Algebraic modal correspondence: Sahlqvist and beyond

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Abstract

This paper provides a bridge in the gap between the model-theoretic and the algebraic side of modal correspondence theory. We give a new, algebraic proof of the classical Sahlqvist correspondence theorem, as well as a new, algebraic proof of the analogous result for the *atomic inductive formulas*, which form a proper extension of the Sahlqvist class.

1 Introduction

Sahlqvist theory: the model-theoretic setting Sahlqvist theory is among the most celebrated and useful results of the classical theory of modal logic, and one of the hallmarks of its success. Traditionally developed in a model-theoretic setting (cf. [15, 1]), it provides an algorithmic, syntactic identification of a class of modal formulas whose associated normal modal logics are *strongly complete* with respect to *elementary* (i.e. first-order definable) classes of frames. The whole theory consists of two parts: *canonicity* and *correspondence*. The *canonicity* [1] of a modal formula φ guarantees the strong completeness of the normal modal logic generated by φ w.r.t. the class $\mathsf{F}(\varphi)$ of frames defined by φ. The fact that φ *corresponds* to some first-order formula α (i.e., that for every frame $\mathcal{F}$, $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \vDash \alpha$) guarantees that $\mathsf{F}(\varphi)$ is elementary. Sahlqvist theory provides sufficient but not

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[1] φ is *canonical* iff φ is valid on the canonical frame for $K\varphi$.

necessary conditions for canonicity and correspondence of modal formulas, and in fact it was shown [4] that the class of elementary modal formulas is undecidable; hence the best one can hope for is achieving decidable approximations of it. The 'Sahlqvist class' has been significantly enlarged over the years, both in syntactic ways (e.g. the definition of *inductive formulas* [13] or the *complex formulas* [17]) and in algorithmic ways, e.g., the algorithm SQEMA introduced in [6].

Sahlqvist theory: the algebraic approach The topo-algebraic approach to Sahlqvist theory was recognized to be useful both for simplifying proofs of existing results [16, 14], and for uniformly extending the Salhqvist theory itself from Boolean to distributive modal logic [11].
The syntactic identification of the Sahlqvist fragment is explained purely in terms of favorable interactions between the topological or order-theoretic properties of the algebraic operations interpreting the logical connectives. The advantages of this approach are, firstly, that the results generalize more easily to different logical signatures; secondly, that canonicity can be treated independently of correspondence; thirdly, that, thanks to this separation, the proof of canonicity can in some cases be achieved constructively [12].

Recently, in [7], SQEMA has been extended to distributive modal logic, thus substantially enlarging the 'Sahlqvist class' of [11]. Also, in [9], a Kripke-style relational semantics has been defined uniformly for logics, such as some substructural logics and their fragments, whose associated algebras are not even based on lattices, but on posets. This latter result paves the way for the development of a Sahlqvist-style canonicity and correspondence theory also for these logics.

Aims and contributions The algebraic approach to Sahlqvist theory is established and fruitful, but alas, not yet very accessible or intuitive: in particular, both the focus over the years on Sahlqvist canonicity (see e.g. [14, 12, 11]) rather than – and independent from – Sahlqvist correspondence, and the general distributive lattice setting it adopts, might be seen as rather arcane. On the model-theoretic side, from most accounts in the current literature, it is not easy to grasp the underlying *mechanism* which endows the Sahlqvist formulas with their excellent behaviour, and devise ways in which the Sahlqvist class can be extended. This paper addresses both these issues: the gist of the algebraic approach is made completely accessible to a readership of non cognoscenti by focusing on *correspondence* rather than canonicity: indeed we prove the best-known *classical* (i.e. Boolean)

Sahlqvist *correspondence* theorem for basic modal logic in the *algebraic* setting of complex algebras of frames. The crucial feature of our account is that the *reduction strategies* (for the elimination of the second order variables) are neatly separated from the *order-theoretic conditions* that guarantee the applicability of these strategies — this is key to a deep understanding of the Sahlqvist mechanism. Moreover, having uncovered the Sahlqvist mechanism, the Sahlqvist correspondence result is then naturally extended to the class of *atomic inductive formulas*, from which more complex reduction strategies can be easily extracted. Finally, the algebraic proofs provided here make it easier to recognize that the Boolean setting in fact plays no essential role in the proof of the Sahlqvist correspondence, and this provides a strong and natural motivation for the algebraic development of Sahlqvist theory in the generalized setting of distributive lattices (see [11, 7]). Further related to this, the algebraic proofs throw light in particular on *adjunction* as a fundamental ingredient of the order-theoretic conditions — a fact which is not easily recognizable in the model-theoretic account. Adjunction is available in complete generality even for posets, and provides the missing link towards the powerful generalizations of the theory to distributive modal logic and to substructural logics.

2 Preliminaries

In this section we recall the basic notions involved in correspondence theory for classical modal logic, and some syntactically specified classes of formulas for which the classical theory effectively provides first-order correspondents.

2.1 Syntax and reformulated modal validity

The *basic modal language*, denoted ML, is defined using a set Prop of propositional variables, also called atomic propositions, and a unary modal operator $\Diamond$ ('diamond'). The well-formed *formulas* of this language are given by the rule

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \psi \vee \varphi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. The connective $\Box$, the dual of $\Diamond$, is defined as $\Box\varphi := \neg\Diamond\neg\varphi$. The boolean connectives $\wedge$, $\rightarrow$, and $\leftrightarrow$ and also the constant $\top$ are defined as usual. We interpret this language on Kripke frames and models in the usual way. Recall that the *complex algebra* of a frame $\mathcal{F} = \langle W, R\rangle$ is the boolean algebra with operator (BAO)

$$\mathcal{F}^+ = (\mathcal{P}(W), \cap, \cup, -_W, \varnothing, W, m_R)$$

where $-_W$ denotes set complementation relative to W, and $m_R : \mathcal{P}(W) \to \mathcal{P}(W)$ is given by

$$m_R(X) := \{w \in W | \text{ there exists } v \in X \text{ such that } Rwv\}.$$

We also let $l_R(X) := \{w \in W | \text{ for all } v \text{ if } Rwv \text{ then } v \in X\}$, or equivalently, $l_{R(X)} := -_W m_R(-_W X)$.

The perspective we develop in Section 3 is based on $(\mathcal{P}(W), \subseteq)$ being a partial order and the operations of $\mathcal{F}^+$ enjoying certain properties w.r.t. this order.

Meaning Function For a formula $\varphi \in$ ML we write $\varphi = \varphi(p_1, \ldots, p_n)$ to indicate that *at most* the atomic propositions $p_1, \ldots p_n$ occur in φ. Every such φ induces an n-ary operation on $\mathcal{P}(W)$,

$$[\![\varphi]\!] : \mathcal{P}(W)^n \longrightarrow \mathcal{P}(W),$$

inductively given by:

$$\begin{array}{rl} [\![\bot]\!] & \text{is the constant function } \varnothing \\ [\![p]\!] & \text{is the identity map } Id_{\mathcal{P}(W)} \\ [\![\neg\varphi]\!] & \text{is the complementation } W \setminus [\![\varphi]\!] \\ [\![\varphi \vee \psi]\!] & \text{is the union } [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\Diamond\varphi]\!] & \text{is the semantic diamond } m_R([\![\varphi]\!]). \end{array}$$

It follows that

$$\begin{array}{rl} [\![\top]\!] & \text{is the constant function } W \\ [\![\varphi \wedge \psi]\!] & \text{is the intersection } [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\Box\varphi]\!] & \text{is the semantic box } l_R([\![\varphi]\!]). \end{array}$$

As to the reformulation mentioned in the title of this subsection, it is based on the following simple observation: for every formula φ, the n-ary operation $[\![\varphi]\!]$ can be also regarded as a map that takes valuations as arguments and gives subsets of $\mathcal{P}(W)$ as its output. Indeed, for every $\varphi \in$ ML, let

$$[\![\varphi]\!](V) := [\![\varphi]\!](V(p_1), \ldots, V(p_n)).$$

Then $[\![\varphi]\!](V)$ is the *extension* of φ under the valuation V, i.e. the set of the states of $(\mathcal{F}, V)$ at which φ is true. Since this happens for all valuations, we can think of $[\![\varphi]\!]$ as the *meaning* function of φ. The local validity clause can be equivalently restated as follows: for every formula φ, frame $\mathcal{F}$ and state $w \in W$,

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad w \in [\![\varphi]\!](V) \text{ for every valuation } V \text{ on } \mathcal{F}.$$

Regarding the meaning functions in this way will be convenient later on in Section 3, when we will develop the discussion on the reduction strategies.

2.2 The standard translation

Let L_0 be the first-order language with $=$, a binary relation symbol R, and denumerably many individual variables $\mathsf{VAR} = \{x_0, x_1, \ldots\}$. Also, let L_1 be the extension of L_0 with a set of unary predicates P, Q, P_0, P_1, etc., corresponding to the propositional variables p, q, p_0, p_1, etc., of Prop. L_2 is the extension of L_1 with universal second-order quantification over the unary predicates $P, Q, P_0, P_1, \ldots$

ML-formulas are translated into L_1 by means of the following *standard translation* ST_x from [3]. Given a first-order variable x and a modal formula φ, this translation yields a first-order formula $\mathrm{ST}_x(\varphi)$ in which x is the only free variable. $\mathrm{ST}_x(\varphi)$ is given inductively by

$$\begin{array}{rcl} \mathrm{ST}_x(p) & = & Px, \\ \mathrm{ST}_x(\bot) & = & x \neq x, \\ \mathrm{ST}_x(\neg\varphi) & = & \neg(ST_x(\varphi)), \\ \mathrm{ST}_x(\varphi \vee \psi) & = & ST_x(\varphi) \vee ST_x(\psi), \\ \mathrm{ST}_x(\Diamond\varphi) & = & \exists y(xRy \wedge ST_y(\varphi)), \text{ where } y \text{ is any fresh variable.} \end{array}$$

The *standard second-order translation* of a modal formula φ is the L_2-formula obtained by universal second-order quantification over all predicates corresponding to proposition letters occurring in φ, that is, the formula $\forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)$.

As is well known, for every model $(\mathcal{F}, V)$ and state w in it, it holds that $(\mathcal{F}, V), w \Vdash \varphi$ iff $(\mathcal{F}, V) \models \mathrm{ST}_x(\varphi)[x := w]$. Moreover, $\mathcal{F}, w \Vdash \varphi$ iff $\mathcal{F} \models \forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)[x := w]$. The analogous global versions of these results are, respectively, $(\mathcal{F}, V) \Vdash \varphi$ iff $(\mathcal{F}, V) \models \forall x \mathrm{ST}_x$ and $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \models \forall P_1 \ldots \forall P_n \forall x \mathrm{ST}_x(\varphi)$.

2.3 Sahlqvist formulas and atomic inductive formulas

For the reader's convenience, we here recall the definition of the Sahlqvist formulas, originally introduced in [15], in a hierarchical form which is conceptually akin to the definition in [3], although it is slightly different from it, to better fit our account in the following section. We also introduce the class of *atomic inductive formulas*, which goes beyond the Sahlqvist class, and is a restriction of the definition of the inductive formulas from [13]. We confine our presentation to the basic modal language.

2.3.1 Sahlqvist implications and atomic inductive implications

Closed and Uniform formulas ([3]). The *closed* modal formulas are those that contain no proposition letter. An occurrence of a proposition letter p in a formula φ is a *positive* (*negative*) if it is under the scope of an even (odd) number of negation signs. (To apply this definition correctly one of course has to bear in mind the negation signs introduced by the defined connectives $\rightarrow$ and $\leftrightarrow$.) A formula φ is *positive* in p (*negative* in p) if all occurrences of p in φ are positive (negative).

A proposition letter occurs *uniformly* in a formula if it occurs only positively or only negatively. A modal formula is *uniform* if all the propositional letters it contains occur uniformly. Let UF be the class of uniform formulas.

Very simple Sahlqvist implications. A *very simple Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and proposition letters, using $\vee$, $\wedge$ and $\Diamond$. A *very simple Sahlqvist implication* is an implication $\varphi \rightarrow \psi$ in which ψ is positive and φ is a very simple Sahlqvist antecedent. Let VSSI be the class of very simple Sahlqvist implications.

Sahlqvist implications ([3]). A *boxed atom* is a propositional variable preceded by a (possibly empty) string of boxes, i.e. a formula of the form $\Box^n p$ where $n \in \mathbb{N}$ and $p \in \mathsf{Prop}$. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using $\vee$, $\wedge$ and $\Diamond$. A *Sahlqvist implication* is an implication $\varphi \rightarrow \psi$ in which ψ is positive and φ is a Sahlqvist antecedent. Let SI be the class of Sahlqvist implications.

Atomic inductive implications. Let $\sharp$ be a symbol not belonging to ML. An *atomic box-form of* $\sharp$ in ML is defined recursively as follows:

1. for every $k \in \mathbb{N}$, $\Box^k \sharp$ is an atomic box-form of $\sharp$;

2. If $B(\sharp)$ is an atomic box-form of $\sharp$, then for any proposition letter p, $\Box(p \rightarrow B(\sharp))$ is an atomic box-form of $\sharp$.

Thus, atomic box-forms of $\sharp$ are of the type

$$\Box(p_0 \rightarrow \Box(p_1 \rightarrow \ldots \Box(p_n \rightarrow \Box^k \sharp) \ldots)),$$

where the p's are are not necessarily different.

By substituting a propositional variable $p \in \mathsf{Prop}$ for $\sharp$ in an atomic box-form $B(\sharp)$ we obtain an *atomic box-formula of* p, namely $B(p)$. The last

occurrence of the variable p is the *head* of $B(p)$ and every other occurrence of a variable in $B(p)$ is *inessential* there. An *atomic regular antecedent* is a formula built up from $\top$, $\bot$, negative formulas, and atomic box-formulas, using $\vee$, $\wedge$ and $\Diamond$.
The *dependency digraph* of a set of box-formulas $\mathcal{B} = \{B_1(p_1), \dots, B_n(p_n)\}$ is the directed graph $G_{\mathcal{B}} = \langle V, E\rangle$ where $V = \{p_1, \dots, p_n\}$ is the set of heads of members of $\mathcal{B}$, and E is a binary relation on V such that $p_i E p_j$ iff p_i occurs as an inessential variable in a box formula in $\mathcal{B}$ with head p_j. A digraph is *acyclic* if it contains no directed cycles or loops. Note that the transitive closure of the edge relation E of an acyclic digraph is a strict partial order, i.e. it is irreflexive and transitive, and consequently also antisymmetric. The dependency digraph of a formula φ is the dependency digraph of the set of box-formulas that occur as subformulas of φ.

An *atomic inductive antecedent* is an atomic regular antecedent with an acyclic dependency digraph. An *atomic regular (resp. inductive) implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is an atomic regular (resp. inductive) antecedent. Let AII be the class of atomic inductive implications.

Example 2.1. Consider the following formulas:

$$\begin{aligned}
\varphi_1 &:= p \wedge \Box(p \to \Box q) \to \Diamond\Box\Box q,\\
\varphi_2 &:= \Diamond\Box p \wedge \Diamond(\Box(p \to q) \vee \Box(p \to \Box\Box r)) \to \Diamond\Box(q \vee \Diamond r)\\
\varphi_3 &:= \Diamond(\Box(p \to \Box\Box q) \vee \Box(q \to \Box p)) \to \Diamond\Box p.
\end{aligned}$$

φ_1 is an atomic inductive implication, which is not a Sahlqvist implication. The antecedent is the conjunction of the atomic box-formulas p and $\Box(p \to \Box q)$. The dependency digraph over the set of heads $\{p, q\}$ has only one edge, from p to q, and thus linearly orders the variables.

φ_2 is an atomic inductive implication. Its dependency digraph has three vertices p, q, and r, and arcs from p to q and from p to r.

φ_3 is an atomic regular but not inductive implication. Its dependency digraph contains a 2-cycle on vertices p and q.

2.3.2 Sahlqvist and atomic inductive formulas

Sahlqvist formulas ([3]). A *Sahlqvist formula* is a formula that is built up from Sahlqvist implications by freely applying boxes, conjunctions and disjunctions[2]. Let SF be the class of Sahlqvist formulas.

[2] Actually, in the definition in [3] the application of disjunction is restricted, and is only allowed between formulas that do not share any propositional variables. Proposition 2.2

Atomic inductive formulas. An *atomic inductive formula* is a formula that is built up from atomic inductive implications by freely applying boxes, conjunctions, and disjunctions. Let AIF be the class of atomic inductive formulas.

The correspondence results for Sahlqvist and atomic inductive formulas can be respectively reduced to the correspondence results for Sahlqvist and atomic inductive *implications*: this is an immediate consequence of the following proposition:

Proposition 2.2. *Let $\Phi \in \{\mathsf{SF}, \mathsf{AIF}\}$. Every $\varphi \in \Phi$ is semantically equivalent to a negated Φ-antecedent, and hence to a Φ-implication*[3].

Proof. Fix a formula $\varphi \in \Phi$, and let φ' be the formula obtained from $\neg\varphi$ by importing the negation over all connectives. Since $\varphi \equiv \varphi' \to \bot$, it is enough to show that φ' is a Φ-antecedent, in order to prove the statement. This is done by induction on the construction of φ from Φ-implications. If φ is a Φ-implication $\alpha \to \mathsf{Pos}$, negating and rewriting it as $\alpha \wedge \neg\mathsf{Pos}$ already turns it into a Φ-antecedent. If $\varphi = \Box\psi$, where ψ satisfies the claim, then $\neg\varphi \equiv \Diamond\neg\psi$ hence the claim follows for φ, because Φ-antecedents are closed under diamonds. Likewise, if $\varphi = \psi_1 \wedge \psi_2$, where ψ_1 and ψ_2 satisfy the claim, then $\neg\varphi \equiv \neg\psi_1 \vee \neg\psi_2$ hence the claim follows for φ, because Φ-antecedents are closed under disjunctions. The case of $\varphi = \psi_1 \vee \psi_2$ is completely analogous. □

2.3.3 Definite implications

In the previous subsubsection, we saw how the correspondence results for formulas in SF and in AIF can be respectively reduced to the correspondence results for formulas in SI and in AII. In their turn, the latter correspondence results can be respectively reduced to the correspondence results for the subclasses of their *definite implications*. These are defined by forbidding the use of disjunction, except within negative formulas, in the building of antecedents. To be precise:

Definition 2.3.
- A *definite very simple Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and propositional letters, using only $\wedge$ and $\Diamond$.

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shows that this restriction is unnecessary in the Boolean case. More about this in Remark 2.6.

[3]By a Φ-*antecedent* we mean a Sahlqvist antecedent if $\Phi = \mathsf{SF}$ or an atomic inductive antecedent if $\Phi = \mathsf{AIF}$; by a Φ-*implication* we mean a Sahlqvist implication if $\Phi = \mathsf{SF}$ or an atomic inductive implication if $\Phi = \mathsf{AIF}$.

- A *definite Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using only $\wedge$ and $\Diamond$.

- A *definite atomic regular antecedent* is a formula built up from $\top$, $\bot$, negative formulas, and atomic box formulas, using only $\wedge$ and $\Diamond$. A *definite atomic inductive antecedent* is a definite atomic regular antecedent with an acyclic dependency digraph.

Let $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}, \mathsf{AII}\}$. Then $\varphi \to \psi \in \Phi$ is a *definite Φ-implication* if φ is a definite Φ-antecedent.

In the next section, we will be able to confine our attention w.l.o.g. to the definite implications in each class, thanks to Fact 2.4 and to Proposition 2.5.

Fact 2.4. *If $\varphi_i \in \mathrm{ML}$ locally corresponds to $\alpha_i(x) \in L_0$ for $1 \leq i \leq n$, then $\bigwedge_{i=1}^{n} \varphi_i$ locally corresponds to $\bigwedge_{i=1}^{n} \alpha_i(x)$.*

Proposition 2.5. *Let $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}, \mathsf{AII}\}$. Every $\varphi \in \Phi$ is equivalent to a conjunction of definite implications in Φ.*

Proof. Note that φ can be equivalently rewritten as a conjunction of definite implications in Φ by exhaustively distributing $\Diamond$ and $\wedge$ over $\vee$ in the antecedent, and then applying the equivalence $A \vee B \to C \equiv (A \to C) \wedge (B \to C)$. $\square$

Remark 2.6. One of the conceptual aims of this paper is showing that the correspondence mechanism is independent of the Boolean setting we are in; however, there is one single point in our presentation in which we took advantage of the specific properties of the classical setting, namely Proposition 2.2. Thanks to classical negation, we are able to pack (i.e. prove the semantical equivalence of) any Sahlqvist/atomic inductive formula in (to) *one* Sahlqvist/atomic inductive implication. In settings where classical negation is not available, this cannot be done. However, Sahlqvist formulas can still be defined as in [3], i.e. allowing the application of disjunction only between formulas that do not share any proposition letters, and the correspondence result for this class can still be reduced to the correspondence result for Sahlqvist implications, thanks to Fact 2.4 and the following facts (cf. [3, Lemma 3.53]):

1. If $\varphi \in \mathrm{ML}$ locally corresponds to $\alpha(x) \in L_0$, then for every $k \in \mathbb{N}$, $\Box^k \varphi$ locally corresponds to $\forall y(xR^k y \to \alpha(y))$.

2. If $\varphi_i \in \mathrm{ML}$ locally corresponds to $\alpha_i(x) \in L_0$ for $1 \leq i \leq n$, and for every $1 \leq i, j \leq n$ if $i \neq j$ then φ_i and φ_j do not have proposition letters in common, then $\bigvee_{i=1}^n \varphi_i$ locally corresponds to $\bigvee_{i=1}^n \alpha(x)_i$.

3 Algebraic correspondence

3.1 The general reduction strategy

Our starting point is the well-known fact, already mentioned above, that any modal formula φ *locally* corresponds to its standard second-order translation, i.e.,

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[x := w]. \tag{1}$$

We are interested in strategies that produce a semantically equivalent first-order condition out of the default local second-order correspondent of φ on the right-hand side of (1).

A large and natural class of formulas for which, by definition, this is possible is introduced in [1] by van Benthem:

Definition 3.1. The class of *van Benthem-formulas*[4] consists of those formulas $\varphi \in \mathrm{ML}$ for which $\forall P_1 \ldots \forall P_n ST_x(\varphi)$ is equivalent to $\forall P'_1 \ldots \forall P'_n ST_x(\varphi)$ where the quantifiers $\forall P'_1 \ldots \forall P'_n$ range, not over all subsets of the domain, but only those that are definable by means of L_0-formulas.

The van Benthem-formulas are the designated targets of the reduction strategy in its most general form. To see this, for every frame $\mathcal{F} = (W, R)$, let

$$\mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}) = \{V' : \mathsf{Prop} \to \mathcal{P}(W) \mid V'(p) \text{ is } L_0\text{-definable for every } p \in \mathsf{Prop}\}.$$

This is the set of the *tame* valuations on $\mathcal{F}$. Using the notation introduced in Subsection 2.1, if $\varphi \in \mathrm{ML}$ is a van Benthem formula, then the following chain of equivalences holds for every $\mathcal{F}$ and every w:

$$\begin{aligned} \mathcal{F}, w \Vdash \varphi \quad & \text{iff} \quad w \in [\![\varphi]\!](V) \text{ for every } V \text{ on } \mathcal{F} \\ & \text{iff} \quad w \in [\![\varphi]\!](V') \text{ for every } V' \in \mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}). \end{aligned}$$

Theorem 3.2. *Every van Benthem-formula has a local first-order frame correspondent.*

[4]this name first appears in [5].

Proof. Let φ be a van Benthem-formula and let Σ be the set of all L_0 substitution instances of $\mathrm{ST}_x(\varphi)$, i.e. the set of all formulas obtained by substituting L_0-formulas $\alpha(y)$ for occurrences $P(y)$ of predicate symbols in $\mathrm{ST}_x(\varphi)$. Clearly, $\forall\overline{P}\mathrm{ST}_x(\varphi) \models \Sigma[x := w]$, where $\overline{P}$ is the vector of all predicate symbols occurring in $\mathrm{ST}_x(\varphi)$. But also, since φ is a van Benthem-formula, $\Sigma \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$.

But then $\Sigma \models \mathrm{ST}_x(\varphi)[x := w]$, and since this is a first-order consequence, we may appeal to the compactness theorem to find some finite subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \mathrm{ST}_x(\varphi)[x := w]$.

We claim that $\Sigma' \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Indeed, let $\mathcal{M}$ be any L_1-model such that $\mathcal{M} \models \Sigma'[x := w]$. Since the predicate symbols in $\overline{P}$ do not occur in Σ', every $\overline{P}$-variant of $\mathcal{M}$ also models Σ', and hence also $\mathrm{ST}_x(\varphi)$. It follows that $\mathcal{M} \models \forall\overline{P}\mathrm{ST}_x(\varphi)[x := w]$.

Thus we may take $\bigwedge \Sigma'$ as a local first-order frame correspondent for φ. □

However, relying on compactness, as it does, Theorem 3.2 is of little use if we want to explicitly calculate the first-order correspondent for a given $\varphi \in \mathrm{ML}$, or devise an algorithm which produces first-order frame correspondents for each member of a given *class* of modal formulas; therefore a more refined strategy is in order, the development of which is the core of correspondence theory.

As mentioned in the beginning, our treatment aims at integrating the model-theoretic and the algebraic accounts of correspondence theory: the model-theoretic one is that each class of modal formulas of Subsection 2.3.1 is defined so as to guarantee that the second crucial 'iff' of the chain above can be proved not just for V' ranging arbitrarily in $\mathsf{Val}_{\mathsf{L_0}}(\mathcal{F})$ but rather ranging in a much more restricted and nicely defined subset of it. And moreover, it is the special way in which each of these subsets of tame valuations is defined that enables the algorithmic generation of the first-order correspondents of the members of each corresponding class of formulas. In this sense, more concretely, we will see that each of the following classes of formulas roughly corresponds to a subset of tame valuations of the form specified in the column on the right:

UF	$V' : \mathsf{Prop} \to \{\varnothing, W\}$
VSSI	$V' : \mathsf{Prop} \to \mathcal{P}_{fin}(W)$
SI	$V' : \mathsf{Prop} \to \{R^n[X] \mid n \in \mathbb{N}, X \in \mathcal{P}_{fin}(W)\}$.

The algebraic account crucially provides an intermediate step which clarifies the model-theoretic account: each class of modal formulas of section

2.3.1 is defined so as to guarantee that, for every formula φ in the given class, $[\![\varphi]\!]$ enjoys certain purely order-theoretic properties that make sure that the second crucial 'iff' can be proved for V' ranging in the nicely defined subclass of tame valuations (the definition of which, as we already mentioned, underlies the algorithmic generation of the first-order correspondent of φ). We start to see how these integrated accounts work together in the next subsection.

3.2 Uniform and Closed formulas

The reduction strategy. Among all the first-order definable valuations V on $\mathcal{F}$, the simplest ones are those which assign W or $\varnothing$ to each propositional variable. Indeed, let V_0 be such a valuation and suppose that the following were equivalent for the modal formula φ:

$$\begin{array}{lll} \mathcal{F}, w \Vdash \varphi & \text{iff} & w \in [\![\varphi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & w \in [\![\varphi]\!](V_0) \end{array}$$

This would in turn mean that

$$\begin{array}{ll} & \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[x := w] \\ \text{iff} & \mathcal{F} \models ST_x(\varphi)[x := w, P_1 := V_0(p_1), \ldots, P_n := V_0(p_n)] \end{array}$$

Therefore, we could equivalently transform the formula above into a first-order formula by replacing each occurrennce $P_i z$ with either $z \neq z$ if $V_0(p_i) = \varnothing$, or with $z = z$ if $V_0(p_i) = W$. This is enough to effectively generate the first-order correspondent of φ.

Order theoretic conditions. For which formulas φ would it be possible to implement the reduction strategy outlined above? The answer to this question can be given in purely order-theoretic terms:

Proposition 3.3. *Let $\langle X_i, \leq \rangle$, $i = 1, \ldots, n$, and $\langle Y, \leq \rangle$ be posets. Let each X_i have a maximum, $\top_i$, and a minimum, $\bot_i$. Let $f : X_1 \times \cdots \times X_n \longrightarrow Y$. If f is either order preserving or order reversing in each coordinate, then the minimum of f exists and is $f(c_1, \ldots c_n)$, where, for every i, $c_i = \bot_i$ if f is order preserving in the i-th coordinate, and $c_i = \top_i$ if f is order reversing in the i-th coordinate.*

Corollary 3.4. For every $\varphi \in$ ML, if $[\![\varphi]\!] : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is order preserving or order reversing in each coordinate, then the following are equivalent:

1. $\forall V[w \in [\![\varphi]\!](V)]$.

2. $w \in [\![\varphi]\!](V_0)$, where $V_0(p_i) = W$ if φ is order reversing in p_i and $V_0(p_i) = \varnothing$ if φ is order preserving in p_i.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) It follows from Proposition 3.3 that $[\![\varphi]\!](V_0)$ is the minimum of $[\![\varphi]\!]$ and hence $[\![\varphi]\!](V_0) \subseteq [\![\varphi]\!](V)$ for every valuation V. □

Syntactic conditions. Now that we have the reduction strategy and sufficient order-theoretic conditions for the strategy to apply, it only remains to verify that these conditions are met by the uniform formulas. And indeed, the following proposition can be easily shown by induction on φ:

Proposition 3.5. *If $\varphi \in$ ML is a uniform formula, then $[\![\varphi]\!]$ is order preserving (reversing) in those coordinates corresponding to propositional variables in which φ is positive (negative).*

Example 3.6. Let us consider the uniform formula $\Box\Diamond p$. The minimal valuation for p is $V_0(p) = \varnothing$, since the formula is positive and hence order-preserving in p. The standard translation of this formula gives

$$\begin{array}{rl} & \mathcal{F} \models \forall P\, \forall y(Rxy \rightarrow \exists z(Ryz \wedge Pz)[x := w] \\ \text{iff} & \mathcal{F} \models \ \forall y(Rxy \rightarrow \exists z(Ryz \wedge P^0 z)[x := w] \end{array}$$

where the predicate $P^0 z$ can be replaced with $z \neq z$ giving a first-order equivalent formula $\forall y(Rxy \rightarrow \exists z(Ryz \wedge z \neq z)$ which simplifies to $\forall y(\neg xRy)$.

To sum up: although the uniform formulas and their accompanying valuations are extremely simple, the key features of our account are already present: first, the subclass of tame valuations is identified, using which the desired first-order correspondent can be effectively computed; second, the order theoretic properties are highlighted, which guarantee the crucial preservation of equivalence; third, the syntactic specification of the formulas φ of the given class guarantees that their associated meaning functions $[\![\varphi]\!]$ meet the required order theoretic properties.

3.2.1 Non-uniform formulas and 'minimal valuation' argument

The discussion above also shows that every uniform formula is locally equivalent on frames to some closed formula (which is obtained by replacing every positive variable with $\bot$ and every negative variable with $\top$). This elimination of variables can in fact be applied not only to uniform formulas but also to formulas that are uniform *with respect to some variables*, so as to eliminate those 'uniform' variables separately. Therefore, modulo this elimination, in the following subsections we are going to assume w.l.o.g. that the formulas we consider are non-uniform in each of their variables. Modulo equivalent rewriting, we can assume w.l.o.g. that every such formula is of the form $\varphi \to \psi$, where ψ is positive, and all the variables occurring in ψ also occur in φ. For such formulas, we have:

$$\begin{array}{lll} \mathcal{F}, w \Vdash \varphi \to \psi & \text{iff} & w \in [\![\varphi \to \psi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & \text{for all } V \text{ on } \mathcal{F}, \text{ if } w \in [\![\varphi]\!](V) \text{ then } w \in [\![\psi]\!](V). \end{array}$$

The well known model-theoretic heuristics for producing the correspondent of formulas of this form is the 'minimal valuation' method (see [2] subsection 9.4): find the (class of) minimal valuation(s) V^* on $\mathcal{F}$ such that $w \in [\![\varphi]\!](V^*)$ (and plug their description in the standard translation of the consequent). The success of this heuristics rests on two conceptually different requirements: first, that 'minimal valuations' exist; second, provided they exists, that they are tame. The account we will present in the next sections, as described in the discussion at the end of Subsection 3.1, will deal with these two requirements separately and in the reverse order. Namely, first we will single out subclasses of L_0-definable valuations (our target class of 'minimal valuations'), restricting the universal quantification to which guarantees that the first-order correspondents can be effectively computed (essentially by way of the 'plug-in' method alluded to above); second, we will show that certain order theoretic properties of the extension maps $[\![\varphi]\!]$, seen as operations on $\mathcal{F}^+$, guarantee that restricting the universal quantification to the target class preserves equivalence (essentially by guaranteeing the existence of a suitable 'minimal valuation' taken in the target class). Third, we will show that the syntactic conditions on φ guarantee that $[\![\varphi]\!]$ satisfies the required order-theoretic properties.

3.3 Very simple Sahlqvist implications

The reduction strategy. Consider the subclass of the tame valuations which assign finite subsets of some bounded size $m \in \mathbb{N}$ to propositional variables, i.e. valuations $V_1 : \mathsf{Prop} \longrightarrow \mathcal{P}_m(W)$, where

$$\mathcal{P}_m(W) := \{S \subseteq W \mid |S| \leq m\},$$

and suppose the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_1(w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1))$.

This would mean that

$$\begin{array}{ll} & \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[x := w] \\ \text{iff} & \mathcal{F} \models \forall P_1^1 \ldots \forall P_n^1 ST_x(\varphi \to \psi)[x := w], \end{array}$$

where the variables P_i^1 would not range over arbitrary subsets of W, but only over those of size at most m. Provided the equivalence between 1 and 2 above holds, we would effectively obtain the local first-order correspondent of $\varphi \to \psi$ by replacing each $\forall P_i^1$ in the prefix with $\forall z_i^1 \forall z_i^2 \ldots \forall z_i^m$ and each atomic formula of the form $P_i^1 y$ with $y = z_i^1 \vee y = z_i^2 \vee \cdots \vee y = z_i^m$, where all the z's are fresh variables.

Order theoretic conditions. An operation $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a *complete operator* if:

(a) f preserves arbitrary joins in each coordinate, i.e., for every $i = 1 \ldots n$, every $\mathcal{X} \subseteq \mathcal{P}(W)$, and all $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \in \mathcal{P}(W)$,

$$f(X_1, \ldots, X_{i-1}, \bigcup \mathcal{X}, X_{i+1}, \ldots, X_n) = \bigcup_{Y \in \mathcal{X}} f(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n);$$

(b) for all $X_1, \ldots, X_n \in \mathcal{P}(W)$, if $X_i = \varnothing$ for some $i \in \{1, \ldots, n\}$, then $f(X_1, \ldots, X_n) = \varnothing$.

Notice that for $n = 1$, (a) implies (b). Indeed,

$$f(\varnothing) = f(\bigcup \{X \in \mathcal{P}(W) \mid X \in \varnothing\}) = \bigcup \{f(X) \mid X \in \varnothing\} = \bigcup \varnothing = \varnothing.$$

Recall that any complete operator is order preserving in each coordinate.

Composition of complete operators will be important for our account: in order to describe their order-theoretic properties, the following definition will be useful.

Definition 3.7. Let $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ be a composition of complete operators.

1. The *degree of g in the ith coordinate*, notation δ_g^i, is defined by induction on g:

   (a) If g is itself a complete operator, then $\delta_g^i = 1$ for every coordinate $1 \leq i \leq n$ whose corresponding variable occurs in g, and $\delta_g^i = 0$ otherwise;

   (b) If $g = f(h_1, h_2, \ldots, h_m)$ for some complete operator f and compositions of complete operators, $h_1, \ldots, h_m$, then $\delta_g^i = \delta_{h_1}^i + \cdots + \delta_{h_m}^i$.

2. The *degree of* g, notation δ_g, is $max\{\delta_g^i \mid 1 \leq i \leq n\}$.

Lemma 3.8. *If* $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ *is a composition of complete operators, then*

1. g *is order preserving in each coordinate, and*
2. *for all* $X_1, \ldots X_n \in \mathcal{P}(W)$*, if* $X_i = \varnothing$ *for some* $1 \leq i \leq n$ *whose corresponding variable occurs in* g*, then* $g(X_1, \ldots, X_n) = \varnothing$*.*

Proof. 1. Every complete operator is order preserving and the composition of order preserving maps is order preserving.
2. By induction on δ_g. □

The composition of *unary* complete operators yields complete operators, but that this is not generally the case for non-unary complete operators:

Example 3.9. Consider the extension map $[\![\varphi]\!]$ for the very simple Sahlqvist antecedent $\varphi(p) = \Diamond p \wedge \Diamond\Diamond p$, defined on the complex algebra of the frame $\mathcal{F} = (W, R)$ with $W = \{w, v, u\}$ and $R = \{(w, v), (v, u)\}$. Then,

$$\begin{array}{rcl}
\varphi(\{v\} \cup \{u\}) & = & m_R(\{v, u\}) \cap m_R(m_R(\{v, u\})) \\
& = & \{w, v\} \cap \{w\} \\
& = & \{w\}. \\
\varphi(\{v\}) \cup \varphi(\{u\}) & = & (m_R(\{v\}) \cap m_R(m_R(\{v\}))) \cup m_R(\{u\}) \cap m_R(m_R(\{u\})) \\
& = & (\{w\} \cap \varnothing) \cup (\{v\} \cap \{w\}) \\
& = & \varnothing \cup \varnothing = \varnothing.
\end{array}$$

However, compositions of complete operators do retain a certain semblance of the join-preservation of the operators from which they are built, as the next lemma shows. We will write $Y \subseteq_k X$ or $Y \in \mathcal{P}_k(X)$ to indicate that $Y \subseteq X$ and $|Y| \leq k$, for $k \in \mathbb{N}$.

Lemma 3.10. *If* $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ *is a composition of complete operators, and* $X_1, \ldots, X_n \in \mathcal{P}(W)$*, then*

$$g(X_1, \ldots, X_n) = \bigcup\{g(S_1, \ldots, S_n) \mid S_i \subseteq_{\delta_g^i} X_i,\ 1 \leq i \leq n\}.$$

Proof. By induction on the degree of g. If $\delta_g = 1$, then g is a complete operator f and hence

$$\begin{array}{rcl}
f(X_1, \ldots, X_n) & = & f(\bigcup\limits_{x_1 \in X_1} \{x_1\}, \ldots, \bigcup\limits_{x_n \in X_n} \{x_n\}) \\
& = & \bigcup\{f(\{x_1\}, \ldots, \{x_n\}) \mid \{x_i\} \subseteq_1 X_i,\ 1 \leq i \leq n\}.
\end{array}$$

If $\delta_g > 1$, then g is of the form $f(h_1, \ldots, h_m)$ where f is a complete operator and each h_i is a composition of complete operators. Then

$$\begin{aligned} g(X_1, \ldots, X_n) &= f(h_1(X_1, \ldots, X_n), \ldots, h_m(X_1, \ldots, X_n)) \\ &= f(\bigcup\{h_i(S_1^i, \ldots, S_n^i) \mid S_j^i \subseteq_{\delta^j_{h_i}} X_j,\ 1 \leq j \leq n\})_{i=1}^m \\ &= \bigcup\{f(h_i(S_1^i, \ldots, S_n^i))_{i=1}^m \mid S_j^i \subseteq_{\delta^j_{h_i}} X_j,\ 1 \leq j \leq n\} \\ &\subseteq \bigcup\{f(h_i(S_1, \ldots, S_n))_{i=1}^m \mid S_j \subseteq_{\delta^j_{h_1} + \cdots + \delta^j_{h_m}} X_j,\ 1 \leq j \leq n\} \\ &= \bigcup\{g(S_1, \ldots, S_n) \mid S_j \subseteq_{\delta^j_g} X_j,\ 1 \leq j \leq n\}. \end{aligned}$$

Here the second equality holds by the inductive hypothesis, and the third since f is a complete operator. The inclusion holds since the set of which the union is taken in the third line is a subset of the corresponding set in the fourth line. The last equality holds by the assumptions on g and by definition of δ_g.
The converse inclusion follows from g being order preserving (Lemma 3.8). □

Consider the following conditions on $[\![\varphi]\!]$:

(a) $[\![\varphi(p_1, \ldots, p_n)]\!] = [\![\varphi']\!](p_1, \ldots, p_n, [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!])$, where

(b) $[\![\varphi'(p_1, \ldots, p_n, s_1, \ldots, s_l)]\!]$ is a composition of complete operators on $\mathcal{P}(W)$,

(c) $[\![\gamma_1]\!]$ to $[\![\gamma_\ell]\!]$ are order reversing in each coordinate.

For every frame $\mathcal{F}$ and every $m \in \mathbb{N}$, let $\mathsf{Val}_1(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ of type $V_1 : \mathsf{Prop} \to \mathcal{P}_m(W)$.

Proposition 3.11. *Let $\varphi \to \psi \in \mathrm{ML}$ be such that $[\![\varphi]\!]$ verifies the conditions (a)-(c) above and $[\![\psi]\!]$ is order-preserving. Let $m = max_{i=1}^n m_i$ where m_i is the degree of $[\![\varphi]\!]$ relative to its ith coordinate. Then the following are equivalent for every frame $\mathcal{F}$:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$
2. $(\forall V_1 \in \mathsf{Val}_1(\mathcal{F}))[w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1)]$.

Proof. (1 ⇒ 2) Clear. (2 ⇒ 1) Let m_i be the degree of $[\![\varphi]\!]$ in the i-th coordinate, for $1 \leq i \leq n$. Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!](V(p_1), \ldots V(p_n), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)).$$

By Lemma 3.8(2), this implies that $V(p_i) \neq \varnothing$ for every $i = 1, \ldots, n$. By Lemma 3.10,

$$[\![\varphi]\!](V) = \bigcup\{[\![\varphi']\!](S_1, \ldots, S_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid S_i \subseteq_{m_i} V(p_i),\ 1 \leq i \leq n\}.$$

Hence, $w \in [\![\varphi]\!](V)$ implies that $w \in [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V))$ for some $T_i \subseteq_{m_i} V(p_i)$, $1 \leq i \leq n$. Let V_1 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_1(p_i) = T_i$: clearly, $V_1 \in \mathsf{Val}_1(\mathcal{F})$; moreover $w \in [\![\varphi]\!](V_1)$: indeed,

$$\begin{array}{rcl} w & \in & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ & \subseteq & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi']\!](V_1(p_1), \ldots, V_1(p_n), [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi]\!](V_1); \end{array}$$

the inclusion in the chain above follows since $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the extensions of the γ's are reversing. Hence, by assumption (2), $w \in [\![\psi]\!](V_1)$. Since $[\![\psi]\!]$ is order preserving in every coordinate, and again $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$, we get $w \in [\![\psi]\!](V_1) \subseteq [\![\psi]\!](V)$, which concludes the proof. $\square$

Syntactic conditions. It remains to verify that the very simple Sahlqvist implications verify the assumptions of Proposition 3.11. The assumptions on ψ are verified because of Proposition 3.5. As to the assumptions on φ:

Proposition 3.12. *If $\varphi = \varphi(p_1, \ldots, p_n)$ is a very simple Sahlqvist antecedent then it verifies the assumptions (a)-(c) of Proposition 3.11. In particular, the maps $[\![\gamma]\!]$'s are exactly the ones induced by the negative formulas occurring in the construction of φ, the map $[\![\varphi']\!]$ is induced by the compound occurrences of $\wedge$ and $\Diamond$, and for every $1 \leq i \leq n$, the degree of $[\![\varphi']\!]$ in the ith coordinate is the number of positive occurrences of p_i in φ.*

Proof. For the first part, note that the identity map $id : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, the intersection $\cap : \mathcal{P}(W) \times \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($\langle X, Y\rangle \mapsto X \cap Y$) and semantic diamond operations $m_R : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($X \mapsto m_R(X)$) are complete operators. The second part is proven by induction on φ. $\square$

Example 3.13. Let us consider the very simple Sahlqvist formula

$$p \wedge \Diamond p \rightarrow \Box p,$$

which locally corresponds to the property of having at most one successor. The variable p occurs twice positively in the antecedent, giving $[\![p \wedge \Diamond p]\!]$

degree 2 in the corresponding coordinate. Hence, according to our reduction strategy, the monadic second order quantification in the second-order translation

$$\forall P[P(x) \wedge \exists y(xRy \wedge P(y)) \to \forall u(xRu \to P(u))]$$

can be equivalently restricted to subsets of size at most 2. Doing this yields the equivalent L_0-formula

$$\forall z_1 \forall z_2[(x = z_1 \vee x = z_2) \wedge \exists y(xRy \wedge (y = z_1 \vee y = z_2)) \to \forall u(xRu \to (u = z_1 \vee u = z_2))].$$

This can be simplified to

$$\forall z_1 \forall z_2[(x = z_1 \vee x = z_2) \wedge (xRz_1 \vee xRz_2)) \to \forall u(xRu \to (u = z_1 \vee u = z_2))],$$

and reasoning a bit further this can be seen to be equivalent to

$$\forall z \forall u(xRu \wedge xRz \to u = z).$$

As seen above, the reduction strategy does not immediately yield the simplest first-order equivalent possible. Some further simplification will usually be possible, as will also be seen in examples 3.20 and 3.25. More optimal equivalents could be produced at the cost of complicating the reduction strategy. This will be further discussed in the conclusion.

3.4 Multiple occurrences of variables

Before moving on to the more general classes of formulas in our hierarchy, let us present some observations that will allow us to significantly simplify the presentation in the following sections.

Definition 3.14. A non-uniform implication $\varphi \to \psi$ is a *1-implication* if every variable $p \in \mathsf{Prop}$ occurs positively in φ at most once.

All the best known examples of Sahlqvist implications in the literature are 1-implications, and from an order theoretic point of view, these formulas are much better behaved: for instance, the following is an easy consequence of Proposition 3.12:

Proposition 3.15. *Let $\varphi \to \psi \in \mathrm{ML}$ be a very simple Sahlqvist implication s.t. φ is a positive formula. If $\varphi \to \psi$ is a 1-implication then $[\![\varphi]\!]$ is a complete operator.*

Moreover, as an immediate consequence of Proposition 3.11, the tame valuations corresponding to the 1-very simple Sahlqvist implications map atomic propositions to singleton subsets. This section is aimed at showing that the correspondence result of any class of implications $\Phi \subseteq \mathrm{ML}$ can be obtained as a consequence of the correspondence result for the 1-implications in Φ.

Given a frame $\mathcal{F} = (W, R)$, let V be the class of all valuations on $\mathcal{F}$, and let V' be a subclass of V. Let $m, k \in \mathbb{N}$, $\varphi = \varphi(r_1, \ldots, r_m, s_1, \ldots, s_k) \in \mathrm{ML}$ be positive in the r-variables, and $\psi = \psi(s_1, \ldots, s_k) \in \mathrm{ML}$. Finally, let $\mathsf{P} \cup \{p\} = \{p_1, \ldots, p_m\} \cup \{p\} \subseteq \mathsf{Prop}$ a subset of fresh variables, i.e. not occurring in φ or ψ, and let V'^p_P be defined as follows: $V'' \in \mathsf{V}'^p_\mathsf{P}$ iff there exists some $V' \in \mathsf{V}'$ s.t. $V'' \sim_p V'$ and $V''(p) = V'(p_1) \cup \cdots \cup V'(p_m)$.

Proposition 3.16. *Suppose that the following are equivalent:*

$$\forall V \in \mathsf{V} \quad [w \in [\![\varphi(p_1/r_1, \ldots, p_m/r_m, \textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V) \\ \Rightarrow w \in [\![\psi(\textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V)] \tag{2}$$

$$\forall V' \in \mathsf{V}' \quad [w \in [\![\varphi(p_1/r_1, \ldots, p_m/r_m, \textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V') \\ \Rightarrow w \in [\![\psi(\textstyle\bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V')] \tag{3}$$

Then the following statements are equivalent:

$$\forall V \in \mathsf{V} \quad [w \in [\![\varphi(p/r_1, \ldots, p/r_m, p/s_1, \ldots, p/s_k)]\!](V) \\ \Rightarrow w \in [\![\psi(p/s_1, \ldots, p/s_k)]\!](V)] \tag{4}$$

$$\forall V' \in \mathsf{V}'^p_\mathsf{P} \quad [w \in [\![\varphi(p/r_1, \ldots, p/r_m, p/s_1, \ldots, p/s_k)]\!](V') \\ \Rightarrow w \in [\![\psi(p/s_1, \ldots, p/s_k)]\!](V')]. \tag{5}$$

Proof. Clearly, (4) implies (5). To prove that (5) implies (4), we will show that (5) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4).

(3) $\Rightarrow$ (2) we have by assumption. Condition (2) implies (4): indeed, since the variables $p_1, \ldots, p_m$ are fresh, then (4) is equivalent to the special case of (2) obtained by imposing the restriction that $V(p) = V(p_1) = \cdots = V(p_m)$ (the details of this proof are left to the reader).

For the sake of the implication from (5) to (3), assume (5) and that

$$w \in [\![\varphi(p_1/r_1, \ldots, p_m/r_m, \bigvee_{i=1}^m p_i/s_1, \ldots, \bigvee_{i=1}^m p_i/s_k)]\!](V') \tag{6}$$

for some $V' \in \mathsf{V}'$. Let V'' be s.t. $V'' \sim_p V'$ and $V''(p) = \bigcup_{i=1}^m V'(p_i)$. Clearly, $V'' \in \mathsf{V}'^p_\mathsf{P}$. Since φ is positive in the r-variables, the following chain holds:

$$\begin{array}{rcl} w & \in & [\![\varphi(p_1/r_1,\ldots,p_m/r_m,\bigvee_{i=1}^m p_i/s_1,\ldots,\bigvee_{i=1}^m p_i/s_k)]\!](V') \\ & \subseteq & [\![\varphi(p/r_1,\ldots,p/r_m,p/s_1,\ldots,p/s_k)]\!](V''). \end{array}$$

Therefore, by assumption (5), $w \in [\![\psi(p/s_1,\ldots,p/s_k)]\!](V'')$. But again, $V'(\bigvee_{i=1}^m p_i) = V''(p)$ implies that

$$[\![\psi(p/s_1,\ldots,p/s_k)]\!](V'') = [\![\psi(\bigvee_{i=1}^m p_i/s_1,\ldots,\bigvee_{i=1}^m p_i/s_k)]\!](V'),$$

which finishes the proof. □

The statement of the Proposition above is more general than we will need: when it is applied to our case of interest, that of the 1-implications, the r-variables and the s-variables in φ are thought of as the place holders for the positive and negative occurrences of the variable p. This statement is also less general than we need, treating just multiple occurrences of *one* variable. However, it is straightforward, modulo introducing another set of indexes, to extend it so as to treat multiple occurrences of n variables.

As we mentioned early on, the proposition above provides us with a uniform way of deriving the correspondence result for any class of implications Φ from the correspondence result for the class 1-Φ of the 1-implications in Φ: indeed, notice that if the class V' is L_0-definable, then so is V'^p_P; therefore if a reduction strategy is available for 1-Φ w.r.t. the class V' of tame valuations, then the Proposition above guarantees that a reduction strategy for any formula $\varphi \to \psi \in \Phi$ is available w.r.t. a class V'_φ of tame valuations that only depends on the multiplicity of occurrences of each variable in φ. Moreover, the reduction algorithm for Φ can be effectively derived from the reduction algorithm for 1-Φ in the following way (here again we just consider multiple occurrences of just *one* variable, leaving the multi-variable case to the reader): if $p \in \mathsf{AtProp}$ occurs positively in φ m times, then consider the *1-formula transform* of $\varphi \to \psi$, i.e. the formula $\varphi^* \to \psi^*$, where φ^* is obtained by replacing each positive occurrence of p in φ by a fresh variable in $\mathsf{P} \subseteq \mathsf{AtProp}$ as above, and each negative occurrence of p in ψ by $\bigvee_{i=1}^m p_i$, and $\psi^* = \psi(\bigvee_{i=1}^m p_i/p)$; consider the standard translation of $\varphi^* \to \psi^*$; by assumption we can eliminate the second order variables P_i from this standard translation by replacing the quantification $\forall P_i$ with $\forall z_i$, where z_i is a fresh variable, and the single occurrences of $P_i y$ with its first-order description $\beta_i(z_i, y)$ derived from the tame valuations. Then the first order correspondent of $\varphi \to \psi$ is effectively obtained by replacing the quantification $\forall P$ with $\forall z_1 \cdots \forall z_m$, where the z_i's are fresh variables, and each occurrence of Py with $\bigvee_{i=1}^m \beta_i(z_i, y)$.

In the remainder of the paper, we will restrict our treatment to the 1-formulas in each class, and will provide more details on the general account only when needed.

3.5 Sahlqvist implications

The reduction strategy. Another promising subclass of tame valuations is formed by those $V_2 \in \mathsf{Val}(\mathcal{F})$ such that for every $p \in \mathsf{Prop}$, $V_2(p) = R[z]$ for some $z \in W$. Indeed, suppose that the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_2(w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2))$.

This would mean that

$$\mathcal{F} \models \forall P_1 \dots \forall P_n ST_x(\varphi \to \psi)[\![w]\!] \quad \text{iff} \quad \mathcal{F} \models \forall P_1^2 \dots \forall P_n^2 ST_x(\varphi \to \psi)[\![w]\!]$$

where the variables P_i^2 would not range on $\mathcal{P}(W)$, but only on $\{R[z] \mid z \in W\}$. Therefore the formula above on the right-hand side can be transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$ and each atomic formula of the form $P_i^2 y$ with $z_i R y \bigvee_{j=1}^{m} z_i^j R y$, where all the z's are fresh variables.

Actually, this argument can be refined and extended to valuations V_2 such that for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[z]$ for some $z \in W$, and some $k \in \mathbb{N}$ relation on W (Notice that the valuations V_1 ranging over singletons are the special case of V_2 where $k = 0$). In this case, the formula above on the right-hand side can be equivalently transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$, and each formula of the form $P_i^2 y$ with an L_0-formula which says 'there exists an R-path from z_i to y in k_i steps', such as:

$$\exists v_0, \dots v_{k_i}[(z_i = v_0 \wedge \bigwedge_{j=0}^{k_i - 1} v_j R v_{j+1} \wedge v_{k_i} = y]$$

This time we are after some conditions on φ and ψ that guarantee that the universal quantification $\forall V$ can be equivalently replaced with the universal quantification $\forall V_2$.

Order theoretic conditions. The maps $f, g : \mathcal{P}(W) \to \mathcal{P}(W)$ form an *adjoint pair* (notation: $f \dashv g$) iff for every $X, Y \in \mathcal{P}(W)$,

$$f(X) \subseteq Y \text{ iff } X \subseteq g(Y).$$

Whenever $f \dashv g$, f is the *left adjoint* of g and g is *the right adjoint* of f. One important property of adjoint pairs of maps is that if a map admits a left (resp. right) adjoint, the adjoint is unique and can be computed pointwise from the map itself and the order (which in our case is the inclusion). This means that admitting a left (resp. right) adjoint is an *intrinsically* order-theoretic property of maps.

Proposition 3.17. *1. Right adjoints between complete lattices are exactly the completely meet-preserving maps, i.e., in the concrete case of powerset algebras $\mathcal{P}(W)$ they are exactly those maps g such that $g(\bigcap S) = \bigcap\{g(X) \mid X \in S\}$ for all $S \subseteq \mathcal{P}(W)$;*

2. Right adjoints on a powerset algebra $\mathcal{P}(W)$ are exactly maps of the form $l_{\mathcal{S}}$ for some binary relation $\mathcal{S}$ on W.

3. For any binary relation $\mathcal{S}$ on W, the left adjoint of $l_{\mathcal{S}}$ is the map $m_{\mathcal{S}^{-1}}$, defined by the assignment $X \mapsto \mathcal{S}[X]$.

Proof. 1. See [8, Proposition 7.34]. 2. We leave to the reader to verify that every map of form $l_{\mathcal{S}}$ is completely meet preserving, hence it is a right adjoint. Conversely, let $g : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ be a right adjoint. Then by item 1 above, g is completely meet preserving. Define $\mathcal{S} \subseteq W \times W$ as follows: for every $x, z \in W$,

$$x\mathcal{S}z \quad \text{iff} \quad x \notin g(W \setminus \{z\}).$$

Hence,

$$x \in l_{\mathcal{S}}(W \setminus \{z\}) \text{ iff } \mathcal{S}[x] \subseteq (W \setminus \{z\}) \text{ iff } z \notin \mathcal{S}[x] \text{ iff } x \in g(W \setminus \{z\}),$$

which shows our claim for all the special subsets of W of type $W \setminus \{z\}$. In order to show it in general, fix $X \in \mathcal{P}(W)$ and notice that $X = \bigcap_{z \notin X}(W \setminus \{z\})$. Using the fact that g is completely meet preserving and the special case shown above, we get:

$$\begin{array}{rcl} g(X) & = & g(\bigcap\{(W \setminus \{z\}) \mid z \notin X\}) \\ & = & \bigcap\{g(W \setminus \{z\}) \mid z \notin X\} \\ & = & \bigcap\{l_{\mathcal{S}}(W \setminus \{z\}) \mid z \notin X\} \\ & = & l_{\mathcal{S}}(\bigcap\{(W \setminus \{z\}) \mid z \notin X\} \quad (*) \\ & = & l_{\mathcal{S}}(X). \end{array}$$

The marked equality can be verified directly, but also follows from the more general fact that $l_{\mathcal{S}}$ is completely meet preserving for every $\mathcal{S}$. 3. Left to the reader. □

Consider the following conditions on $[\![\varphi]\!]$:

(a) $\varphi(p_1,\ldots,p_n) = \varphi'(\chi_1(p_1),\ldots,\chi_n(p_n),\gamma_1,\ldots,\gamma_\ell)$, moreover

(b) $[\![\varphi']\!]$ is a complete operator;

(c) for $1 \le i \le n$, $[\![\chi_i]\!] : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ is a right adjoint, i.e. there exists some $f_i : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ such that for every $X, Y \in \mathcal{P}(W)$, $f_i(X) \subseteq Y$ iff $X \subseteq [\![\chi_i]\!](Y)$;

(d) for every $1 \le i \le n$, f_i is defined by $X \mapsto R^{k_i}[X]$

(e) $[\![\gamma_1]\!]$ to $[\![\gamma_\ell]\!]$ are order reversing in each coordinate.

Notice that, by Proposition 3.17, condition (c) already guarantees that for every i, f_i is defined by $X \mapsto \mathcal{S}_i[X]$ for some *arbitrary* binary relation $\mathcal{S}_i$ on W; however, since $\mathcal{S}_i$ is arbitrary, this is not yet enough to guarantee that valuations defined by $p_i \mapsto f_i(\{z_i\})$ be L_0-definable. Condition (d) above guarantees this last point.

For every frame $\mathcal{F}$, let $\mathsf{Val}_2(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ such that, for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[x]$ for some $x \in W$ and some $k \in \mathbb{N}$.

Proposition 3.18. *Let $\varphi \to \psi \in$ ML be such that $[\![\varphi]\!]$ verifies the conditions (a)-(d) above and $[\![\psi]\!]$ is order preserving in each coordinate. Then the following are equivalent:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$

2. $(\forall V_2 \in \mathsf{Val}_2(\mathcal{F}))[w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2)]$.

Proof. (1 ⇒ 2) Clear. (2 ⇒ 1) Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!]([\![\chi_1]\!](V(p_1)),\ldots,[\![\chi_n]\!](V(p_n)),[\![\gamma_1]\!](V),\ldots,[\![\gamma_\ell]\!](V)),$$

and since by assumption (b) $[\![\varphi']\!]$ is a complete operator, $[\![\chi_i]\!](V(p_i)) \neq \varnothing$ for every $1 \le i \le n$. Moreover, because every set is the union of the singletons of its elements and complete operators preserve arbitrary unions in each coordinate, the following chain of equalities holds:

$$
\begin{array}{rl}
& w \in [\![\varphi]\!](V) \\
= & [\![\varphi']\!]([\![\chi_1]\!](V(p_1)), \ldots, [\![\chi_n]\!](V(p_n)), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\
= & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V(p_i))\}_{i=1}^n \\
= & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid f_i(\{x_i\}) \subseteq V(p_i)\}_{i=1}^n
\end{array}
$$

where the last equality is a consequence of assumption (c). Then $w \in [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)))$ for some $z_i \in W$, $1 \leq i \leq n$, such that $f_i(\{z_i\}) \subseteq V(p_i)$. Let V_2 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_2(p_i) = f_i(\{z_i\})$. By assumption (d), $V_2 \in \mathsf{Val}_2(\mathcal{F})$. Let us show that $w \in [\![\varphi]\!](V_2)$: indeed,

$$
\begin{array}{rl}
w \in & [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\
\subseteq & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid f_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\
\subseteq & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid f_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\
= & \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V_2(p_i))\}_{i=1}^n \\
= & [\![\varphi']\!]([\![\chi_1]\!](V_2(p_1)), \ldots, [\![\chi_n]\!](V_2(p_n)), [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \\
= & [\![\varphi]\!](V_2).
\end{array}
$$

The second inclusion holds since $V_2(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the $[\![\gamma]\!]'s$ are order reversing. By assumption (2), we can conclude that $w \in [\![\psi]\!](V_2)$. Now, since $[\![\psi]\!]$ is order preserving in every coordinate, and $V_2(p_i) = f_i(\{z_i\}) \subseteq V(p_i)$ for every $1 \leq i \leq n$, we get $w \in [\![\psi]\!](V_2) \subseteq [\![\psi]\!](V)$, which concludes the proof. □

Syntactic conditions.

Proposition 3.19. *If* $\varphi \to \psi \in$ ML *is a Sahlqvist 1-implication, then* $\varphi \to \psi$ *verifies the hypotheses of Proposition 3.18. In particular, the maps* $[\![\chi_i]\!]$ *are exactly the ones induced by the boxed atoms.*

Proof. It follows from Propositions 3.12 and 3.17, and the additional fact that, for every $R_1, R_2 \subseteq W \times W$, $l_{R_2} \circ l_{R_1} = l_{R_1 \circ R_2}$. □

Example 3.20. Consider the (definite) 1-Sahlqvist implication $\Diamond\Box p \wedge \Box q \to \Box\Diamond(p \wedge q)$. This has standard second-order frame equivalent

$$\forall P \forall Q[(\exists y(xRy \wedge \forall u(yRu \to P(u))) \wedge \forall v(xRv \to Q(v))) \to \forall w(xRw \to \exists s(Rws \wedge P(s) \wedge Q(s)))].$$

The reduction strategy prescribes that in the above we replace $\forall P \forall Q$ with $\forall z_1 \forall z_2$, and substitute $P(y)$ and $Q(y)$ with $\exists v_0 \exists v_1(v_0 = z_1 \wedge v_0 R v_1 \wedge v_1 = y)$

and $\exists v_0 \exists v_1(v_0 = z_2 \wedge v_0 R v_1 \wedge v_1 = y)$, respectively, which simplify to $z_1 R y$ and $z_2 R y$, respectively. Doing this we obtain the first-order frame equivalent

$$\forall z_1 \forall z_2[(\exists y(xRy \wedge \forall u(yRu \rightarrow z_1 Ru)) \wedge \forall v(xRv \rightarrow z_2 Rv)) \rightarrow \forall w(xRw \rightarrow \exists s(Rws \wedge z_1 Rs \wedge z_2 Rs))],$$

Using the well-known fact that for any first-order formula $\beta(x, y)$ it holds that $\forall x \forall y \beta(x, y) \models \forall x \forall x \beta(x, x)$, we see (by pulling out quantifiers and setting $z_1 = y$ and $z_2 = x$) that the above has as consequence

$$\forall y \forall w(xRy \wedge xRw \rightarrow \exists s(xRs \wedge yRs \wedge wRs)).$$

An easy semantic argument shows that the converse also holds, and hence that the last formula is actually a local first-order frame correspondent for $\Diamond \Box p \wedge \Box q \rightarrow \Box \Diamond (p \wedge q)$.

3.6 Atomic inductive formulas

The reduction strategy. Let us introduce the following piece of notation: For every $i + 1$-ary relation $\mathcal{S}_i$ on W and all $X_1, \ldots, X_i \subseteq W$, let

$$\begin{array}{ll} & \mathcal{S}_i[X_1, \ldots, X_i] \\ := & \{y \in W \mid \exists x_1 \cdots \exists x_i[\bigwedge_{h=1}^{i} x_h \in X_h \ \wedge \ \mathcal{S}_i(x_1, \ldots, x_i, y)]\}. \end{array}$$

The last subclass of tame valuations we are going to consider in this paper can be described as follows: Given an arbitrary finite set $\{p_1, \ldots, p_n\} \subseteq \mathsf{Prop}$ strictly and linearly ordered by $p_i < p_j$ iff $i < j$, assume that there exist some $k_1, \ldots, k_n \in \mathbb{N}$ and some $m_i \in \mathbb{N}$ for $1 < i \leq n$, such that V_3 can be inductively defined as follows:

1. $V_3(p_1) = \mathcal{S}_1[w_1]$ for some $w_1 \in W$, where each $\mathcal{S}_1$ is the composition R^{k_1} of R with itself k_1 times;

2. for every $1 < i \leq n$, $V_3(p_i) = \mathcal{S}_i[\{w_i\}, V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}})]$ for some $w_i \in W$, $p_{i_1}, \ldots, p_{i_{m_i}} \in \{p_0, p_1, \ldots, p_{i-1}\}$, and where $\mathcal{S}_i$ is such that for all $x_0, \ldots, x_{m_i}, y \in W$,

$$\mathcal{S}_i(x_0, \ldots, x_{m_i}, y) \ \text{ iff } \ (\bigwedge_{0 \leq h < m_i} x_h R x_{h+1}) \wedge x_{m_i} R^{k_i} y, \qquad (7)$$

where, as before, R^{k_i} is the composition of R with itself k_i times. (Notice that when $n = 1$, the valuations V_3 reduce to special valuations V_2 of the previous section on simple Sahlqvist formulas).

Suppose that the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_3(w \in [\![\varphi]\!](V_3) \Rightarrow w \in [\![\psi]\!](V_3))$.

This would mean that

$$\mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[\![w]\!] \quad \text{iff} \quad \mathcal{F} \models \forall P_1^3 \ldots \forall P_n^3 ST_x(\varphi \to \psi)[\![w]\!], \tag{8}$$

where the variables P_i^3 would not range on $\mathcal{P}(W)$, but only on sets as described in the first enumeration above. In this case, the right-hand side of (8) can be equivalently transformed into a first-order formula by the following procedure, that we define inductively:

1. Replace $\forall P_1^3$ in the prefix with $\forall z_1$, and substitute each subformula of the form $P_1^3 y$ with $z_1 R^{k_1} y$, where the latter is an abbreviation for $\exists u_1 \ldots \exists u_{k_1+1}(z_1 = u_1 \wedge y = u_{k_1+1} \wedge \bigwedge_{j=1}^{k_1} u_j R u_{j+1})$.
2. Suppose that, for each $1 \leq h < i$, $\forall P_h^3$ in the prefix has been replaced by first-order quantifiers and in the matrix each subformula of the form $P_h^3 y$ has been substituted with an L_0-formula $\alpha_h(y)$. Then, replace $\forall P_i^3$ in the prefix by $\forall z_i$ and substitute each subformula of the form $P_i^3 y$ with

$$\exists v_0, \ldots \exists v_{m_i}[z_i = v_0 \wedge (\bigwedge_{j=0}^{m_i-1} v_j R v_{j+1}) \wedge (\bigwedge_{h=1}^{m_i} \alpha_h(v_h)) \wedge v_{m_i} R^{k_i} y],$$

where $v_{m_i} R^{k_i} y$ is defined similarly to $z_1 R^{k_1} y$ in clause 1 above.

This time we are after some conditions on φ and ψ that guarantee that the universal quantification $\forall V$ can be replaced with the universal quantification $\forall V_3$.

Order theoretic conditions. The notion of adjunction of monotone maps can be generalized to j-ary maps in a component-wise fashion: a j-ary map $f : \mathcal{P}(W)^j \to \mathcal{P}(W)$ is *residuated* if there exists a collection of maps

$$\{g_h : \mathcal{P}(W)^j \to \mathcal{P}(W) \mid 1 \leq h \leq j\}$$

s.t. for every $1 \leq h \leq j$ and for all $X_1, \ldots, X_j, Y \in \mathcal{P}(W)$,

$$f(X_1, \ldots, X_j) \subseteq Y \quad \text{iff} \quad X_h \subseteq g_h(X_1, \ldots, X_{h-1}, Y, X_{h+1}, \ldots, X_j).$$

The map g_h is the *h-th residual* of f. The facts stated in the following example and proposition are well known in the literature in their binary instance (cf. [10, Subsection 3.1.3]):

Example 3.21. For every $j+1$-ary relation $\mathcal{S}$ on W and every $(X_1, \ldots, X_j) \in \mathcal{P}(W)^j$, let

$$\begin{array}{rl} & \mathcal{S}[X_1, \ldots, X_j] \\ := & \{y \in W \mid \exists x_1 \cdots \exists x_j[\bigwedge_{h=1}^{j} x_h \in X_h \ \wedge \ \mathcal{S}(x_1, \ldots, x_j, y)]\}. \end{array}$$

The j-ary operation on $\mathcal{P}(W)$ defined by the assignment

$$(X_1, \ldots, X_j) \mapsto \mathcal{S}[X_1, \ldots, X_j] \tag{9}$$

is residuated and its h-th residual is the map

$$g_h : \mathcal{P}(W)^j \to \mathcal{P}(W)$$

which maps every j-tuple $(X_1, \ldots, X_{h-1}, Y, X_{h+1}, \ldots, X_j)$ to the set

$$\{w \in W \mid \alpha^h_{\mathcal{S}}(w)\},$$

where $\alpha^h_{\mathcal{S}}(w)$ is the following first-order formula:

$$\forall x_1 \cdots \forall y \cdots \forall x_j[(\bigwedge_{k\in \mathbf{j}_h} x_k \in X_k \ \&\ \mathcal{S}(x_1, \ldots, w, \ldots, x_j, y)) \Rightarrow y \in Y],$$

and for every $j \in \mathbb{N}$ and $1 \leq h \leq j$, $\mathbf{j}_h = \{1, \ldots, j\} \setminus \{h\}$.

Proposition 3.22. *If $f : \mathcal{P}(W)^j \to \mathcal{P}(W)$ is residuated and $\{g_h : \mathcal{P}(W)^j \to \mathcal{P}(W) \mid 1 \leq h \leq j\}$ is the collection of its residuals, then:*

1. *f is order preserving in each coordinate and for $1 \leq h \leq j$, g_h is order preserving in its h-th coordinate;*
2. *f preserves arbitrary joins in each coordinate;*
3. *f coincides with the map defined by the assignment (9), for some $j+1$-ary relation $\mathcal{S}$ on W.*

Proof. 1. Fix $1 \leq h \leq j$, let $X_1, \ldots, X_j, Y, Z \in \mathcal{P}(W)$, and assume that $Y \subseteq Z$. By residuation, the "tautological" inclusion

$$f(X_1, \ldots, Z, \ldots, X_j) \subseteq f(X_1, \ldots, Z, \ldots, X_j)$$

is equivalent to the second inclusion in the following chain:

$$Y \subseteq Z \subseteq g_h(X_1, \ldots, f(X_1, \ldots, Z, \ldots, X_j), \ldots, X_j),$$

which yields, again by residuation,

$$f(X_1, \ldots, Y, \ldots, X_j) \subseteq f(X_1, \ldots, Z, \ldots, X_j).$$

The proof that g_h is monotone in the h-th coordinate goes likewise.

2. Fix $\mathcal{Y} \subseteq \mathcal{P}(W)$, $X_1, \ldots, X_{h-1}, X_{h+1}, \ldots X_j \in \mathcal{P}(W)$ and let us show that

$$f(X_1, \ldots, \bigcup \mathcal{Y}, \ldots, X_j) = \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}.$$

The right-to-left inclusion follows by f being order preserving in each coordinate. By residuation, the converse inclusion is equivalent to

$$\bigcup \mathcal{Y} \subseteq g_h(X_1, \ldots, \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}, \ldots, X_j),$$

to prove which, the following chain suffices:

$$\begin{array}{rcl} \bigcup \mathcal{Y} & \subseteq & \bigcup \{g_h(X_1, \ldots, f(X_1, \ldots, Y, \ldots, X_j), \ldots, X_j) \mid Y \in \mathcal{Y}\} \\ & \subseteq & g_h(X_1, \ldots, \bigcup \{f(X_1, \ldots, Y, \ldots, X_j) \mid Y \in \mathcal{Y}\}, \ldots, X_j). \end{array}$$

The first inclusion readily follows by applying residuation to the "tautological" inclusions

$$f(X_1, \ldots, Y, \ldots, X_j) \subseteq f(X_1, \ldots, Y, \ldots, X_j)$$

for every $Y \in \mathcal{Y}$. The second one follows by the monotonicity of g_h in its h-th coordinate.

3. Let $\mathcal{S} \subseteq W^{j+1}$ be defined as follows: for all $y, x_1, \ldots, x_j \in W$,

$$(x_1, \ldots, x_j, y) \in \mathcal{S} \ \text{ iff } \ \{y\} \subseteq f(\{x_1\}, \ldots \{x_j\}).$$

Then $f(\{x_1\}, \ldots \{x_j\}) = \mathcal{S}[\{x_1\}, \ldots \{x_j\}]$. Using this observation and the fact that, by item 2 above, every residuated map is completely join preserving in each argument, then it is easy to show that for all $X_1, \ldots, X_j \in \mathcal{P}(W)$,

$$f(X_1, \ldots, X_j) = \mathcal{S}[X_1, \ldots, X_j].$$

□

Consider the following conditions on $[\![\varphi]\!]$: There exist $k_1, \ldots, k_n, m_1, \ldots, m_n \in \mathbb{N}$, such that

(a) $[\![\varphi]\!](p_1, \ldots, p_n)$
$= [\![\varphi']\!]([\![\chi_1]\!](p_1), \ldots, [\![\chi_n]\!](p_{n_1}, \ldots, p_{n_{m_n}}, p_n), [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!])$,
where

(b) $[\![\varphi']\!]$ is a complete operator on $\mathcal{P}(W)$,

(c) for $1 \le i \le n$, $[\![\chi_i]\!](p_{i_1}, \ldots, p_{i_{m_i}}, p_i)$ is the (m_i+1)-th residual of some $(m_i{+}1)$-ary residuated operation f_i, and $p_{i_1}, \ldots, p_{i_{m_i}} \in \{p_1, \ldots, p_{i-1}\}$;

(d) the binary relations $\mathcal{S}_1$ corresponding to f_1 is of the form R^{k_1};

(e) for every $1 < i \le n$, the (m_i+2)-ary relations $\mathcal{S}_i$ on W corresponding to f_i is given, for all $x_0, \ldots, x_{m_i}, y \in W$, by

$$\mathcal{S}_i(x_0, \ldots, x_{m_i}, y) \text{ iff } (\bigwedge_{0 \le h < m_i} x_h R x_{h+1}) \wedge x_{m_i} R^{k_i} y.$$

(f) γ_1 to γ_ℓ are negative formulas.

For every frame $\mathcal{F}$, every formula $\varphi(p_1, \ldots, p_n)$ satisfying condition (a) to (f), let Val_3 be the set of valuations V_3 on $\mathcal{F}$ which map any $q \in \mathsf{Prop} \setminus \{p_1, \ldots, p_n\}$ to $\varnothing$ and are defined inductively on $\{p_1, \ldots, p_n\}$ as follows:

- $V_3(p_1) = \mathcal{S}_1[\{w_1\}]$ for some $w_1 \in W$, and where $\mathcal{S}_1$ is as specified in (d), above.
- For every $1 < i \le n$, $V_3(p_i) = \mathcal{S}_i[\{w_i\}, V_3(pi_1), \ldots, V_3(p_{i_{m_i}})]$, and where $\mathcal{S}_i$ is as specified in (e), above.

Proposition 3.23. *Let $\varphi \to \psi \in \mathrm{ML}$ be such that $[\![\varphi]\!]$ verifies the conditions (a)-(f) above and $[\![\psi]\!]$ is order preserving in each coordinate. Then the following are equivalent:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$
2. $(\forall V_3 \in \mathsf{Val}_3(\mathcal{F}))[w \in [\![\varphi]\!](V_3) \Rightarrow w \in [\![\psi]\!](V_3)]$.

Proof. $(1 \Rightarrow 2)$ Clear. $(2 \Rightarrow 1)$ Fix $V \in \mathsf{Val}(\mathcal{F})$ and let $w \in [\![\varphi]\!](V)$. Hence $\varnothing \ne [\![\varphi']\!]([\![\chi_1]\!](V), \ldots, [\![\chi_n]\!](V), \ldots, [\![\chi_\ell]\!](V))$, and since by assumption (b) $[\![\varphi']\!]$ is a complete operator, we get, $[\![\chi_i]\!](V) \ne \varnothing$ for every $1 \le i \le n$. Then

$$\begin{aligned} w &\in [\![\varphi']\!](V(\chi_1), \ldots, V(\chi_n), V(\gamma_1), \ldots, V(\gamma_\ell)) \\ &= \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \mid z_i \in V(\chi_i), 1 \le i \le n\} \\ &= \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \mid f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{z_i\}) \subseteq V(p_i), 1 \le i \le n\} \end{aligned}$$

where the first equality follows from the fact that $[\![\varphi']\!]$ is a complete operator and the second from assumption (c).

Then $w \in [\![\varphi']\!](w_1, \ldots, w_n, V(\gamma_1), \ldots, V(\gamma_\ell))$ for some $w_1, \ldots, w_n \in W$ such that, for all $1 \leq i \leq n$,

$$f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{w_i\}) \subseteq V(p_i). \tag{10}$$

Let V_3 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and is defined inductively on $\{p_1, \ldots, p_n\}$ as follows:

- $V_3(p_1) = f_1(\{w_1\})$, and
- for $1 < i \leq n$, $V_3(p_i) = f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{w_i\})$.

By assumptions (d) and (e), $V_3 \in \mathsf{Val}_3$. Moreover, $V_3(p_i) \subseteq V(p_i)$, for all $1 \leq i \leq p$. Indeed, the latter can be shown by induction on i. For $i = 1$, since w_1 satisfies (10), we get $V_3(p_1) = f_1(\{w_1\}) \subseteq V(p_1)$. Let $i > 1$ and suppose that the inclusion holds for the first $i - 1$ variables. Since w_i satisfies(10) and, by Proposition 3.22, f_i is order preserving in each coordinate, we get

$$\begin{array}{rcl} V_3(p_i) & = & f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{w_i\}) \\ & \subseteq & f_i(V(p_{i_1}), \ldots, V(p_{i_{m_i}}), \{w_i\}) \subseteq V(p_i). \end{array}$$

Let us now show that $w \in [\![\varphi]\!](V_3)$:

$$\begin{array}{rl} w \in & [\![\varphi']\!](w_1, \ldots, w_n, V(\gamma_1), \ldots, V(\gamma_\ell)) \\ \subseteq & [\![\varphi']\!](w_1, \ldots, w_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \\ \subseteq & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \mid f_i(V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), \{z_i\}) \subseteq V_3(p_i), 1 \leq i \leq n\} \\ = & \bigcup\{[\![\varphi']\!](z_1, \ldots, z_n, V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \mid z_i \in [\![\chi_i]\!](V_3(p_{i_1}), \ldots, V_3(p_{i_{m_i}}), V_3(p_i)), 1 \leq i \leq n\} \\ = & [\![\varphi']\!](V_3(\chi_1), \ldots, V_3(\chi_n), V_3(\gamma_1), \ldots, V_3(\gamma_\ell)) \\ = & [\![\varphi]\!](V_3). \end{array}$$

Here the first inclusion holds since the $[\![\gamma]\!]$'s are order reversing by assumption (f), $[\![\varphi']\!]$ is order-preserving, and as mentioned above $V_3(p_i) \subseteq V(p_i)$ for all $1 \leq i \leq n$. The second inclusion holds by the definition of V_3. By assumption 2., from $w \in [\![\varphi]\!](V_3)$ we have that $w \in [\![\psi]\!](V_3)$. Now, from the facts that $[\![\psi]\!]$ is order preserving in each coordinate and that $V_3(p_i) \subseteq V(p_i)$ for all $1 \leq i \leq n$, we conclude that that $w \in [\![\psi]\!](V)$. □

Syntactic conditions.

Proposition 3.24. *If φ is definite 1-atomic inductive antecedent, then $[\![\varphi]\!]$ verifies the hypotheses (a) to (f) of Proposition 3.23. In particular, the maps $[\![\chi_i]\!]$ are exactly the ones induced by the atomic box-formulas.*

Proof. (Sketch) As far as conditions (a), (b), and (f) are concerned, we note the following. If φ is a definite 1-atomic inductive antecedent, then the atomic box formulas and negative formulas used in its construction correspond to $\chi_1, \dots \chi_n$ and $\gamma_1, \dots, \gamma_\ell$, respectively, in

$$[\![\varphi']\!]([\![\chi_1]\!](p_1), \dots, [\![\chi_n]\!](p_{n_1}, \dots, p_{n_{m_n}}, p_n), [\![\gamma_1]\!], \dots, [\![\gamma_\ell]\!]).$$

The 'skeleton' consisting of the composition of $\Diamond$'s and $\wedge$'s used in the construction corresponds to the complete operator $[\![\varphi']\!]$.

As for conditions (c), (d) and (e), we note that since the dependency digraph of φ is acyclic, its transitive closure is a strict partial order. We can therefore assume, without loss of generality, that the variables are ordered $p_1 < p_2 < \cdots < p_n$ by some linear extension of this partial order. Hence we will have that $p_{i_1}, \dots, p_{i_{m_i}} \in \{p_1, \dots, p_{i-1}\}$ in each $[\![\chi_i]\!](p_{i_1}, \dots, p_{i_{m_i}}, p_i)$. By Propositions 3.12, 3.17, and 3.19, in order to complete the proof, it is enough to show that for every atomic box-formula

$$\chi_i(p_{i_1}, \dots, p_{i_{m_i}}, p_i) = \Box(p_{i_1} \to \Box(p_{i_2} \to \dots \Box(p_{i_{m_i}} \to \Box^k p_i) \dots)),$$

$[\![\chi_i]\!]$ is the $(m_i + 1)$-th residual of the map $f_i : \mathcal{P}(W)^{m_i+1} \to \mathcal{P}(W)$ defined as

$$f(X_1, \dots, X_{m_i}, Y) = R^{k_i}[X_{m_i} \cap R[X_{m_i-1} \cap \cdots R[X_2 \cap R[X_1 \cap Y] \cdots]]].$$

The details of the proof are left to the reader. $\Box$

Example 3.25. Let us consider the 1-atomic inductive formula

$$\varphi := p \wedge \Box(p \to q) \to \Diamond q,$$

which locally corresponds to the property of being a reflexive state. The dependency digraph induces the order $p < q$ on the variables. We have $k_1 = 0$, $m_1 = 0$, $k_2 = 0$ and $m_2 = 1$. The standard local second-order translation is

$$\forall P \forall Q[P(x) \wedge \forall y(xRy \to (P(y) \to Q(y))) \to \exists u(xRu \wedge Q(u))].$$

The reduction strategy prescribes that we replace $\forall P \forall Q$ in the prefix with $\forall z_1 \forall z_2$ and that we substitute occurrences of the form $P(y)$ with $\alpha_1(y) := \exists u_1(z_1 = u_1 \wedge y = u_1)$ which is equivalent to $y = z_1$. It further prescribes that occurrences of the form $Q(y)$ should be substituted with $\alpha_2(y) := \exists v_0 \exists v_1(z_2 = v_0 \wedge v_0 R v_1 \wedge v_1 = z_1 \wedge v_1 = y)$, where we have already used the simplified version of α_1. Now $\alpha_2(y)$ can be further simplified to $z_2 R z_1 \wedge z_1 = y$. Doing the substitution we obtain

$$\forall z_1 \forall z_2 [x = z_1 \wedge \forall y(xRy \to (y = z_1 \to z_2 R z_1 \wedge y = z_1)) \to \exists u(xRu \wedge z_2 R z_1 \wedge u = z_1)].$$

This is equivalent to

$$\begin{aligned} & \forall z_1 \forall z_2 [\forall y(xRy \to (y = x \to z_2 Rx)) \to \exists u(xRu \wedge z_2 Rx \wedge u = x)] \\ \equiv\ & \forall z_1 \forall z_2 [(xRx \to z_2 Rx) \to \exists u(xRx \wedge z_2 Rx)] \\ \equiv\ & xRx. \end{aligned}$$

4 Conclusions

In this paper, the Sahlqvist-style syntactic identification of classes of modal formulas that are endowed with local first order correspondents has been explained in terms of certain order-theoretic properties of the extension maps corresponding to the formulas of these classes. These properties and the resulting methodology hold beyond the Sahlqvist class, as the example of atomic inductive formulas shows. Further features which we would like to emphasize are:

Generalizing the signatures. Our treatment is modular: in particular, we neatly divided the correspondence proof for each class of formulas in three stages. Although, for simplicity, we confined our treatment to the basic modal signature, the most important stage, i.e. the one referred to as 'order theoretic conditions' is intrinsically independent from any algebraic signature. Therefore, it can be applied to any one, and in particular to any modal signature.

Modifying the description of tame valuations. We have showed that at the core of the correspondence mechanism there are special classes of (tame) valuations, the members of which can be described uniformly in the language L_0. Of course the classes $\mathsf{Val}_1(\mathcal{F})$, $\mathsf{Val}_2(\mathcal{F})$ and $\mathsf{Val}_3(\mathcal{F})$, on which we settled as a compromise between simplicity of presentation and generality, are just three instances. A plethora of variations on them is possible. On the one hand, by complicating the definitions of these classes, using more parameters and taking into account the positions where boxed atoms / box formulas occur within the antecedent, the shape of the first-order correspondent obtained can be improved. On the other hand, the class $\mathsf{Val}_3(\mathcal{F})$, for example, is designed to be targeted by *atomic* inductive implications, and as such is too small for the whole class of inductive implications. The essential difference lies in the definition of the (general, not necessarily atomic) box formulas, which yields shapes such as the following:

$$\Box(A_0 \to \Box(A_1 \to \ldots \Box(A_n \to \Box^k p)\ldots)),$$

where the A_i's are arbitrary positive formulas. The correspondence result for the whole class of inductive formulas can be obtained by the same methodology we proposed in Section 3.6, applied to a suitably larger class of special tame valuations.

From the Boolean to the distributive setting. The statements and proofs about the order-theoretic conditions only use the following two features of powerset algebras: that they are complete distributive lattices and that they are completely join-generated by their completely join prime elements[5] (the singleton subsets). Therefore, these proofs go through virtually unchanged in the more general setting of distributive lattices enjoying these two properties. In fact, it is well known that the modal expansion of each of these special lattices is isomorphic to the complex algebra of some frame for distributive modal logic. This observation motivates the development of the algorithmic correspondence theory for distributive modal logic in [7] in the purely algebraic setting of lattices of such kind.

Duality and correspondence. Although it was not strictly needed for our exposition, in this paper correspondence is regarded as a byproduct of the duality between Kripke frames and complete and atomic BAO's. For instance, results such as Proposition 3.17.2 and .3 and Proposition 3.22.3 are essentially characterizations of objects across a duality. More in general, the relational interpretation of modal logic can be obtained by dualizing its canonical algebraic interpretation on BAO's. This modus operandi is not confined to modal logic: any duality involving the class of algebras canonically associated with a given propositional logic provides the appropriate setting for correspondence results.

[5]an element c of a complete lattice L is *completely join prime* if, for every $S \subseteq L$, $c \leq s$ for some $s \in S$ whenever $c \leq \bigvee S$.

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