  Task: calculating social welfare as a summation of individual utility functions.

  Assumptions: 1) utility can be measured for every individual (cardinal utility)
  2) interpersonal comparability of individual utility functions

  The second assumption, in particular, became the target of increased criticism. This opened the search for more 'realistic' assumptions.

- since 1930s, new welfare economics (Pareto, Hicks, Kaldor):

  The most we are allowed to assume is that each individual can produce preference rankings of alternatives (ordinal utility).

  => notions of Pareto improvement & Pareto efficiency [cf. *passim*]

  => operational test for Pareto efficiency: Kaldor-Hicks-efficiency ("compensation test")
  [cf. p. 330]

Social welfare functions

In 1938, Abram Bergson introduced the notion of social welfare function and showed how much of welfare economics (i.e. those aspects regarding allocative efficiency) could be preserved without assuming cardinal utility. (This approach was subsequently refined by Paul Samuelson.)

=> Problem: In general, many different states are Pareto improvements and Pareto efficient. Can we say more than that?

Cf. Arrow's remark about excise taxation on p. 330:

All we can really say is that society ought to abolish the excise tax and make some redistribution of income and tax burdens; [which would be, according to Arrow, a Pareto improvement] but this is no prescription for action unless there is some principle by which society can make its choice among attainable income distributions, i.e., a social indifference map.
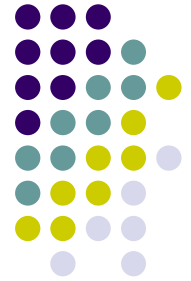
=> Arrow's theorem shows that this is a general problem.

Overview of the presentation

1. Preliminaries: surveying the development and state of
    welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps
    toward the (General) Possibility Theorem

3. Blau (1957): sketching the historical evolution of the
    General Possibility Theorem

4. The theorem's significance according to Arrow &
    some pointers to further discussion

Overview of the presentation

1. Preliminaries: surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. Blau (1957): sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

Basic concepts & axioms

- Basic ingredients:
  - a (finite) set of *individuals* (each denoted by some $n \in \mathbf{N} \backslash \{0\}$);
  - a (finite) set of mutually exclusive *social states*, denoted by $x, y, z, \ldots$;
    [Note: Arrow leaves open what factors enter into the constitution of alternative states: commodity bundles, labour legislations, collective activities etc.];
  - a subset S of the set of social states;
  - three sets of binary relations, denoted by $R, R_1, \ldots, R_n$; $P, P_1, \ldots, P_n$; and $I, I_1, \ldots, I_n$ respectively. Intuitively, the second set expresses *preference* relations, the third *indifference* relations, and the first *preference-or-indifference* relations;
  - a *n*-ary social welfare function SWF: $(R_1, \ldots, R_n) \rightarrow R$ satisfying two axioms and five 'natural' conditions.

- Axioms (inducing a non-strict weak order):
  - Axiom 1: for all x', y', $x'Ry'$ or $y'Rx'$ (totality)
  - Axiom 2: for all x', y', $x'Ry'$ & $y'Rz' \Rightarrow x'Rz'$ (transitivity).

Five 'natural' conditions – part I

- <u>Condition 1:</u>

  SWF is defined for every admissible n-tuple of individual orderings $(R_1, \ldots, R_n)$.

  [<u>Note:</u> The domain of SWF does/need not comprise every logically possible n-tuple of individual orderings. It only includes "some sufficiently wide range of sets of individual orderings" (cf. p. 336 and the example at pp. 339-40)]

- <u>Condition 2:</u>

  If x' is preferred to y' in the social ordering R and x' is raised or does not fall in any of the individual orderings $R_1, \ldots, R_n$ (other things being equal), then x' is preferred to y' in the social ordering R'.

- <u>Condition 3</u>: (independence of irrelevant alternatives)

  Let $(R_1, \ldots, R_n)$, $(R_1', \ldots, R_n')$ be two n-tuples of individual orderings. If for all $x', y' \in S$ and every $0 \leq i \leq n$: $x' R_i y'$ iff $x' R_i' y'$, then the social choice made from S is the same whether we consider $(R_1, \ldots, R_n)$ or $(R_1', \ldots, R_n')$.

Elaborating on condition 1:

Arrow's example for the range of admissible individual orderings (p. 340):

ALTERNATIVE	INDIVIDUAL 1		INDIVIDUAL 2	
	Com- modity 1	Com- modity 2	Com- modity 1	Com- modity 2
1	5	1	5	9
2	4	2	6	8
3	3	3	7	7

Five 'natural conditions' – part II

- <u>Condition 4:</u> (citizens' sovereignty)

  SWF is not imposed.

  [According to Definition 4, SWF is *imposed* iff there are x', y' such that $x'Ry'$ for any n-tuple $(R_1, \ldots, R_n)$.]

- <u>Condition 5:</u> (non-dictatorship)

  SWF is not dictatorial.

  [According to Definition 5, SWF is *dictatorial* iff there exists (an individual) $1 \leq i \leq n$ such that for all x', y': $x'P_i y' \rightarrow x'Py'$.]

Proof framework – part I

- <u>To show:</u> There is no SWF satisfying the two axioms and five 'natural' conditions.

- <u>Strategy:</u> Find some admissible n-tuple of individual orderings $(R_1, …, R_n)$ and show that there is no social ordering R for it, without violating the axioms and conditions.

- <u>Arrow's counterexample:</u>
  Consider a situation with two individuals (denoted by 1 and 2) and three social states (denoted by x, y, z). Consider (R_1, R_2), where $R_1 : x \geq y \geq z$ and $R_2 : z \geq x \geq y$.

Arrow first proceeds to prove two lemmas:

- <u>Consequence 1:</u> If (R_i, R_j) such that $x'P_i y'$ and $x'P_j y'$, then $x'Py'$.
  [by using conditions 2, 3 & 4]

Proof framework – part II

➢ <u>Consequence 3:</u>   If (R_i, R_j) such that $x'P_i y'$ and $y'P_j x'$, then $x'Iy'$.

[by using conditions 2 and 3 and deriving a contradiction with
condition 5]

=> Applying consequences 1 and 3 to the counterexample, we can infer
xPy and yIz and thus, xPz. But we have xIz, too. Contradiction with
weak ordering.

▪ <u>Possibility Theorem:</u>

"If there are at least three alternatives among which the members of the society are free
to choose in any way, then every social welfare function satisfying Conditions 2 and 3
and yielding a social ordering satisfying Axioms I and II must be either imposed or
dictatorial." [p. 342]

Overview of the presentation

1. <u>Preliminaries:</u> surveying the development and state of
   welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps
   toward the (General) Possibility Theorem

3. <u>Blau (1957):</u> sketching the historical evolution of the
   General Possibility Theorem

4. The theorem's significance according to Arrow &
   some pointers to further discussion

Overview of the presentation

1. <u>Preliminaries:</u> surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. <u>Blau (1957):</u> sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

Blau (1957): correcting an 'error'

- Julian H. Blau estimates the contribution of his paper "The existence of social welfare functions" as follows:

  "We study here his [i.e., Arrow's] celebrated theorem that five plausible conditions on the method of aggregation are inconsistent. This theorem is in fact false in general, as a counterexample shows. *When we increase the amount of disagreement which is allowed to occur*, then the inconsistency is restored." [p. 302; my emphasis]

- Blau adopts Arrow's conditions 2-5 (ignoring differences in wording), but replaces condition 1 by:

Condition 1*:

The domain D [of SWF] is sufficiently extensive so that there exists at least one free triple of alternatives [= states]. (A triple is called *free* iff all conceivable combinations of individual orderings of this triple actually occur in D.)

=> since Arrow's notion of admissibility is vague, a comparison is not evident

Blau's counterexample
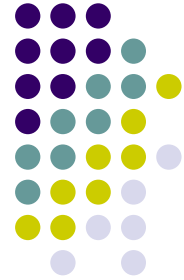
Consider a situation with at least three individuals (one of whom is called Glutton) and four social states (denoted by a, b, c, d).

Let the domain D be described as follows:

a) Each individual must rank all of a, b, c either above or below d (but is otherwise unrestricted.

b) If Glutton ranks d first, then the others must rank d last. If Glutton ranks d last, then the others must rank d first.

Then let SWF be the function whose ordering always coincides with Glutton on the ranking of a, b, c and with the majority (i.e., the others) on d.

It can be verified that this SWF satisfies all axioms and conditions. Thus, Arrow's Possibility Theorem fails if the domain is restricted according to Condition 1*.

Some further results from Blau (1957)
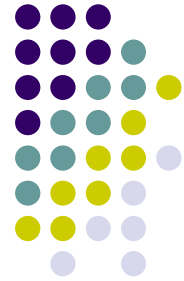
Blau's paper contains a number of results connecting Arrow's original statement of the theorem to modern formulations thereof:

- <u>General Possibility Theorem:</u>

  No SWF can satisfy Conditions 1*, 2-5. [FAILS]

- <u>Unanimity Rule for Preference</u> (URP):

  For a given n-tuple $(R_1, \ldots, R_n)$, if xP_iy for every $1 \leq i \leq n$, then xP_iy.

- <u>Theorem:</u>

  Assume that Conditions 2 and 3 hold, and that D is universal. Then URP is equivalent to Condition 4 (citizens' sovereignty).

- <u>Theorem:</u>

  If D is universal, no SWF can satisfy Conditions 2*, 3 and 5, and URP.

Overview of the presentation

1. <u>Preliminaries:</u> surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. <u>Blau (1957):</u> sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

Overview of the presentation

1. <u>Preliminaries:</u> surveying the development and state of welfare economics up to 1950

2. Basic concepts, axioms/conditions, and key proof steps toward the (General) Possibility Theorem

3. <u>Blau (1957):</u> sketching the historical evolution of the General Possibility Theorem

4. The theorem's significance according to Arrow & some pointers to further discussion

Arrow's own conclusions

- First, consider another statement of the theorem:

  "If we exclude the possibility of interpersonal comparisons of utility, then the only methods of passing from individual tastes to social preferences which will be satisfactory and which will be defined for a wide range of sets of individual orderings are either imposed or dictatorial" (p. 342)

- This statement appears to be interesting for several reasons:

  - "the only methods […] are either imposed or dictatorial":
  Conditions 4 and 5 intended to model social choice by social customs/codes and by dictatorship respectively; cf. discussion at the beginning of the paper.

  Question: Does Condition 5 provide a good formal modeling of dictatorship?
  After all, dictatorship describes a method of decision-making process, not so much its outcome. Might there not be cases where the outcome would be dictatorial in Arrow's sense, but intuitively not?

Arrow's conclusions continued

➢ <u>excluding the possibility of interpersonal comparisons of utility:</u>
In the absence of such interpersonal comparability, Arrow considers his theorem to show that any study of *maximal states* is pointless.

=> Some conditions need to be modified. Several options:

1) Accepting non-transitivity of social orderings (e.g. Sen); for Arrow explicitly not an option.

2) Weakening one or more of the Conditions 2-5?

3) Weakening Condition 1 by limiting the admissible sets of individual orderings? Can such a restriction be achieved by principled means? (Compare, in this context, Arrow's discussion of individual *tastes* and *values* as well as of individualistic assumptions.)

4) Any combination of the above?
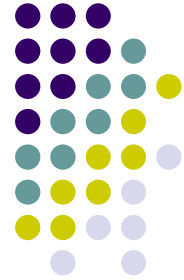
Literature

Arrow, Kenneth J. (1950): "A Difficulty in the Concept of Social Welfare" in: *The Journal of Political Economy*, vol. 58, no. 4, pp. 328-46.

Barberá, Salvador (1980): "A New Proof of Arrow's Theorem" in: *Economic Letters*, vol. 6, pp. 13-16.

Blau, Julian H. (1957): "The Existence of Social Welfare Functions" in: *Econometrica*, vol. 25, no. 2, pp. 302-13.

Geanakoplos, John (2005): "Three brief proofs of Arrow's Impossibility Theorem" in: *Economic Theory*, vol. 26, pp. 211-15.

Goodin, Robert E. & Pettit, Philip (ed.) (1995): *A Companion to Contemporary Political Philosophy* (Oxford: Blackwell).

"The history of Utilitarianism" in: *Stanford Encyclopedia of Philosophy*.