Universal Models and the finite model property for closed formulas

Dick de Jongh

Institute for Logic, Language and Computation
Universiteit van Amsterdam

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Outline

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Universal Models

Closed formulas and the finite model property

OVERVIEW

- In cooperations with:
- Nick Bezhanishvili, Cigdem Gençer, Fan Yang, Gerard Renardel de Lavalette, Lex Hendriks.
- in Modal Logic
  - Finite Model Property.
  - Unifiability
- In Intuitionistic Logic
  - Finite fragments
  - Jankov theorems
- Others: Provability Logic, Logics for Topology
- Today: Finite model property (fmp) for closed formulas in modal logic, work with Gençer. Closed formulas are formulas without propositional variables, i.e. containing only $\bot$ and $\top$.

General Idea 1

- For logics with the finite model property:
- Collection of all finite rooted n-models up to bisimulation.
- n-models are models for the language of $p_1, \ldots, p_n$.
- Combined into one model, the n-universal model $\mathcal{U}(n)$.
- Such a model is universal if the logic has the fmp in the sense that it has counterexamples to all formulas that need one, all non-theorems of the logic.
- The canonical model (also: Henkin model) has this property as well, but the advantage of the n-universal model is that from one world the accessible points are finite in number, in other words, submodels generated by 1 point are finite, in still other words each point has finite depth.
- Often the universal model is identical with the points in the Henkin model of finite depth.

General Idea 2

- For logics with the finite model property:
- Collection of all finite rooted n-models up to bisimulation.
- n-models are models for the language of $p_1, \ldots, p_n$.
- Combined into one model, the n-universal model $\mathcal{U}(n)$.
- We look at the 0-universal model of **K4** and extensions(, and n-universal models of **IPC**).
- Construction starts with all the non-isomorphic 1-point models and downwards (against R) avoiding p-morphisms.
- Universal models are in a way, not explained today the smallest models which have all the desired counter-models.
- No formal definition today.

The logic K4

K4 has the axioms

- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (**K**-axiom)
- $\Box\varphi \to \Box\Box\varphi$

Its models are the transitive Kripke models.

P-MORPHISMS (BOUNDED MORPHISMS)

A p-morphism from a model $< W, R, V >$ onto (we are only interested in surjective ones here) a model $< W', R', V' >$ is a function f such that

- $w \in V(p)$ iff $f(w) \in V'(p)$.
- If wRv, then $f(w)R'f(v)$.
- If $f(w)R'v'$, then, for some w' with wRw', $f(w') = v'$.

Theorem

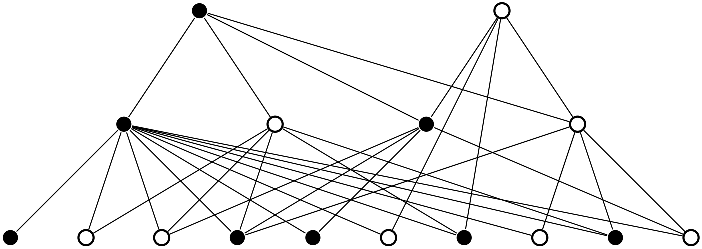
If f is a p-morphism from $< W, R, V >$ onto $< W', R', V' >$, then $w \models \varphi$ iff $f(w) \models' \varphi$. Validity is preserved by p-morphisms.

The 0-universal model of **K4**

Definition

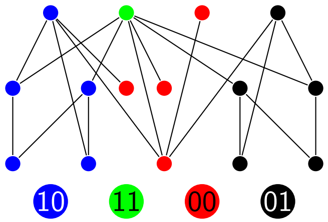
The 0-Universal model $\mathcal{U}_{\mathbf{K4}}(0)$ of **K4** (simultaneously 0-Universal Frame) is constructed as follows: It contains two maximal elements, a reflexive and an irreflexive element. Under any finite proper anti-chain A in $\mathcal{U}_{\mathbf{K4}}(0)$ we put a new reflexive element that is covered by A (i.e. A is exactly the set of its immediate successors), and a new irreflexive element that is covered by A. Under each irreflexive element w we put a reflexive v_1 such that $v_1 \prec w$ (v_1 is covered by $\{w\}$) and an irreflexive v_2 such that $v_2 \prec w$. $\mathcal{U}_{\mathbf{K4}}(0)$ is the result of iterating this procedure.

Drawing the 0-universal model of **K4**



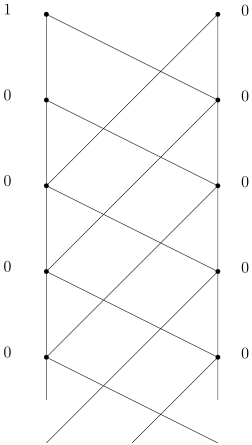
In this figure irreflexive nodes are indicated by a dot, reflexive points by a small circle. The two nodes of level one and the four nodes of level two have all been given, but of the nodes of level three only ones have been drawn that are connected to the leftmost node of level two (and only half of those, either the reflexive ones, or the irreflexive ones). No nodes of higher levels have been drawn.

n-universal models of IPC (drawing n=2)



The top elements are the 2^n different elements of distinct colors (color = classical valuation). Under any finite proper anti-chain A in $\mathcal{U}_{\mathbf{IPC}}(n)$ and any fitting (i.e. respecting **IPC**-persistency) color we put a new element that is covered by A. Under each element w we put a node v such that $v \prec w$ (v is covered by $\{w\}$) with a fitting color that is distinct from the color of w.

The Rieger-Nishimuraladder $(U)_{\mathsf{IPC}}(1)$



Properties of the universal model

Theorem

1. *Each finite Kripke frame for* **K4** *can be mapped p-morphically onto a generated submodel of* $\mathcal{U}_{\mathbf{K4}}(0)$ *in a unique manner.*
2. *For each closed formula* α, $\mathbf{K4} \vdash \alpha$ *iff* $\mathcal{U}_{\mathbf{K4}}(0) \Vdash \alpha$.
3. *For each node* w *of* $\mathcal{U}_{\mathbf{K4}}(0)$ *there exists a (closed) formula* φ_w *such that* $v \Vdash \varphi_w$ *iff* $v = w$.

This generalizes to $\mathcal{U}_\lambda(0)$ for 0-fmp extensions of **K4**.

Outline

Universal Models

Closed formulas and the finite model property

Closed formulas and the fmp

- Definition
  A logic λ has the 0-fmp property (is 0-fmp) if λ has the finite model property with respect to closed formulas, i.e. if $\lambda \nvdash \sigma$ with σ closed, then there is a finite λ-frame on which σ can be falsified.
  - It was not known whether all extensions of **K4** by closed formulas are 0-fmp.
  - We will show that this is not so. But first we will give some extensions of **K4** for which it is the case.

0-universal models of 0-fmp extensions of **K4**

Definition

1. For a 0-fmp logic λ extending **K4** the 0-universal model and frame $\mathcal{U}_\lambda(0)$ is the restriction of the 0-universal model $\mathcal{U}_{\mathbf{K4}}(0)$ to those nodes w for which the upward closed set generated by w is a λ-frame.
2. A subset $A \subseteq \mathcal{U}_\lambda(0)$ is called definable or admissible in $\mathcal{U}_\lambda(0)$ iff there exists a (closed) formula α such that $A = \{x \in \mathcal{U}_\lambda(0) | x \Vdash \alpha\}$.
   Property (3) of universal models:
   For each node w of $\mathcal{U}_{\mathbf{K4}}(0)$ there exists a (*closed*) formula φ_w such that $v \Vdash \varphi_w$ iff $v = w$
   says that each singleton set is admissible, and a corollary is that each finite and each cofinite set is admissible. There may be more of course. Cofinite = complement of finite.

Upsets and downsets

1. Let $< W, R >$ be a frame. A subset $U \subseteq W$ is an upset (upward closed set) if $w \in U$ and wRv imply $v \in U$. A subset $U \subseteq W$ is a downset (downward closed set) if $w \in U$ and vRw imply $v \in U$.
2. The upset generated by a single point is called an upward cone, the downset generated by a single point is called a downward cone.

Upward cone and downward cone

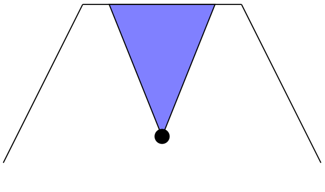


Figure: Upward cone and downward cone

$\Box$- AND $\Diamond$-FORMULAS

Let X_γ denote the set $X_\gamma = \{w | w \Vdash \gamma\}$ for any closed formula γ. If Γ is a set of formulas, $X_\Gamma = \{w | w \Vdash \Gamma\}$, and we write $\Box\Gamma$ for $\Box\Gamma := \{\Box\gamma | \gamma \in \Gamma\}$.

FACT

1. $X_{\Box\psi}$ *is an upward closed set.*
2. $X_{\boxdot\psi} = X_{\psi\wedge\Box\psi}$ *is an upward closed set.*
3. $X_{\Diamond\psi}$ *is a downward closed set.*
4. $X_{\blacklozenge\psi} = X_{\psi\vee\Diamond\psi}$ *is a downward closed set.*

Upsets of logics

Fact

1. *Let Γ be a finite set of closed formulas. For any formula φ the logic $K4 + \Gamma \vdash \varphi$ iff $\boxdot\Gamma \vdash_{\mathbf{K4}} \varphi$.*
2. *Let Γ be a finite set of closed formulas. For any formula φ, $\boxdot\Gamma \vdash_{K4} \varphi$ iff $\boxdot\Gamma \vdash_{K4} \boxdot\varphi$.*

This means that to determine whether $\mathbf{K4} + \Gamma$ has the 0-fmp it suffices to consider formulas φ with an upward closed set X_φ.

Finite extensions of **K4**

Corollary

1. *For any finite set of closed formulas* Γ, $\mathbf{K4} + \Gamma$ *has the* 0*-fmp.*
2. *If* λ *is an extension of* **K4** *that has the* 0*-fmp, then for any finite set of closed formulas* Γ, $\lambda + \Gamma$ *has the* 0*-fmp.*

Arbitrary closed extensions of K4

Theorem

1. *For any (infinite) set of closed formulas* Γ, $\mathbf{K4} + \Gamma$ *has the* 0*-fmp iff, whenever in* $\mathcal{U}_{\mathbf{K4}}(0)$, $X_{\boxdot\Gamma} \subseteq X_\varphi$, *then already* $X_{\boxdot\Delta} \subseteq X_\varphi$ *for some finite* $\Delta \subseteq \Gamma$.
2. *If* λ *is an extension of* $\mathbf{K4}$ *that has the* 0*-fmp, then for any (infinite) set of closed formulas* Γ, $\lambda + \Gamma$ *has the* 0*-fmp iff, whenever in* $\mathcal{U}_\lambda(0)$, $X_{\boxdot\Gamma} \subseteq X_\varphi$, *then already* $X_{\boxdot\Delta} \subseteq X_\varphi$ *for some finite* $\Delta \subseteq \Gamma$.

Sketch of the Proof

Γ does not have the fmp w.r.t. φ iff, on all w in $\mathcal{U}_{\mathbf{K4}}(0)$ where $\boxdot\Gamma$ is true φ is true as well but $\boxdot\Gamma \nvdash_{\mathbf{K4}} \varphi$.
We first note that, on all w in $\mathcal{U}_{\mathbf{K4}}(0)$ where $\boxdot\Gamma$ is true φ is true as well, iff $X_{\boxdot\Gamma} \subseteq X_\varphi$
Next we see that $\mathbf{K4} + \Gamma \nvdash \varphi$ iff, for each finite $\Delta \subseteq \Gamma$, $\mathbf{K4} + \Delta \nvdash \varphi$ iff for each finite $\Delta \subseteq \Gamma$, $X_{\boxdot\Delta} \not\subseteq X_\varphi$.

THE 0-UNIVERSAL MODEL OF **GL**

The 0-universal model $\mathcal{U}_{\mathbf{GL}}(0)$ of **GL** consists of the set of irreflexive worlds $\{w_i \mid i \in \mathbb{N}\backslash\{0\}\}$ where $w_i R w_j$ iff $j < i$.

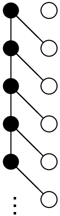
0-FMP FOR GL AND ITS EXTENSIONS

THEOREM

Any modal logic λ extending **GL** *has the* 0*-fmp.*

Sketch of the Proof: The admissible sets in the 0-universal model for **GL** are the finite and cofinite sets. That means that the upward closed admissible sets are besides the whole set the finite ones (defined by $\Box^n\bot$). It also means that any collection of φ_n that define upward closed sets are equivalent to the strongest one of them, $\Box^n\bot$ with n smallest, so since 0-fmp is always true for finite sets of closed formulas, it follows for infinite sets of closed formulas.

The 0-universal model of **K4.3**



Result for K4.3

Theorem

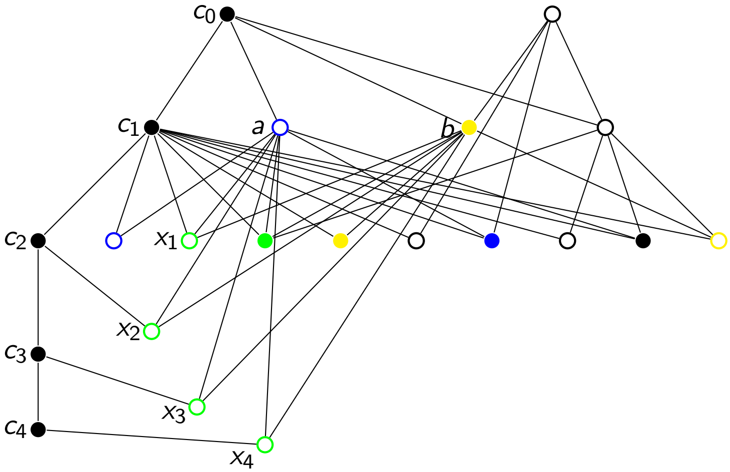
Any modal logic λ extending **K4.3** *has the* 0*-fmp.*

Sketch of the Proof: It is not difficult to see that the admissible sets in $\mathcal{U}_{\mathbf{K4.3}}(0)$ are the finite and cofinite sets. So, the admissible upsets are the finite and cofinite upsets. The upward closed truth sets of infinite sets of closed formulas in $\mathcal{U}_{\mathbf{K4.3}}(0)$ can have arbitrary sets of reflexive nodes but the sets of irreflexive nodes remain the same as in the case of **GL**. Assume we have an upset X_φ and a subset $X_{\Gamma\wedge\Box\Gamma}$ of it. If X_φ is cofinite, the finitely many elements that are not in it are already missed by a finite subset $\Delta \wedge \Box\Delta$ of $\Gamma \wedge \Box\Gamma$, and Theorem 3 can be applied. If X_φ is finite, as in the case of **GL**, $\Gamma \wedge \Box\Gamma$ can only be equivalent to a finite subset, and again Theorem 3 can be applied.

Not all closed extensions of **K4** have the fmp

- 0-fmp closed extensions of **K4** are given by upsets (upward closed sets) in the 0-universal model.
- Consider b. Recall the formula φ_b which is true only in b.
- $\Diamond\varphi_b$ defines the downward cone from b (excluding b).
- Its complement $\Box\neg\varphi_b$ is an admissible upset, containing b.
- Because of the well-foundedness of universal models upsets are uniquely determined by the set of maximal elements of their complement. We call these maximal points the border points of the upset.

Infinitely many border points



Definability of downward cones and their complements

Fact

1. *If x is irreflexive then $\Diamond\varphi_x$ defines the downward cone of x excluding x. The complement of the downward cone of x, including x is defined by $\Box\neg\varphi_x$.*
2. *$\blacklozenge\varphi_x$ always defines the downward cone of x including x. The complement of the downward cone of x, excluding x is defined by $\boxdot\neg\varphi_x$.*

0–FMP FAILS FOR INFINITE SETS OF CLOSED FORMULAS

Not all extensions of $K4$ by closed formulas have the 0-fmp.
Sketch of the Proof:

- It is easy to see that $\Diamond\varphi_b$ has infinitely many maximal elements. Take the lefmost chain $c_0, c_1, c_2, \ldots$ (which is in $\Box\neg\varphi_b$) and consider the nodes x_n that cover $\{a, b, c_n\}$. This is a set of incomparable maximal elements of $\Diamond\varphi_b$.
- This means that $\Box\neg\varphi_b$ has infinitely many border points $\{y_n | n \in \mathbb{N}\}$ including the nodes x_n, and that $\Box\neg\varphi_b$ and $\{\boxdot\neg\varphi_{y_n} | n \in \mathbb{N}\}$ define the same set in the 0-universal model.
- This shows that **K4** $+ \{\boxdot\neg\varphi_{y_n} | n \in \mathbb{N}\}$ does not have the finite model property w.r.t. closed formulas.