

Representations of $SU(2)$ and Jacobi polynomials *

Tom H. Koornwinder, thk@science.uva.nl

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1 Preliminaries about representation theory

Literature for this section is for instance Sugiura [3, Ch. 1].

Let G be a group. Representations of G can be defined on any vector space (possibly infinite dimensional) over any field, but we will only consider representations on finite dimensional complex vector spaces. Let V be a finite dimensional complex vector space. Let $GL(V)$ be the set of all invertible linear transformations of V . This is a group under composition. If V has dimension n and if we choose a basis e_1, \dots, e_n of V then the map $x = x_1 e_1 + \dots + x_n e_n \mapsto (x_1, \dots, x_n): V \rightarrow \mathbb{C}^n$ is an isomorphism of vector spaces. There is a corresponding group isomorphism $GL(V) \rightarrow GL(\mathbb{C}^n)$ which sends each invertible linear transformation of V to the corresponding invertible matrix with respect to this basis. We denote $GL(\mathbb{C}^n)$ by $GL(n, \mathbb{C})$: the group of all invertible complex $n \times n$ matrices. Here the group multiplication is by multiplication of matrices.

Definition 1.1. A *representation* of a group G on a finite dimensional complex vector space V is a group homomorphism $\pi: G \rightarrow GL(V)$. A linear subspace W of V is called *invariant* (with respect to the representation π) if $\pi(g)W \subset W$ for all $g \in G$. The representation π on V is called *irreducible* if V and $\{0\}$ are the only invariant subspaces of V .

Definition 1.2. Let π be a representation of a group G on a finite dimensional complex vector space V . Choose a basis e_1, \dots, e_n of V . Then, for $g \in G$, the linear map $\pi(g)$ has a matrix $(\pi_{i,j}(g))_{i,j=1,\dots,n}$ with respect to this basis, which is determined by the formula

$$\pi(g) e_j = \sum_{i=1}^n \pi_{i,j}(g) e_i.$$

The $\pi_{i,j}$ are complex-valued functions on G which are called the *matrix elements* of the representation π with respect to the basis e_1, \dots, e_n .

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Remark 1.3. Let $\text{End}(V)$ be the space of all linear transformations $A: V \rightarrow V$. If π is a map of the group G into $\text{End}(V)$ such that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for all g_1, g_2 and $\pi(e) = \text{id}$, then π maps into $GL(V)$ and π is a representation of G on V (proof left as an exercise).

Definition 1.4. A *topological group* is a set G which is both a group and a topological space such that the maps $(g_1, g_2) \mapsto g_1g_2: G \times G \rightarrow G$ and $g \mapsto g^{-1}: G \rightarrow G$ are continuous.

Example 1.5. $GL(n, \mathbb{C})$ can be considered as a subset of \mathbb{C}^{n^2} by associating with the element $T = (t_{i,j})_{i,j=1,\dots,n} \in GL(n, \mathbb{C})$ the n^2 complex coordinates $t_{i,j}$. Then the group $GL(n, \mathbb{C})$, with the topology inherited from \mathbb{C}^{n^2} , is a topological group (proofs as an exercise).

Let V be an n -dimensional complex vector space. With any basis of V the group $GL(V)$ is isomorphic with $GL(n, \mathbb{C})$. Give a topology to $GL(V)$ such that this isomorphism is also a homeomorphism. Then $GL(V)$ is a topological group and the topology is independent of the choice of the basis (proofs as an exercise).

Definition 1.6. A *representation* of a topological group G on a finite dimensional complex vector space V is a continuous group homomorphism $\pi: G \rightarrow GL(V)$.

Remark 1.7. Let G be a topological group, V a finite dimensional complex vector space and $\pi: G \rightarrow GL(V)$ a group homomorphism. Let e_1, \dots, e_n a basis for V . Then the following five properties are equivalent:

- a) π is continuous;
- b) for all $v \in V$ the map $g \mapsto \pi(g)v: G \rightarrow V$ is continuous;
- c) for all j the map $g \mapsto \pi(g)e_j: G \rightarrow V$ is continuous;
- d) for all $v \in V$ and for all complex linear functionals f on V the map $g \mapsto f(\pi(g)v): G \rightarrow \mathbb{C}$ is continuous.
- e) The matrix elements $\pi_{i,j}$ of π with respect to the basis e_1, \dots, e_n are continuous functions on G .

The proofs are left as an exercise. Be aware that these equivalences are not necessarily true if V is an infinite dimensional topological vector space.

Remark 1.8. If π is a representation of G on V and if H is a subgroup of G then the restriction of the group homomorphism $\pi: G \rightarrow GL(V)$ to H is a group homomorphism $\pi: H \rightarrow GL(V)$, so it is a representation of H on V .

If G is moreover a topological group then H with the topology inherited from G becomes a topological group (exercise!).

If, furthermore, π is a representation of G as a topological group on V then the restriction of π to H is a representation of H as a topological group on V .

Definition 1.9. Let V be a finite dimensional complex vector space with hermitian inner product $\langle \cdot, \cdot \rangle$. A representation π of a group G on V is called *unitary* if $\pi(g)$ is a unitary operator on V for all $g \in G$, i.e., if

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in V \text{ and for all } g \in G.$$

Remark 1.10. Let V and G be as in Definition 1.9 and let π be a representation of G on V . Let e_1, \dots, e_n be an orthonormal basis of V and let $\pi(g)$ have matrix $(\pi_{i,j}(g))$ with respect to this basis. Then the representation π is unitary iff the matrix $(\pi_{i,j}(g))$ is unitary for each $g \in G$. One of the ways to characterize unitarity of the matrix $(\pi_{i,j}(g))$ is that

$$\overline{\pi_{i,j}(g)} = \pi_{j,i}(g^{-1}) \quad (i, j = 1, \dots, n).$$

The proof is left as an exercise.

Proposition 1.11. (Complete reducibility of unitary representations)

Let V and G be as in Definition 1.9 and let π be a unitary representation of G on V . Then:

- a) If W is an invariant subspace of V then the orthoplement W^\perp of W is also an invariant subspace.
- b) V can be written as an orthogonal direct sum of subspaces V_i such that the representation π , when restricted to V_i , is irreducible.

The proof is left as an exercise.

2 A class of representations of $SU(2)$

References for this and following sections are for instance Sugiura [3, Ch. 2] and Vilenkin & Klimyk [4, Ch. 6].

Fix $l \in \{0, \frac{1}{2}, 1, \dots\}$. Let \mathcal{H}_l be the space of homogeneous polynomials of degree $2l$ in two complex variables z_1, z_2 . So the monomials $z_1^{l-n} z_2^{l+n}$ ($n = -l, -l+1, \dots, l$) form a basis of \mathcal{H}_l , and \mathcal{H}_l has dimension $2l+1$. For reasons which will become clear later, we will work with a renormalized basis

$$\psi_n^l(z_1, z_2) := \binom{2l}{l-n}^{\frac{1}{2}} z_1^{l-n} z_2^{l+n} \quad (n = -l, -l+1, \dots, l). \quad (2.1)$$

For $A \in GL(2, \mathbb{C})$ and $f \in \mathcal{H}_l$ define the function $t^l(A)f$ on \mathbb{C}^2 by

$$(t^l(A)f)(z) := f(A'z) \quad (z = (z_1, z_2) \in \mathbb{C}^2), \quad (2.2)$$

where A' is the transpose of the matrix A . So

$$\left(t^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} f \right) (z_1, z_2) = f(az_1 + cz_2, bz_1 + dz_2), \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}).$$

From this it is clear that $(t^l(A)f)(z_1, z_2)$ is again a homogeneous polynomial of degree $2l$ in z_1, z_2 . Moreover, t^l is a representation of $GL(2, \mathbb{C})$ on \mathcal{H}_l , since $t^l(I)f = f$ and

$$\begin{aligned} (t^l(AB)f)(z) &= f((AB)'z) = f(B'A'z) = (t^l(B)f)(A'z) \\ &= (t^l(A)(t^l(B)f))(z) = \left((t^l(A)t^l(B))f \right) (z). \end{aligned}$$

The matrix elements $t_{m,n}^l$ ($m, n = -l, -l+1, \dots, l$) of t^l with respect to the basis (2.1) are determined by

$$t^l(g) \psi_n^l = \sum_{m=-l}^l t_{m,n}^l(g) \psi_m^l \quad (g \in GL(2, \mathbb{C})), \quad (2.3)$$

Since

$$\left(t^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi_n^l \right) (z_1, z_2) = \binom{2l}{l-n}^{\frac{1}{2}} (az_1 + cz_2)^{l-n} (bz_1 + dz_2)^{l+n}, \quad (2.4)$$

(2.3) can be written more explicitly as

$$\binom{2l}{l-n}^{\frac{1}{2}} (az_1 + cz_2)^{l-n} (bz_1 + dz_2)^{l+n} = \sum_{m=-l}^l \binom{2l}{l-m}^{\frac{1}{2}} t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} z_1^{l-m} z_2^{l+m}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}). \quad (2.5)$$

From (2.5) we see that $t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a homogeneous polynomial of degree $2l$ in a, b, c, d , so $t_{m,n}^l$ is continuous on $GL(2, \mathbb{C})$. By Remark 1.7 t^l is then also a representation of $GL(2, \mathbb{C})$ considered as a topological group.

For fixed n we can consider (2.5) as a *generating function* for the matrix elements $t_{m,n}^l$ with $m = -l, \dots, l$: the matrix elements are obtained as the coefficients in the power series expansion of the elementary function in z_1, z_2 on the left-hand side.

From (2.5) for $n = l$ elementary expressions for the matrix elements $t_{n,l}^l$ can be obtained (exercise):

$$t_{m,l}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{2l}{l-m}^{\frac{1}{2}} b^{l-m} d^{l+m}. \quad (2.6)$$

From (2.5) we can derive a double generating function for the matrix elements $t_{m,n}^l$: Multiply both sides of (2.5) with

$$\binom{2l}{l-n}^{\frac{1}{2}} w_1^{l-n} w_2^{l+n},$$

and sum over n . Then we obtain

$$\begin{aligned} (az_1 w_1 + bz_1 w_2 + cz_2 w_1 + dz_2 w_2)^{2l} &= \sum_{m,n=-l}^l \binom{2l}{l-m}^{\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &\quad \times z_1^{l-m} z_2^{l+m} w_1^{l-n} w_2^{l+n}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}). \end{aligned} \quad (2.7)$$

Formula (2.7) implies the symmetry

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = t_{n,m}^l \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad (2.8)$$

while (2.5) implies that

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = t_{-m,-n}^l \begin{pmatrix} d & c \\ b & a \end{pmatrix}. \quad (2.9)$$

From (2.8) and (2.9) we obtain a third symmetry

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = t_{-n,-m}^l \begin{pmatrix} d & b \\ c & a \end{pmatrix}. \quad (2.10)$$

The details of the proofs of (2.8)–(2.10) are left as exercises.

Let $SU(2)$ denote the set of all 2×2 unitary matrices of determinant 1. This is clearly a subgroup of $GL(2, \mathbb{C})$ (exercise!). Note that $SU(2)$ consists of all matrices

$$\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \quad \text{with } a, c \in \mathbb{C} \text{ and } |a|^2 + |c|^2 = 1. \quad (2.11)$$

Prove this as an exercise. Hence, as a topological space, $SU(2)$ is homeomorphic with $\{(a, c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$, which is the unit sphere in \mathbb{C}^2 , i.e., the sphere S^3 . In particular, $SU(2)$ is compact.

The representation t^l of $GL(2, \mathbb{C})$ given by (2.2), becomes by restriction a representation of $SU(2)$. Put a hermitian inner product on \mathcal{H}_l such that the basis of functions ψ_n^l ($n = -l, -l+1, \dots, l$) is orthonormal.

Proposition 2.1. *The representation t^l of $SU(2)$ is unitary.*

Proof The inverse of $\begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix} \in SU(2)$ is $\begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix}$. In view of Remark 1.10 we have to show that

$$\overline{t_{m,n}^l \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}} = t_{n,m}^l \begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix}.$$

Since, by (2.5), $t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a polynomial with real coefficients in a, b, c, d , we have

$$\overline{t_{m,n}^l \begin{pmatrix} a & -\bar{c} \\ c & \bar{a} \end{pmatrix}} = t_{m,n}^l \begin{pmatrix} \bar{a} & -c \\ \bar{c} & a \end{pmatrix}.$$

Hence we have to show that

$$t_{m,n}^l \begin{pmatrix} \bar{a} & -c \\ \bar{c} & a \end{pmatrix} = t_{n,m}^l \begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix}.$$

This last identity follows from (2.8). □

3 Computation of matrix elements of representations of $SU(2)$

We can use the generating function (2.5) in order to compute the matrix elements $t_{m,n}^l$. First we expand the two powers on the left-hand side of (2.5) by the binomial formula:

$$\begin{aligned} (az_1 + cz_2)^{l-n} &= \sum_{j=0}^{l-n} \binom{l-n}{j} a^j z_1^j c^{l-n-j} z_2^{l-n-j}, \\ (bz_1 + dz_2)^{l+n} &= \sum_{k=0}^{l+n} \binom{l+n}{k} b^k z_1^k d^{l+n-k} z_2^{l+n-k}. \end{aligned}$$

Hence the left-hand side of (2.5) can be rewritten as

$$\binom{2l}{l-n}^{\frac{1}{2}} \sum_{j=0}^{l-n} \sum_{k=0}^{l+n} \binom{l-n}{j} \binom{l+n}{k} a^j b^k c^{l-n-j} d^{l+n-k} z_1^{j+k} z_2^{2l-j-k}. \quad (3.1)$$

In this double sum we make a change of summation variables $(j, k) \mapsto (m, j)$, where $j+k = l-m$. Hence

$$(j, k) \mapsto (l-k-j, j) \text{ with inverse map } (m, j) \mapsto (j, l-m-j). \quad (3.2)$$

Now we have

$$\begin{aligned} 0 \leq j \leq l-n \text{ and } 0 \leq k \leq l+n &\iff \\ -l \leq m \leq l \text{ and } 0 \leq j \leq l-n \text{ and } -m-n \leq j \leq l-m. &\quad (3.3) \end{aligned}$$

Indeed, the inequalities to the left of the equivalence sign in (3.3) imply that $0 \leq j+k \leq 2l$, hence $0 \leq l-m \leq 2l$, hence $-l \leq m \leq l$. Also, $0 \leq k \leq l+n$ implies $0 \leq l-m-j \leq l+n$, hence $-m-n \leq j \leq l-m$. Conversely, $-m-n \leq j \leq l-m$ implies (substitute $m = l-k-j$) that $-l-n+k+j \leq j \leq k+j$, hence $0 \leq k \leq l+n$. (Note that $-l \leq m \leq l$ to the right of the equivalence sign in (3.3) is not strictly needed because it is implied by the other inequalities on the right.)

We conclude that the double sum (3.1) can be rewritten by the substitution $j+k = l-m$ as follows:

$$\binom{2l}{l-n}^{\frac{1}{2}} \sum_{m=-l}^l \sum_{j=0 \vee (-m-n)}^{(l-m) \wedge (l-n)} \binom{l-n}{j} \binom{l+n}{l-m-j} a^j b^{l-m-j} c^{l-n-j} d^{n+m+j} z_1^{l-m} z_2^{l+m}. \quad (3.4)$$

Here the first summation is by convention over all $m \in \{-l, -l+1, \dots, l\}$. In the second summation the symbol \vee means maximum and the symbol \wedge means minimum. The range of the double summation in (3.4) is justified by the equivalence (3.3). Note that the second summation is an inner summation since its summation bounds depend on m , which is the summation variable for the outer summation. The summand in (3.4) is obtained from the summand in (3.1) by the substitution $k = l-m-j$.

Since (3.4) is a rewritten form of the left-hand side of (2.5), it must be equal to the right-hand side of (2.5). Both (3.4) and the right-hand side of (2.5) are polynomials in z_1, z_2 with explicit coefficients. Hence the corresponding coefficients must be equal. We conclude:

Proposition 3.1.

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{2l}{l-m}^{-\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} \sum_{j=0 \vee (-m-n)}^{(l-m) \wedge (l-n)} \binom{l-n}{j} \binom{l+n}{l-m-j} a^j b^{l-m-j} c^{l-n-j} d^{n+m+j}. \quad (3.5)$$

Note that the summation bounds in (1.12) reduce to one of four alternatives depending on the signs of $m+n$ and $m-n$:

$$\begin{aligned} 0 \leq j \leq l-m & \quad \text{if } m+n \geq 0 \text{ and } m-n \geq 0; \\ 0 \leq j \leq l-n & \quad \text{if } m+n \geq 0 \text{ and } m-n \leq 0; \\ -m-n \leq j \leq l-m & \quad \text{if } m+n \leq 0 \text{ and } m-n \geq 0; \\ -m-n \leq j \leq l-n & \quad \text{if } m+n \leq 0 \text{ and } m-n \leq 0. \end{aligned}$$

These four alternatives correspond to four subsets of the set $\{(m,n) \mid m,n \in \{-l, -l+1, \dots, l\}\}$, which have triangular shape, overlapping boundaries, and together span the whole set. These four subsets are mapped onto each other by the symmetries (2.8)–(2.10).

Hence it is sufficient to compute $t_{m,n}^l$ if $m+n \geq 0$, $m-n \geq 0$. For a while we only assume $m+n \geq 0$ and not yet $m-n \geq 0$. Then (3.5) takes the form

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{2l}{l-m}^{-\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} \sum_{j \geq 0} \binom{l-n}{j} \binom{l+n}{l-m-j} a^j b^{l-m-j} c^{l-n-j} d^{n+m+j}. \quad (3.6)$$

We will rewrite the right-hand side of (3.6) first as a Gauss hypergeometric function (with some elementary factors in front) and next as a Jacobi polynomial. For this derivation remember the *Pochhammer symbol*

$$(a)_0 := 1, \quad (a)_k := a(a+1) \dots (a+k-1) \quad (k = 1, 2, \dots).$$

In particular, note that

$$\frac{(n+k)!}{n!} = (n+1)_k, \quad \frac{n!}{(n-k)!} = (-1)^k (-n)_k.$$

Now we have

$$\begin{aligned}
& \binom{2l}{l-m}^{-\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} \sum_{j \geq 0} \binom{l-n}{j} \binom{l+n}{l-m-j} a^j b^{l-m-j} c^{l-n-j} d^{m+m+j} \\
&= \left(\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!} \right)^{\frac{1}{2}} \sum_{j \geq 0} \frac{(l-n)!}{j!(l-n-j)!} \frac{(l+n)!}{(l-m-j)!(n+m+j)!} a^j b^{l-m-j} c^{l-n-j} d^{m+m+j} \\
&= \left(\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!} \right)^{\frac{1}{2}} \frac{(l+n)! b^{l-m} c^{l-n} d^{m+n}}{(l-m)!(m+n)!} \sum_{j \geq 0} \frac{(l-m)!}{(l-m-j)!} \frac{(l-n)!}{(l-n-j)!} \frac{(m+n)!}{(m+n+j)! j!} \left(\frac{ad}{bc} \right)^j \\
&= \left(\frac{(l+m)!(l+n)!}{(l-m)!(l-n)!} \right)^{\frac{1}{2}} \frac{b^{l-m} c^{l-n} d^{m+n}}{(m+n)!} \sum_{j \geq 0} \frac{(-l+m)_j (-l+n)_j}{(m+n+1)_j j!} \left(\frac{ad}{bc} \right)^j \\
&= \left(\frac{(l+m)!(l+n)!}{(l-m)!(l-n)!} \right)^{\frac{1}{2}} \frac{b^{l-m} c^{l-n} d^{m+n}}{(m+n)!} {}_2F_1 \left(\begin{matrix} -l+m, -l+n \\ m+n+1 \end{matrix}; \frac{ad}{bc} \right). \tag{3.7}
\end{aligned}$$

Here we used the definition of Gauss hypergeometric series, see [2] or [1]. Note that the two upper parameters $-l+m, -l+n$ of the hypergeometric function in (3.7) are both non-positive, and that the series will terminate after the term with $j = (l-m) \wedge (l-n)$.

Pfaff's transformation

$${}_2F_1 \left(\begin{matrix} a, b \\ c \end{matrix}; z \right) = (1-z)^{-a} {}_2F_1 \left(\begin{matrix} a, c-b \\ c \end{matrix}; \frac{z}{z-1} \right) \tag{3.8}$$

implies for the hypergeometric function in (3.7) that

$${}_2F_1 \left(\begin{matrix} -l+m, -l+n \\ m+n+1 \end{matrix}; \frac{ad}{bc} \right) = b^{m-l} c^{m-l} (bc-ad)^{l-m} {}_2F_1 \left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc} \right).$$

Hence we arrive at the following rewritten form of (3.6) (from now on assume $m+n \geq 0$, $m-n \geq 0$):

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(l+m)!(l+n)!}{(l-m)!(l-n)!} \right)^{\frac{1}{2}} \frac{c^{m-n} d^{m+n} (bc-ad)^{l-m}}{(m+n)!} {}_2F_1 \left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc} \right). \tag{3.9}$$

Now use the expression of *Jacobi polynomials* in terms of the Gauss hypergeometric function (see [2], [1]):

$$P_n^{(\alpha, \beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1 \left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix}; \frac{1-x}{2} \right). \tag{3.10}$$

This implies for the hypergeometric function in (3.9) that

$${}_2F_1 \left(\begin{matrix} -l+m, l+m+1 \\ m+n+1 \end{matrix}; \frac{ad}{ad-bc} \right) = \frac{(l-m)!(m+n)!}{(l+n)!} P_{l-m}^{(m+n, m-n)} \left(\frac{bc+ad}{bc-ad} \right).$$

Hence we can further rewrite (3.9) (if $m + n \geq 0$, $m - n \geq 0$) as follows:

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!} \right)^{\frac{1}{2}} c^{m-n} d^{m+n} (bc - ad)^{l-m} P_{l-m}^{(m+n, m-n)} \left(\frac{bc + ad}{bc - ad} \right). \quad (3.11)$$

We are in particular interested in (3.11) if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(2)$. Note that by (2.11) a general element of $SU(2)$ can be written as

$$\begin{pmatrix} \sin \theta e^{i\phi} & -\cos \theta e^{-i\psi} \\ \cos \theta e^{i\psi} & \sin \theta e^{-i\phi} \end{pmatrix} \quad \text{with } 0 \leq \theta \leq \pi/2 \text{ and } \phi, \psi \in [0, 2\pi).$$

Hence we obtain:

Theorem 3.2. *If $m + n \geq 0$, $m - n \geq 0$ then*

$$t_{m,n}^l \begin{pmatrix} \sin \theta e^{i\phi} & -\cos \theta e^{-i\psi} \\ \cos \theta e^{i\psi} & \sin \theta e^{-i\phi} \end{pmatrix} = (-1)^{l-m} \left(\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!} \right)^{\frac{1}{2}} \\ \times e^{-i(m+n)\phi} e^{i(m-n)\psi} (\sin \theta)^{m+n} (\cos \theta)^{m-n} P_{l-m}^{(m+n, m-n)}(\cos 2\theta). \quad (3.12)$$

4 Orthogonality of matrix elements

We introduce a special Borel measure μ on $SU(2)$ such that

$$\int_{SU(2)} f d\mu = \frac{1}{2\pi^2} \int_{\phi=0}^{2\pi} \int_{\psi=0}^{2\pi} \int_{\theta=0}^{\pi/2} f \begin{pmatrix} \sin \theta e^{i\phi} & -\cos \theta e^{-i\psi} \\ \cos \theta e^{i\psi} & \sin \theta e^{-i\phi} \end{pmatrix} \sin \theta \cos \theta d\theta d\psi d\phi. \quad (4.1)$$

for all continuous functions f on $SU(2)$. Note that

$$\int_{SU(2)} d\mu = 1. \quad (4.2)$$

The matrix elements $t_{m,n}^l$ satisfy a remarkable orthogonality relation with respect to this measure:

$$\int_{SU(2)} t_{m,n}^l \overline{t_{m',n'}^{l'}} d\mu = \frac{1}{2l+1} \delta_{l,l'} \delta_{m,m'} \delta_{n,n'}. \quad (4.3)$$

For $(m, n) \neq (m', n')$ this follows immediately from (3.12), (4.1) and the symmetries (2.8)–(2.10). (exercise). For $(m, n) = (m', n')$ with $m + n, m - n \geq 0$ we have to show that

$$\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!} \int_0^{\pi/2} P_{l-m}^{(m+n, m-n)}(\cos 2\theta) P_{l-m}^{(m+n, m-n)}(\cos 2\theta) (\sin \theta)^{2m+2n+1} (\cos \theta)^{2m-2n+1} d\theta \\ = \frac{1}{2l+1} \delta_{l,l'}.$$

By the substitution $x = \cos 2\theta$ this can be rewritten as

$$\frac{(l+m)!(l-m)!}{(l+n)!(l-n)!2^{2m+1}} \int_{-1}^1 P_{l-m}^{(m+n,m-n)}(x) P_{l-m}^{(m+n,m-n)}(x) (1-x)^{m+n}(1+x)^{m-n} dx = \frac{\delta_{l,l'}}{2l+1}. \quad (4.4)$$

In order to show this identity we remember the orthogonality relations for Jacobi polynomials (see [2] or [1]):

$$\int_{-1}^1 P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx = h_n^{(\alpha,\beta)} \delta_{m,n}$$

with

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1} (n+\alpha+\beta+1)_n \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n! \Gamma(2n+\alpha+\beta+2)}.$$

Now observe that

$$h_{l-m}^{(m+n,m-n)} = \frac{2^{2m+1} (l+n)!(l-n)!}{(2l+1)(l+m)!(l-m)!}.$$

This settles (4.4) and hence (4.3).

The orthogonality relation (4.3) is a special case of *Schur's orthogonality relations* for the matrix elements of the irreducible unitary representations of a compact group. For the formulation of this theorem we need the concept of the *Haar measure* on a compact group.

Theorem 4.1. *Let G be a compact group. There is a unique Borel measure μ on G , called **Haar measure**, such that $\mu(G) = 1$ and, for all Borel sets $E \subset G$ and for all $g \in G$,*

$$\mu(gE) = \mu(E) = \mu(Eg).$$

For a continuous function f on G this implies:

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g) = \int_G f(gh) d\mu(g) \quad (h \in G).$$

We also need the concept of equivalence of representations:

Definition 4.2. Let π_1 and π_2 be representations of a group G on finite dimensional complex vector spaces V_1 and V_2 , respectively. Then π_1 and π_2 are called *equivalent* to each other if there is a linear bijection $A: V \rightarrow W$ such that

$$A \pi_1(g) = \pi_2(g) A \quad \text{for all } g \in G.$$

The representations π_1 and π_2 are called *inequivalent* to each other if they are not equivalent to each other.

The relation of equivalence of representations is an equivalence relation on the set of all finite dimensional representations of a group G (exercise).

Theorem 4.3. (Schur's orthogonality relations)

Let G be a compact group with Haar measure μ . Let π and ρ be finite dimensional complex irreducible unitary representations of G which are inequivalent to each other. Let π resp. ρ have matrix elements $(\pi_{i,j})_{i,j=1,\dots,d_\pi}$ and $(\rho_{k,l})_{k,l=1,\dots,d_\rho}$ with respect to certain orthonormal bases of their representation spaces. Then

$$\int_G \pi_{i,j}(g) \overline{\rho_{k,l}(g)} d\mu(g) = 0$$

and

$$\int_G \pi_{i,j}(g) \overline{\pi_{k,l}(g)} d\mu(g) = \frac{1}{d_\pi} \delta_{i,k} \delta_{j,l}.$$

Let $(\pi^\alpha)_{\alpha \in A}$ be a maximal set of mutually inequivalent finite dimensional complex irreducible unitary representations of G , and put $d_\alpha := d_{\pi^\alpha}$. Then the orthogonality relations in Theorem 4.3 can be written as

$$\int_G \pi_{i,j}^\alpha(g) \overline{\pi_{k,l}^\beta(g)} d\mu(g) = \frac{1}{d_\alpha} \delta_{\alpha,\beta} \delta_{i,k} \delta_{j,l}. \quad (4.5)$$

Write $L^2(G)$ for $L^2(G, \mu)$. The functions $\pi_{i,j}$ are continuous on G , so they are certainly in $L^2(G)$. By (4.5) the functions $d_\alpha^{-\frac{1}{2}} \pi_{i,j}^\alpha$ ($\alpha \in A$, $i, j = 1, \dots, d_\alpha$) form an orthonormal system in $L^2(G)$.

Theorem 4.4. *With notation as above, the functions $d_\alpha^{-\frac{1}{2}} \pi_{i,j}^\alpha$ form a complete orthonormal system in $L^2(G)$.*

In order to match the orthogonality relation (4.3) to (4.5) we need an explicit form of the Haar measure on $SU(2)$ and we have to prove that the representations t^l are irreducible. First we deal with the Haar measure. By (2.11) the group $SU(2)$ is homeomorphic with the unit sphere $S^3 = \{(a, c) \in \mathbb{C}^2 \mid |a|^2 + |c|^2 = 1\}$. Let $S \in SU(2)$. A left multiplication $T \mapsto ST: SU(2) \rightarrow SU(2)$ corresponds to some rotation of S^3 (exercise). Thus a rotation invariant measure on S^3 will provide, after suitable normalization, the Haar measure on $SU(2)$. There exists, up to a constant factor, a unique rotation invariant measure ω on S^3 . This measure is such that, for all continuous functions on \mathbb{R}^4 of compact support and with λ Lebesgue measure on \mathbb{R}^4 ,

$$\int_{\mathbb{R}^4} f d\lambda = \int_{r=0}^{\infty} \int_{\xi \in S^3} f(r\xi) d\omega(\xi) r^3 dr. \quad (4.6)$$

Now take coordinates

$$x = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta \cos \psi, r \cos \theta \sin \psi)$$

on \mathbb{R}^4 , which means for $r = 1$ that $x_1 + ix_2 = \sin \theta e^{i\phi}$, $x_3 + ix_4 = \cos \theta e^{i\psi}$. These are just the coordinates chosen in (4.1) for $(a, c) \in \mathbb{C}^2$ with $|a|^2 + |c|^2 = 1$. A straightforward computation

of the Jacobian yields:

$$\begin{aligned} & \int_{\mathbb{R}^4} f(x_1, x_2, x_3, x_4) dx_1 dx_2 dx_3 dx_4 \\ &= \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \int_{\psi=0}^{2\pi} f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta \cos \psi, r \cos \theta \sin \psi) \\ & \quad \times r^3 \sin \theta \cos \theta dr d\theta d\phi d\psi. \end{aligned} \quad (4.7)$$

Comparison of (4.6) and (4.7) gives, for continuous functions F on $S^3 \subset \mathbb{C}^2$, that

$$\int_{S^3} F d\omega = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \int_{\psi=0}^{2\pi} F(\sin \theta e^{i\phi}, \cos \theta e^{i\psi}) \sin \theta \cos \theta d\theta d\phi d\psi. \quad (4.8)$$

In view of the previous observations we have thus shown that the Haar measure on $SU(2)$ is given by (4.1).

Now we will show the irreducibility of the representations t^l as representations of $SU(2)$.

Put $a_\phi := \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}$. Then $a_\phi a_\psi = a_{\phi+\psi}$ and $a_{\phi+2\pi} = a_\phi$. The group $A := \{a_\phi \mid 0 \leq \phi < 2\pi\}$ is a closed abelian subgroup of $SU(2)$. It is isomorphic and homeomorphic with the group $U(1)$ of complex numbers of absolute value 1, which has multiplication of complex numbers as the group multiplication. It follows from (2.4) that

$$t^l(a_\phi) \psi_n^l = e^{-2in\phi} \psi_n^l. \quad (4.9)$$

Lemma 4.5. *Let V be an invariant subspace of \mathcal{H}_l with respect to the representation π^l of $SU(2)$. If $v \in V$ and $\langle v, \psi_m^l \rangle \neq 0$ then $\psi_m^l \in V$.*

Proof We have

$$\begin{aligned} v &= \sum_{n=-l}^l \langle v, \psi_n^l \rangle \psi_n^l, \\ t^l(a_\phi) v &= \sum_{n=-l}^l \langle v, \psi_n^l \rangle t^l(a_\phi) \psi_n^l = \sum_{n=-l}^l \langle v, \psi_n^l \rangle e^{-2in\phi} \psi_n^l. \end{aligned}$$

Hence

$$\int_0^{2\pi} e^{2im\phi} t^l(a_\phi) v d\phi = \sum_{n=-l}^l \langle v, \psi_n^l \rangle \left(\int_0^{2\pi} e^{2im\phi} e^{-2in\phi} d\phi \right) \psi_n^l = 2\pi \langle v, \psi_m^l \rangle \psi_m^l.$$

The integral on the left should be interpreted as a Riemann integral of vectors, which can be approximated by Riemann sums of vectors. Since $v \in V$, each approximating Riemann sum is in V , and hence also their limit, the Riemann integral, is in V . Hence $2\pi \langle v, \psi_m^l \rangle \psi_m^l \in V$. So $\psi_m^l \in V$ if $\langle v, \psi_m^l \rangle \neq 0$. \square

This Lemma implies the following Proposition, the proof of which is left as an exercise.

Proposition 4.6. *Let V be an invariant subspace of \mathcal{H}_l with respect to the representation π^l of $SU(2)$. Then there is a subset \mathcal{A} of $\{-l, \dots, l\}$ such that $V = \text{Span}\{\psi_n^l \mid n \in \mathcal{A}\}$. Let W be the orthoplement of V and \mathcal{B} the complement of \mathcal{A} . Then W is also an invariant subspace and $W = \text{Span}\{\psi_n^l \mid n \in \mathcal{B}\}$.*

Theorem 4.7. *The representation t^l of $SU(2)$ is irreducible.*

Proof Suppose t^l is not irreducible. By Proposition 4.6 \mathcal{H}_l is the orthogonal direct sum of invariant subspaces $V = \text{Span}\{\psi_n^l \mid n \in \mathcal{A}\}$ and $W = \text{Span}\{\psi_n^l \mid n \in \mathcal{B}\}$, where $\{-l, \dots, l\}$ is the disjoint union of certain nonempty subsets \mathcal{A} and \mathcal{B} . One of these subsets, say \mathcal{A} , will contain l . Then some m will be in \mathcal{B} . Then $t^l(T)\psi_l^l$ will be in V for all $T \in SU(2)$, and therefore orthogonal to ψ_m^l . Hence $t_{m,l}^l(T) = 0$ for all $T \in SU(2)$. In particular, also using (2.6), we obtain

$$0 = t_{m,l}^l \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} = (-1)^{l-m} (\cos \theta)^{l-m} (\sin \theta)^{l+m},$$

which gives a contradiction. □

So finally we have matched (4.3) to (4.5). We could have started with (4.5) and have derived from this (4.3) and hence (4.4). Thus a proof of the orthogonality relations for Jacobi polynomials with nonnegative integer parameters is possible from the interpretation of Jacobi polynomials in connection with $SU(2)$.

5 Exercises

1. (An interpretation of Krawtchouk polynomials as matrix elements of irreducible representations of $SU(2)$)

a) Prove that

$$P_n^{(\alpha, \beta)}(-x) = (-1)^n P_n^{(\beta, \alpha)}(x). \quad (5.1)$$

(Use the orthogonality relations for Jacobi polynomials and the explicit expression for the leading coefficient of $P_n^{(\alpha, \beta)}(x)$.)

b) Prove that

$${}_2F_1 \left(\begin{matrix} -n, b \\ c \end{matrix}; x \right) = \frac{(c-b)_n}{(c)_n} {}_2F_1 \left(\begin{matrix} -n, b \\ b-c-n+1 \end{matrix}; 1-x \right) \quad (n = 0, 1, 2, \dots). \quad (5.2)$$

(Use (3.10) and (5.1).)

c) Prove that

$${}_2F_1 \left(\begin{matrix} -n, -m \\ c \end{matrix}; x \right) = \frac{(c)_{m+n}}{(c)_n (c)_m} {}_2F_1 \left(\begin{matrix} -n, -m \\ -c-n-m+1 \end{matrix}; 1-x \right) \quad (n, m = 0, 1, 2, \dots). \quad (5.3)$$

(Use (5.2).)

d) Prove that, for $m + n \geq 0$,

$$t_{m,n}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \binom{2l}{l-m}^{\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} b^{l-m} c^{l-n} d^{m+n} {}_2F_1 \left(\begin{matrix} -l+m, -l+n \\ -2l \end{matrix}; \frac{bc-ad}{bc} \right). \quad (5.4)$$

(Use (5.3) and (3.7).)

e) Prove that, for $m + n \geq 0$,

$$t_{m,n}^l \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} = \binom{2l}{l-m}^{\frac{1}{2}} \binom{2l}{l-n}^{\frac{1}{2}} (-1)^{l-m} (\cos \theta)^{2l-m-n} (\sin \theta)^{m+n} \\ \times K_{l-m}(l-n; \cos^2 \theta, 2l), \quad (5.5)$$

where the *Krawtchouk polynomials* are given by

$$K_n(x; p, N) := {}_2F_1 \left(\begin{matrix} -n, -x \\ -N \end{matrix}; p^{-1} \right) \quad (n = 0, 1, \dots, N). \quad (5.6)$$

f) Prove that

$$K_n(x; p, N) = (1 - p^{-1})^{x+n-N} K_{N-n}(N-x; p, N). \quad (5.7)$$

(Use (5.6) and Euler's transformation for Gauss hypergeometric functions.)

g) Prove that (5.5) remains valid for all m, n .

(Use (2.9) and (5.7).)

h) Show that

$$\sum_{n=-l}^l t_{m,n}^l \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} t_{m',n}^l \begin{pmatrix} \sin \theta & -\cos \theta \\ \cos \theta & \sin \theta \end{pmatrix} = \delta_{m,m'}, \quad (5.8)$$

and that this matches with the orthogonality relation for the Krawtchouk polynomials occurring on the right-hand side of (5.5).

2. (Addition formula and product formula for Legendre polynomials) Let $l = 0, 1, 2, \dots$

a) Prove that, for $ad - bc = 1$,

$$t_{0,0}^l \begin{pmatrix} a & b \\ c & d \end{pmatrix} = P_l(2ad - 1), \quad (5.9)$$

where P_l is the Legendre polynomial. (Use (3.11).)

b) Prove that

$$t_{0,0}^l(T) = P_l(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) \quad (5.10)$$

if

$$T = \begin{pmatrix} \sin \frac{1}{2}\theta_1 & -\cos \frac{1}{2}\theta_1 \\ \cos \frac{1}{2}\theta_1 & \sin \frac{1}{2}\theta_1 \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}i\phi} & 0 \\ 0 & e^{-\frac{1}{2}i\phi} \end{pmatrix} \begin{pmatrix} \sin \frac{1}{2}\theta_2 & \cos \frac{1}{2}\theta_2 \\ -\cos \frac{1}{2}\theta_2 & \sin \frac{1}{2}\theta_2 \end{pmatrix}. \quad (5.11)$$

c) Prove that

$$P_l(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) = P_l(\cos \theta_1) P_l(\cos \theta_2) + \sum_{0 < |k| \leq l} t_{0,k}^l \begin{pmatrix} \sin \frac{1}{2} \theta_1 & -\cos \frac{1}{2} \theta_1 \\ \cos \frac{1}{2} \theta_1 & \sin \frac{1}{2} \theta_1 \end{pmatrix} t_{k,0}^l \begin{pmatrix} \sin \frac{1}{2} \theta_2 & \cos \frac{1}{2} \theta_2 \\ -\cos \frac{1}{2} \theta_2 & \sin \frac{1}{2} \theta_2 \end{pmatrix} e^{-ik\phi}. \quad (5.12)$$

d) Prove that

$$P_l(\cos \theta_1) P_l(\cos \theta_2) = \frac{1}{2\pi} \int_0^{2\pi} P_l(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \phi) d\phi. \quad (5.13)$$

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