A proof-theoretic view on individual and collective preference

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- Proof-theoretic methods in logic for social choice.
- Logic as axiomatic method.
- Logic beyond axioms: rule-based calculi.
- Method of proof analysis.
- Formalize the proofs of impossibility theorems.
- "Inferentialize" social choice theory.

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Hilbert-style proof theory for individual preference

- First-order language where atoms

  $x \geqslant y$ are interpreted as x is at least good as y

- First-order axiomatization

  Axioms for $\forall, \wedge, \rightarrow, \bot$
  Modus Ponens

$\forall x(x \geqslant x)$	$\geqslant$ is reflexive
$\forall x \forall y \forall z(x \geqslant y \wedge y \geqslant z \rightarrow x \geqslant z)$	$\geqslant$ is transitive
$\forall x \forall y(x \geqslant y \vee y \geqslant x)$	$\geqslant$ is total

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Hilbert-style proof theory for individual preference

- Definitions of $>$ (strict preference) and $\sim$ (indifference)

$$
\begin{aligned}
x > y &\quad =_{df} \quad x \geqslant y \text{ and } y \not\geqslant x \\
x \sim y &\quad =_{df} \quad x \geqslant y \text{ and } y \geqslant x
\end{aligned}
$$

- Theorems

$\vdash$	$\forall x(x \sim x)$	$\geqslant$ is reflexive
$\vdash$	$\forall x \forall y \forall z(x \sim y \wedge y \sim z \rightarrow x \sim z)$	$\sim$ is transitive
$\vdash$	$\forall x \forall y(x \sim y \rightarrow y \sim x)$	$\sim$ is symmetric
$\vdash$	$\forall x(x \not> x)$	$\geqslant$ is irreflexive
$\vdash$	$\forall x \forall y \forall z(x > y \wedge y > z \rightarrow x > z)$	$\sim$ is transitive
$\vdash$	$\forall x \forall y(x > y \rightarrow y \not> x)$	$\sim$ is asymmetric

Hilbert-style proof theory for individual preference

- Definitions of $>$ (strict preference) and $\sim$ (indifference)

$$\begin{array}{lll} x > y & =_{df} & x \geqslant y \text{ and } y \ngeqslant x \\ x \sim y & =_{df} & x \geqslant y \text{ and } y \geqslant x \end{array}$$

- Theorems

$$\begin{array}{lll} \vdash & \forall x (x \sim x) & \geqslant \text{ is reflexive} \\ \vdash & \forall x \forall y \forall z (x \sim y \wedge y \sim z \to x \sim z) & \sim \text{ is transitive} \\ \vdash & \forall x \forall y (x \sim y \to y \sim x) & \sim \text{ is symmetric} \\ \vdash & \forall x (x \not> x) & \geqslant \text{ is irreflexive} \\ \vdash & \forall x \forall y \forall z (x > y \wedge y > z \to x > z) & \sim \text{ is transitive} \\ \vdash & \forall x \forall y (x > y \to y \not> x) & \sim \text{ is asymmetric} \end{array}$$

Gentzen-style proof theory for individual preference

- Systematic proof-search procedure.
- Sequent calculi

Γ, Δ	multisets (lists without order) of formulas
$\Gamma \Rightarrow \Delta$	interpreted as $\bigwedge \Gamma \to \bigvee \Delta$

- One axiom.
- Logical rules.
- Structural rules.

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Gentzen-style proof theory for individual preference

- Weakening, Contraction and Cut

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \varphi}\ \text{W} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{W}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi, \varphi}{\Gamma \Rightarrow \Delta, \varphi}\ \text{C} \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \text{C}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \varphi, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}\ \text{CUT}$$

- Admissibility of the structural rules.

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Gentzen-style proof theory for individual preference

$$P, \Gamma \Rightarrow \Delta, P \qquad \overline{\bot, \Gamma \Rightarrow \Delta}$$

$$\frac{\varphi, \psi, \Gamma \Rightarrow \Delta}{\varphi \wedge \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \wedge \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \psi, \Gamma \Rightarrow \Delta}{\varphi \to \psi, \Gamma \Rightarrow \Delta} \qquad \frac{\varphi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

$$\frac{\varphi(x), \forall x \varphi(x), \Gamma \Rightarrow \Delta}{\forall x \varphi(x), \Gamma \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi(y)}{\Gamma \Rightarrow \Delta, \forall x \varphi(x)} \; {\scriptstyle y \notin \Gamma, \Delta}$$

G3c

where P is either $x \geqslant y$ or $x > y$ or else $x \sim y$.

Cut admissibility in presence of axioms

- Rules for $\geqslant$, $\sim$ and $>$ s.t. admissibility results preserved
- **G3c** $+$

$$\Rightarrow x \sim x \qquad (\sim \text{ is reflexive})$$
$$x \sim y \Rightarrow y \sim x \qquad (\sim \text{ is symmetric})$$
$$x \sim y, y \sim z \Rightarrow x \sim z \qquad (\sim \text{ is transitive})$$

- Counter-example to cut admissibility.

$$\frac{x \sim y \Rightarrow y \sim x \qquad y \sim x, x \sim z \Rightarrow y \sim z}{x \sim y, x \sim z \Rightarrow y \sim z} \text{CUT}$$

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Cut admissibility in presence of axioms

- How can we restore cut admissibility?
- Systematic approaches: cut admissibility once and for all.
- Criteria for a new rule to be "good" w.r.t. cut admissibility.

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Axioms as inference rules

- Extension by inference rules
- **G3c** +

$$\frac{x \sim x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \; Ref$$

$$\frac{y \sim x, x \sim y, \Gamma \Rightarrow \Delta}{x \sim y, \Gamma \Rightarrow \Delta} \; Sym$$

$$\frac{x \sim z, x \sim y, y \sim z, \Gamma \Rightarrow \Delta}{x \sim y, y \sim z, \Gamma \Rightarrow \Delta} \; Trans$$

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Axioms as inference rules

- $x \sim y, x \sim z \Rightarrow y \sim z$ has a cut-free derivation

$$\dfrac{\dfrac{y \sim z, y \sim x, x \sim y, x \sim z \Rightarrow y \sim z}{y \sim x, x \sim y, x \sim z \Rightarrow y \sim z}\ \mathit{Trans}}{x \sim y, x \sim z \Rightarrow y \sim z}\ \mathit{Sym}$$

- The new rules are
  - applied bottom-up
  - logic-free
  - left-hand side only
  - cumulative

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Axioms as inference rules

- What class of axioms can be rearranged into rules?
- Regular axioms, *i.e.* universal closure of

$$P_1 \wedge \cdots \wedge P_m \to Q_1 \vee \cdots \vee Q_n$$

- corresponds to

$$\frac{Q_1, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta \quad \ldots \quad Q_n, P_1, \ldots, P_m, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta} \; Reg$$

- *Reg* preserves admissibility results (Negri & von Plato)

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- Rules for $>$

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Gentzen-style proof theory for individual preference

- Let **GP** be **G3c** + rules for $\geqslant$, $>$ and $\sim$
- In **GP**
  - Weakening is admissible
  - Contraction is admissible
  - Cut is admissible

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Gentzen-style proof theory for collective preference

- From individual to collective preference
- Fix $B = \{1 \ldots n\}$ a set of voters
- Indexed preferences: $\geqslant_i$, $>_i$ and $\sim_i$, for $i \in B$
- Social choice rules as rules of inference.

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- Paretian collective preference
- Everybody considers x as good as y but somebody strictly prefers x to y
- Formally,

$$x \geqslant_B y \quad =_{df} \quad \bigwedge_{i=1}^{n} x \geqslant_i y$$

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- Assume $B = \{1,2\}$. The rules for $\geqslant_B$ and $>_B$ are

$$\frac{x \geqslant_{12} y, x \geqslant_B y, \Gamma \Rightarrow \Delta}{x \geqslant_B y, \Gamma \Rightarrow \Delta} \qquad \frac{x \geqslant_B y, x \geqslant_{12} y, \Gamma \Rightarrow \Delta}{x \geqslant_{12} y, \Gamma \Rightarrow \Delta}$$

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- Some known results:

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Gentzen-style proof theory for collective preference

- $\geqslant_B$ is transitive, if each $\geqslant_i$ is transitive too.

$$
\begin{array}{c}
x \geqslant_B z, x \geqslant_2 z, x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z \\
\hline
x \geqslant_2 z, x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z \\
\hline
x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z \\
\hline
y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z \\
\hline
x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z \\
\hline
x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z
\end{array}
$$

Gentzen-style proof theory for collective preference

- $\geqslant_B$ is transitive, if each $\geqslant_i$ is transitive too.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x \geqslant_B z, x \geqslant_2 z, x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}{x \geqslant_2 z, x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}}{x \geqslant_1 z, y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}}{y \geqslant_1 z, y \geqslant_2 z, x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}}{x \geqslant_1 y, x \geqslant_2 y, x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}}{x \geqslant_B y, y \geqslant_B z \Rightarrow x \geqslant_B z}
$$

Gentzen-style proof theory for collective preference

- $\geqslant_B$ is not total, although each $\geqslant_i$ is.
- The sequent $\Rightarrow x \geqslant_B y \vee y \geqslant_B x$ is not derivable
- Interestingly, counter example is found from failed proof search.

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- Majority voting can be expressed by

$$xMy \quad =_{df} \quad \bigvee_{A \subseteq B : |A| \geq \frac{n}{2}} \bigwedge_{i \in A} x \geqslant_i y$$

- With 3 voters

$$xMy \quad =_{df} \quad (x \geqslant_{12} y) \vee (x \geqslant_{31} y) \vee (x \geqslant_{23} y)$$

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- The rules for M are

$$\frac{xMy, x \geqslant_{12} y, \Gamma \Rightarrow \Delta}{x \geqslant_{12} y, \Gamma \Rightarrow \Delta} \quad \frac{xMy, x \geqslant_{31} y, \Gamma \Rightarrow \Delta}{x \geqslant_{31} y, \Gamma \Rightarrow \Delta} \quad \frac{xMy, x \geqslant_{23} y, \Gamma \Rightarrow \Delta}{x \geqslant_{23} y, \Gamma \Rightarrow \Delta}$$

$$\frac{x \geqslant_{12} y, xMy, \Gamma \Rightarrow \Delta \quad x \geqslant_{31} y, xMy, \Gamma \Rightarrow \Delta \quad x \geqslant_{23} y, xMy, \Gamma \Rightarrow \Delta}{xMy, \Gamma \Rightarrow \Delta}$$

Conclusions

- Cut-free sequent calculus for individual preference:
  - decidability and complexity issues.
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