Logical Themes in Social Choice Theory

Daniele Porello

ILLC, University of Amsterdam

LoLaCo - 29/11/2010

Overview

- Part I: Ordinal Welfare and Preference Aggregation:
  - Condorcet's paradox,
  - Aggregating ordinal preferences: Arrow's impossibility theorem,
  - Circumventing impossibilities.
  - Computational themes in preference aggregation.
- Part II: Cardinal Welfare and Resource allocation:
  - Ordinal and Cardinal Welfare,
  - Aggregating cardinal preferences (utility functions),
  - Resource allocation problems,
  - Computational aspects of resource allocation problems.
- Part III: Logic and Resource Allocation:
  - Logical representations of preferences,
  - Multi-sets of goods and Linear Logic.

PART I: ORDINAL WELFARE AND PREFERENCES AGGREGATION

Condorcet's paradox

Given three voters, 1, 2, 3, and three alternatives, r, s, t.

$<$: strong preference relation (transitive, irreflexive, complete)

Suppose individuals rank the alternatives in the following way:

$$1 : r < s < t$$
$$2 : s < t < r$$
$$3 : t < r < s$$

We use *pairwise comparisons* by majority to build up a collective ordering: so $r < s$ and $s < t$. Then, $r < t$, by transitivity. But by majority, also $t < r$. Then, $r < r$ against irreflexivity.

Pairwise comparisons do not define a collective preference order, neither always elect a winner.

Why pairwise comparisons? We look for a *Condorcet-winner*, namely an alternative that beats all the other alternatives in pairwise comparisons.

Marquise de Condorcet "Essai sur l'application de l'analyse à la probabilté des décisions rendues à la pluralité des voix", 1785

Arrow's theorem [1951/1963][1]

Given a (finite) set of voters $\mathcal{N}$ and a (finite) set of alternatives $\mathcal{A}$.
Each individual i has a strong preference order $<_i$,
A *profile of preference orderings*: $\mathbf{P} = (<_\mathbf{1}, \ldots, <_\mathbf{n}) \in \mathcal{L}(\mathcal{A})^\mathbf{n}$
A *social welfare function*:

$$F : \mathcal{L}(\mathcal{A})^n \rightarrow \mathcal{L}(\mathcal{A})$$

$$\mathbf{P} = (<_\mathbf{1}, \ldots, <_\mathbf{n}) \longmapsto <^\mathbf{P}$$

We write $x <_i^\mathbf{P} y$ voter i prefers x over y in the profile $\mathbf{P}$,
we write $x <^\mathbf{P} y$ to say that x is collectively preferred to y according to $F(\mathbf{P})$.

[1] K.J. Arrow. Social Choice and Individual Values. 2nd edition. Cowles Foundation, Yale University Press, 1963.

Formalization of normative constraints

- **Universal domain** (U): the domain of F is the set of all preference profiles $\mathbf{P}$ over alternatives in $\mathcal{A}$.
- **Weak Pareto principle** (P): if for every i in $\mathcal{N}$, $x <_i^{\mathbf{P}} y$, then $x <^{\mathbf{P}} y$.
- **Independence of irrelevant alternatives** (IIA): If $\mathbf{P}$ and $\mathbf{P}'$ are two profiles of preference orders, and x, y are two alternatives in $\mathcal{A}$ such that, for all individuals i in $\mathcal{N}$, $x <_i^{\mathbf{P}} y$ if and only if $x <_i^{\mathbf{P}'} y$, then $x <^{\mathbf{P}} y$ if and only if $x <^{\mathbf{P}'} y$.
- **Non-Dictatorship** (D): There does not exist an individual j in $\mathcal{N}$ such that, for all profiles of preference orders, if $x <_j^{\mathbf{P}} y$, then $x <^{\mathbf{P}} y$.

Arrow's theorem

Theorem [Arrow, 1951]

For at least three individuals and at least three alternatives, there is no social welfare function F satisfying (U), (P), (I) and (D), which generates a collective preference order.

Axiomatic method

- Provides a precise definition of ordinal preferences.
- The general definition of the normative constraints allows for specifying classes of function (instead of considering just particular functions).
- Characterization theorems
- Plurality rule:

Theorem [May, 1952]

For two alternatives, a voting rule is anonymous, neutral, and monotonic iff it is the plurality rule.

- Manipulation problems: an individual knowing others preferences can make his top alternative win submitting a vote which is not his favorite alternative.
  Strategy-proof: not possible to manipulate.

Theorem [Gibbard, 1973 - Satterthwaite, 1975]

Every strategy-proof voting rule for three or more candidates must be dictatorial

Circumventing Impossibility: weakening Arrow's conditions.

Weakening Independence of Irrelevant Alternatives:

- E.g. *Borda rule* (scoring rule):
  Submit a ranking of the m alternatives and give $m-1$ points to 1st ranked, $m-2$ points to 2nd ranked, etc.
- It violates IIA, the result depends not just on the ranking of x and y but also on their "distance".
- E.g.

Profile **P**

a	b	**c**	**d**
3	2	1	0
3	2	1	0
3	2	1	0
1	3	0	2
0	2	1	3
		4	5

Profile **P′**

a	b	**c**	**d**
3	2	1	0
3	2	1	0
3	2	1	0
0	1	2	3
0	1	2	3
		7	6

$c >_i^{\mathbf{P}} d$ iff $c >_i^{\mathbf{P}'} d$ but different outcomes

Possibility and Feasibility

Computational point of view:

- Manipulating Borda outcomes can be done in polynomial time[2].
- Decision problem

MANIPULATION(F):

Instance: Preferences for each voter but one, say j; an alternative c
Question: Is there a preference for j such that c wins?

Sketch of the procedure:

1. Place the alternative c you want to make win on top;
2. Then, check if you can put the remaining alternatives next to it without preventing c from winning. If you cannot do it, then you cannot manipulate on that profile.

[2] J.J. Bartholdi III, C.A. Tovey, and M. A. Trick. The Computational Difficulty of Manipulating an Election. Soc. Choice and Welfare, 1989

Restricting the domain: single peaked profiles

- *Single-peaked profiles*:
  A profile is single-peaked iff there exists a "left-to-right" ordering $\succ$ on the alternatives such that any voter prefers x to y if x is between y and her top alternative wrt $\succ$.
- On single-peaked profiles everything works fine:

Black Median Voter's Theorem:

Given an odd number of voters, if a profile is single-peaked, then there exists a Condorcet winner.

D. Black (1958). *The Theory of Committees and Elections*. Cambridge: Cambridge University Press.

- Remark: checking whether a profile is single-peaked can be done in polynomial time[3]

[3] Bruno Escoffier, Jérôme Lang, "Single-peaked consistency and its complexity". *ECAI, 2008*.

K. Arrow, *Social Choice and Individual Values*, 1951, 2nd ed. 1963

An important book:

- Precise representation of preferences, axiomatization of normative constrains: a foundational work for the formal theory of social choice.
- A foundation of ordinal welfare and of voting theory. Discussion of the assumptions involved (both normative and epistemological).
- Arrow's book and the normative theory of democracy: William Riker, *Liberalism Against Populism. A Confrontation Between the Theory of Democracy and the Theory of Social Choice*, 1982.
- Social choice theory results show that populist justifications of democracy are not grounded: there is no (Rousseauvian) *will of the people*, social decisions depends on the aggregation procedure.

PART II: CARDINAL WELFARE AND RESOURCE ALLOCATION

Ordinal and Cardinal welfare

- Ordinal welfare and voting theory:
  "Because preference relations can only be defined within an explicit set of feasible outcomes, the aggregation method amounts to a decision process very much like voting"
  H. J. Moulin, *Fair Division and Collective Welfare*, MIT Press, 2003.
- *Cardinal welfare*:
- *Utilitarian* point of view: "the greatest *good* for the greatest number of people" (J. Bentham, *The Principles of Morals and Legislation*, 1789.)
  Individuals preferences are given by utility functions (measurability of welfare) and collective welfare is measured aggregating utility functions (interpersonal comparisons).
- The objection ordinalists cast on the cardinal models usually relies on the idea that individual conceptions of *good* are not commensurable.
- Against utilitarianism: "Persons do not count as individuals in this anymore than individual petrol tanks do in the analysis of the national consumption of petroleum"
  A. Sen and B. Williams (eds), *Utilitarianism and Beyond*, Cambridge, 1982 (Introduction).

Cardinal Welfare and Arrow's theorem

- Assumptions: Individual welfare is measurable; the level of welfare of different individuals is comparable;
- This radical change of the framework, allows for circumventing Arrow's impossibility result:

Depending on the definition of interpersonal comparisons, we obtain different social choice procedures matching the *desiderata*.
J. E. Roemer, *Theories of distributive justice*, Harvard University Press, 1996.

- In the next part, we are going to see from a closer perspective an application of cardinal welfare model to *resource allocation*. Then we will point at some logical and computational aspects.

Resource allocation: a framework

- A finite set $\mathcal{N}$ of individuals, a finite set of goods A;
- Individuals express their preferences on *combinations* of goods as a utility function:

$$u_i : \mathcal{P}(A) \to \mathbb{R}$$

- Various preferences can be modelled using different types of utility functions (e.g. monotone, additive, k-additive, ...).
- A resource allocation problem is finding an allocation $\alpha : A \to \mathcal{N}$ satisfying some *desiderata*.
- Qualitative indexes (modelling normative assumptions):
  - utilitarian social welfare: $\sum_{i \in \mathcal{N}} u_i(\alpha_i)$
  - egalitarian social welfare: $\min\{u_i(\alpha_i) \mid i \in \mathcal{N}\}$
- *Two architectures*: centralized vs distributed: (auctions vs negotiation): submitting preferences and evaluating the allocation vs transforming allocations by means of deals.

Combinatorial auctions

- Given a finite set of goods $A = \{a_1, ..., a_n\}$ and a finite set of bidders $N = \{1, ..., n\}$;
- Bidders evaluate different bundles of goods $S \subseteq A$ offering (atomic) bids of the form (S, w) where w is the price associated to the bundle S.
- Each atomic bid (T, w) defines a (simple) *valuation* (an utility function) $v_T : \mathcal{P}(A) \to W$, where W is a set of weights (usually $\mathbb{R}$):
  Given $S \subseteq A$, $v_T(S) = w$ if $T \subseteq S$, $v_T(S) = 0$ otherwise.
- *Bidding languages*: languages to define *complex* bids.
  E.g. OR-language: bid_1 OR bid_2 $OR \ldots OR$ bid_n (pay the sum of the non-overlapping satisfied bids), XOR languages (pay the highest satisfied bid).
  (N. Nisan, "Bidding Languages for Combinatorial Auctions", in *Combinatorial Auctions*, MIT, 2006.)
- Given a collection of bids, the auctioneer looks for an allocation of goods to bids that maximizes the sum of the revenue (utilitarian social welfare).
- The *value* of an allocation α is given by $v(\alpha) = \sum_i \{w_i \ : \ (S_i, w_i) \in \alpha\}$ (utilitarian social welfare)
- *Winner determination problem* (WDP): find an allocation that maximizes the revenue given a set of bids.

Combinatorial Auctions: complexity results

In general, WDP is NP complete [4] .

Theorem

Let $K \in \mathbb{Z}$. The problem of checking whether there exists a solution to a given combinatorial auction instance generating a revenue exceeding K is NP-complete.
(Also NP-complete if bidders submit just atomic bids (S, w))

---

[4] M.H. Rothkopf, A. Pekec, and R.M. Harstad. Computationally Manageable Combinatorial Auctions. Management Science, 44(8):1131-1147, 1998

PART III: LOGIC AND RESOURCE ALLOCATION

Logical languages for preference representation

- Intuitive idea: logical formulas (syntax) can describe utility functions on a set of goods A (semantics).
- *Goal bases*[5]: $G = \{(\phi_1, w_1), \ldots, (\phi_n, w_n)\}$ where ϕ_i is a formula in propositional logic and w_i is a weight.
- Valuations induced by a goal base G, $u_G : \mathcal{P}(A) \to \mathbb{R}$ are defined:

$$X \longmapsto \sum \{w_i \mid (\phi_i, w_i) \in G \text{ and } X \models \phi_i\}$$

- Remark: resources are viewed as propositional atoms and the satisfaction of bids is modelled by the classical logic consequence, e.g. $X = \{a, b, c\}$, $\{a, b, c\} \models a \wedge b$.
- Expressivity issues: which classes of formulas define which classes of functions?
  Succinctness of the representation: is the representation of utility functions via a certain class of formulas essentially more succinct that another?

[5] Joel Uckelman, Yann Chevaleyre, Ulle Endriss, and Jérôme Lang. Representing Utility Functions via Weighted Goals. Mathematical Logic Quarterly, 2009

Multi-sets of goods

- We assume that goods constitute a multi-set, namely that we deal with several indistinguishable copies of a good and we want to model such situations.
  For example, $\{a, a, b\}$ and $\{a, b\}$ are different multi-sets.
- Classical entailment doesn't work well with copies:

Weakening	Contraction
$\dfrac{a, b \vdash a \wedge b}{a, a, b \vdash a \wedge b}$	$\dfrac{a, a \vdash a \wedge a}{a \vdash a \wedge a}$

- We need to use an entailment relation that keeps track of the number of copies of a given propositional letter.

Linear Logic: resource-sensitive account of proofs

(J. Y. Girard, Linear logic, *Theoretical Computer Science*, 1987)
In classical logic sequent calculus, *structural rules* of contraction and weakening define how to deal with hypotheses in a proof:

$$\frac{\Gamma, A, A, \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\mathrm{C}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\,(\mathrm{W})$$

W and *C* determine the behavior of logical connectives, in particular they make the following two presentations of logical rules are equivalent:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\,\wedge \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\,\wedge$$

(*multiplicative and additive presentation*)

- Rejecting structural rules, we are lead to define two conjunctions with different behavior: copying contexts ($\otimes$) or identifying them ($\&$). Linear logic rejects the global validity of structural rules providing a resource sensitive account of proofs.

Intuitive meaning of linear logic connectives:

An analogy: provability as deal between buyers and sellers, *modus ponens*:

Sell one A	*buy one A for one B*	*get one B*
A	$A \multimap B$	$\vdash B$

- $A \multimap B$: buy A for B;
- $A \otimes B$: sell A and B; ($A \otimes B \nvdash A$, we miss someone who buys B)
- $A \,\&\, B$: sell one of A or B; seller's choice $A \,\&\, B \vdash A$;
- $A \oplus B$: sell one of A or B; buyer's choice $A \vdash A \oplus B$;
- $!A$: sell any copy of A *ad libitum* (! reintroduces structural rules.)

Example:

Meaning of linear logic connectives:

- Price: 27 euros,
- Appetizer: Prosciutto e melone/fichi (depending on season)
- Primo: Spaghetti/Gnocchi,
- Drink: Water (as much as you like)

$$Pz \multimap ((P \otimes M) \oplus (P \otimes F)) \otimes (S \,\&\, G) \otimes !W$$

A model of (multi-unit) combinatorial auctions

- An auctioneer wants to sell elements of a finite multiset of goods $\mathcal{M}$ (with finite multiplicity) to a group of bidders.
- We define *Atoms* $\mathcal{A} = \{p_1, \ldots, p_m\}$ as the elements of $\mathcal{M}$ ignoring their multiplicity. Then, multi-sets of goods can be defined as tensor formulas in LL.
  E.g. $p \otimes p \otimes q$.
- Bids are expressions $\langle B_i, w_i \rangle$, with $B_i \subseteq \mathcal{M}$ and a price w_i.
- Bids generate valuations: $v_{\langle B,w \rangle} : \mathcal{P}(\mathcal{M}) \longrightarrow W$, $v_{\langle B,w \rangle}(X) = w$ if $B \subseteq X$, $v_{(B,w)}(X) = 0$ otherwise.
- We can define bidding languages formula in linear logic, with or without weakening.

Bids as formulas

We model atomic bids as formulas of the form $B \multimap u^k$, where B is a tensor product of atoms in $\mathcal{A}$ and u^k is used to model prices symbolically: prices are tensors of a given unit symbol u: $u^k = u \otimes \underbrace{\ldots}_{k-times} \otimes u$

$$\underbrace{p, q, r}_{goods}, \underbrace{p \otimes q \otimes r \multimap u^k}_{bid} \vdash u^k$$

Free disposal, a bidder is willing to obtain *at least* what she demands, is modelled assuming linear logic plus weakening (W):

$$\underbrace{p, q, r, s, t}_{goods}, \underbrace{p \otimes q \otimes r \multimap u^k}_{bid} \vdash_W u^k$$

Valuations as formulas, Allocations as proofs

- In this way, we can define several classes of bidding languages just considering fragments of linear logic.
- *Valuations as formulas*:
  Every bid formula BID generates a valuation v_{BID} mapping multi-sets $X \subseteq \mathcal{M}$ to prices:

$$v_{\mathrm{BID}}(X) = \max\{k \mid X, \mathrm{BID} \vdash u^k\}$$

- *Allocations as proofs*:
  The allocation problem and winner determination can be interpreted as proof-search problems.

Theorem [Porello and Endriss, *KR 2010*]

A proof in (fragments of) linear logic correspond to an allocation of goods and *vice versa*.

(Complexity results corresponds as well)

Example

Decision problem:
Given goods: p, q, r, bids: $p \otimes q \multimap u^4$ and $r \multimap u^2$, can we get a social welfare of 6 units (u^6)?
This is equivalent to ask: can we prove the following sequent?

$$p, q, r, p \otimes q \multimap u^4, r \multimap u^2 \vdash u^6$$

$$\dfrac{\dfrac{p, q \vdash p \otimes q \qquad u^4 \vdash u^4}{p, q, p \otimes q \multimap u^4 \vdash u^4} \qquad \dfrac{r \vdash r \qquad v \vdash v}{r, r \multimap u^2 \vdash u^2}}{p, q, r, p \otimes q \multimap u^4, r \multimap u^2 \vdash u^4 \otimes u^2 (= u^6)}$$

- The proof shows that a social welfare of 6 is achievable.
- The multi-set of allocated goods can be read form the atoms used in the proof.

Summing up

- Ordinal welfare is related to voting theory and preferences aggregation. Thought it relies on lighter methodological assumptions, impossibility results are (almost) everywhere (Arrow, Gibbard- Sattherthwaite).
- We mentioned some complexity considerations since they provides a new type of arguments for choosing a social welfare procedure (possibility vs feasibility).
- We moved to a cardinal welfare model that provides notions of fairness and efficiency without inconsistencies, thought the assumptions are demanding.
- We discussed a framework to deal with a more specific aspect of cardinal welfare, i.e. resource allocation. We saw a centralized mechanism for resource allocation (combinatorial auction) and we discussed its modelling from a more computational perspective.
- We saw how logic can be used to model languages to encode preferences as utility functions.
- We saw how non-classical (sub-structural) logic can be used to model bids when we assume that goods are multi-sets.