Orthogonal polynomials in several variables potentially useful in pde

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Orthogonal polynomials

Definition

μ a positive measure on $\mathbb{R}$ such that $\int_{\mathbb{R}} |x^n|\,d\mu(x) < \infty \quad (n \geq 0)$.
Call $\{p_n\}_{n=0,1,2,\ldots}$ a *system of orthogonal polynomials* (OP's) with respect to μ if p_n is polynomial of degree n and if

$$\int_{\mathbb{R}} p_n(x)\,q(x)\,d\mu(x) = 0 \quad (q \text{ polynomial of degree } < n).$$

Then p_n is unique up to constant factor and for certain $h_n > 0$:

$$\int_{\mathbb{R}} p_n(x)\,p_m(x)\,d\mu(x) = h_n\,\delta_{n,m}.$$

Special case: μ is absolutely continuous

$d\mu(x) = w(x)\,dx$ on (a,b) and $\mu = 0$ outside $[a,b]$. Then

$$\int_a^b p_n(x)\,p_m(x)\,w(x)\,dx = h_n\,\delta_{n,m}\,.$$

Three-term recurrence relation

Orthogonal polynomials p_n satisfy a three-term recurrence relation

$$xp_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x)$$

with $A_n C_{n+1} > 0$. Thus OP's $p_n(x)$ are eigenfunctions of a seond order difference equation in n with eigenvalue x.

Conversely, solutions of such a recurrence relation wih starting values $p_{-1}(x) = 0$, $p_0(x) = 1$ are orthogonal polynomials with respect to a certain positive measure (Favard's theorem).

Classical orthogonal polynomials

	$p_n(x)$	$w(x)$	(a,b)	
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha(1+x)^\beta$	$(-1,1)$	$\alpha,\beta>-1$
Laguerre	$L_n^\alpha(x)$	$x^\alpha e^{-x}$	$(0,\infty)$	$\alpha>-1$
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty,\infty)$	

Theorem (Bochner, 1929)

Above are essentially the only systems of OP's p_n for which $Lp_n = \lambda_n p_n$ (L some second order differential operator).

	$(Lf)(x)$	λ_n
Jacobi	$(1-x^2)f''(x)+(\beta-\alpha-(\alpha+\beta+2)x)f'(x)$	$-n(n+\alpha+\beta+1)$
Laguerre	$xf''(x)+(\alpha+1-x)f'(x)$	$-n$
Hermite	$\frac{1}{2}f''(x)-xf'(x)$	$-n$

Special cases of Jacobi polynomials

Legendre polynomials

$$P_n(x) := P_n^{(0,0)}(x), \qquad w(x) = 1 \text{ on } (-1,1).$$

Chebyshev polynomials of first kind

$$T_n(x) := \text{const. } P_n^{(-\frac{1}{2},-\frac{1}{2})}(x), \quad w(x) = \frac{1}{\sqrt{1-x^2}}, \quad T_n(\cos\theta) = \cos(n\theta).$$

Chebyshev polynomials of second kind

$$U_n(x) := \text{const. } P_n^{(\frac{1}{2},\frac{1}{2})}(x), \; w(x) = \sqrt{1-x^2}, \; U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}.$$

Orthogonal polynomials in d variables

Definition

μ a positive measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}} |x^\alpha| \, d\mu(x) < \infty$ $(\alpha \in (\mathbb{Z}_{\geq 0})^d)$. Call $\{\mathcal{P}_n\}_{n=0,1,2,\ldots}$ a *system of orthogonal polynomials* (OP's) with respect to μ if $\mathcal{P}_n$ is a linear space of dimension $\binom{n+d-1}{n}$ consisting of polynomials of degree n and if

$$\int_{\mathbb{R}^d} p(x)\, q(x)\, d\mu(x) = 0 \quad (p \in \mathcal{P}_n,\; q \in \mathcal{P}_m,\; n \neq m).$$

Note $\binom{n+d-1}{n}$ is the dimension of the space of homogeneous polynomials of degree n in d variables.

Refinement Choose an orthogonal basis $\{p_\alpha\}_{\alpha_1+\cdots+\alpha_d=n}$ for each space $\mathcal{P}_n$. Then

$$\int_{\mathbb{R}^d} p_\alpha(x)\, p_\beta(x)\, d\mu(x) = 0 \quad (\alpha, \beta \in (\mathbb{Z}_{\geq 0})^d,\; \alpha \neq \beta).$$

Recurrence relation for OP's in d variables

Let $\{\mathcal{P}_n\}_{n=0,1,2,\ldots}$ be a system of OP's. By the orthogonality measure μ there is an inner product on each space $\mathcal{P}_n$. Denote by $\mathcal{M}_i$ the operator of multiplication by x_i. Then

$$\mathcal{M}_i\big|_{\mathcal{P}_n} = \mathcal{A}_{n,i} + \mathcal{B}_{n,i} + \mathcal{C}_{n,i}\,,$$

where
$\mathcal{A}_{n,i}\colon \mathcal{P}_n \to \mathcal{P}_{n+1}$, injective,
$\mathcal{B}_{n,i}\colon \mathcal{P}_n \to \mathcal{P}_n$, symmetric,
$\mathcal{C}_{n,i}\colon \mathcal{P}_n \to \mathcal{P}_{n-1}$, transpose of $\mathcal{A}_{n-1,i}$.
Moreover, $\mathcal{P}_{n+1}$ is spanned by the images of $\mathcal{A}_{n,1},\ldots\mathcal{A}_{n,d}$.

There are also converse Favard type theorems (Xu, 1993, 1994).

Classical orthogonal polynomials in d variables

Special case of general OP's: μ is absolutely continuous.
$d\mu(x) = w(x)\,dx$ on some open domain U in $\mathbb{R}^d$ and $\mu = 0$ outside the closure of U. Then

$$\int_U p(x)\,q(x)\,w(x)\,dx = 0 \quad (p \in \mathcal{P}_n,\ q \in \mathcal{P}_m,\ n \neq m).$$

Definition

A system $\{\mathcal{P}_n\}$ of orthogonal polynomials in d variables is called *classical* if there is a second order pdo L such that

$$L\,p = \lambda_n\,p \quad (p \in \mathcal{P}_n).$$

Refinement Apart from $L = L_1$ there are $d-1$ further pdo's $L_2, \ldots, L_d$ such that $L_1, L_2, \ldots, L_d$ commute, are self-adjoint with respect to μ, and have one-dimensional joint eigenspaces. Then we have OP's p_α with $L_j p_\alpha = \lambda_\alpha^{(j)} p_\alpha$.

Classical orthogonal polynomials in 2 variables

	$w(x,y)$	U	L	λ_n
1.	$e^{-x^2-y^2}$	$\mathbb{R}^2$	$\frac{1}{2}(\partial_{xx}+\partial_{yy}) - x\partial_x - y\partial_y$	$-n$
2.	$x^\alpha y^\beta e^{-x-y}$	$(0,\infty)^2$	$x\partial_{xx} + y\partial_{yy} + (1+\alpha-x)\partial_x$ $+(1+\alpha-y)\partial_y$	$-n$
3.	$y^\beta e^{-x^2-y}$	$\mathbb{R}\times(0,\infty)$	$\frac{1}{2}\partial_{xx} + y\partial_{yy} - x\partial_x$ $+(1+\beta-y)\partial_y$	$-n$
4.	$x^\alpha y^\beta 1-x-y)^\gamma$	$x,y>0,$ $x+y<1$	L	λ_n
5.	$(1-x^2-y^2)^\alpha$	$x^2+y^2<1$	L	λ_n

In case 4 (triangular region):

$$L = x(1-x)\partial_{xx} + y(1-y)\partial_{yy} - 2xy\partial_{xy}$$
$$+\big(\alpha+1-(\alpha+\beta+\gamma+3)x\big)\partial_x + \big(\beta+1-(\alpha+\beta+\gamma+3)y\big)\partial_y,$$
$$\lambda_n = -n(n+\alpha+\beta+\gamma+2).$$

In case 5 (disk):

$$L = (1-x^2)\partial_{xx} + (1-y^2)\partial_{yy} - 2xy\partial_{xy} - (2\alpha+3)(x\partial_x + y\partial_y),$$
$$\lambda_n = -n(n+2\alpha+2).$$

Classical orthogonal polynomials in 2 variables, cntd.

Theorem (Krall & Sheffer, 1967; Kwon, Lee & Littlejohn, 2001)

The five cases just listed are essentially all classical OP's in two variables.

Special orthogonal basis $\{p_{n,k}\}_{k=0,1,\ldots,n}$ *for* $\mathcal{P}_n$ *in these 5 cases*:

	$w(x,y)$	U	$p_{n,k}(x,y)$
1.	$e^{-x^2-y^2}$	$\mathbb{R}^2$	$H_{n-k}(x)H_k(y)$
2.	$x^\alpha y^\beta e^{-x-y}$	$(0,\infty)^2$	$L_{n-k}^\alpha(x)L_k^\beta(y)$
3.	$y^\beta e^{-x^2-y}$	$\mathbb{R}\times(0,\infty)$	$H_{n-k}(x)L_k^\beta(y)$
4.	$x^\alpha y^\beta 1-x-y)^\gamma$	$x,y>0,$ $x+y<1$	$P_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(1-2x)$ $\times(1-x)^k P_k^{(\beta,\gamma)}(1-2y/(1-x))$
5.	$(1-x^2-y^2)^\alpha$	$x^2+y^2<1$	$P_{n-k}^{(\alpha+k+\frac12,\alpha+k+\frac12)}(x)$ $\times(1-x^2)^{k/2}P_k^{(\alpha,\alpha)}\big(y(1-x^2)^{-1/2}\big)$

History of triangle polynomials

The orthogonal polynomials of case 4 (on a triangular region) were introduced by Proriol (1957), and included in a survey paper by K (1975). Their special case $\alpha = \beta = \gamma = 0$ (constant weight function) also occurred in Munschy & Pluvinage (1957), and much later again in Dubiner (1991), who was not aware of the earlier results and was motivated by applications to finite elements. Dubiner's paper was much quoted in the context of finite elements. The book *Orthogonal polynomials of several variables* by Dunkl & Xu (2001) finally referred to Dubiner's paper. Conversely, Proriol's paper was observed by Hesthaven & Teng (2000), and later in the book *Spectral/hp element methods for computational fluid dynamics* by Karniadakis & Sherwin (second ed., 2005).

The orthogonal polynomials of case 5 (on the unit disk) seem to go back to Hermite. Again, the case $\alpha = 0$ (constant weight function) was independently observed by Dubiner (1991).

Classical orthogonal polynomials in 2 variables, cntd.

Put $L_1 := L$. There is a pdo L_2 commuting with L_1 and splitting up the eigenspaces of L_1.

$$L_1 p_{n,k} = \lambda_n p_{n,k}, \quad L_2 p_{n,k} = \mu_k p_{n,k}.$$

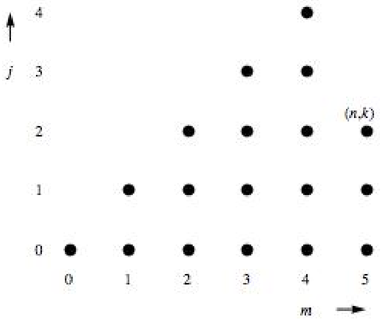
	$p_{n,k}(x,y)$	L_2	μ_k
1.	$H_{n-k}(x)H_k(y)$	$\frac{1}{2}\partial_{yy} - y\partial_y$	$-k$
2.	$L_{n-k}^{\alpha}(x)L_k^{\beta}(y)$	$y\partial_{yy} + (1+\alpha-y)\partial_y$	$-k$
3.	$H_{n-k}(x)L_k^{\beta}(y)$	$y\partial_{yy} + (1+\alpha-y)\partial_y$	$-k$
4.	$P_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(1-2x)$ $\times(1-x)^k$ $\times P_k^{(\beta,\gamma)}(1-2y/(1-x))$	$y(1-x-y)\partial_{yy}$ $+((\beta+1)(1-x)$ $-(\beta+\gamma+2)y)\partial_y$	$-k(k+\beta+\gamma+1)$
5.	$P_{n-k}^{(\alpha+k+\frac{1}{2},\alpha+k+\frac{1}{2})}(x)$ $\times(1-x^2)^{k/2}$ $\times P_k^{(\alpha,\alpha)}(y(1-x^2)^{-1/2})$	$(1-x^2-y^2)\partial_{yy}$ $-2(\alpha+1)y\partial_y$	$-k(k+2\alpha+1)$

Orthogonalization

In general, for OP's in 2 variables with respect to a measure μ, orthogonal bases $\{p_{n,k}\}_{k=0,1,\ldots,n}$ for the successive spaces $\mathcal{P}_n$ ($n = 0, 1, 2, \ldots$) can be obtained by Gram-Schmidt orthogonalization of the monomials

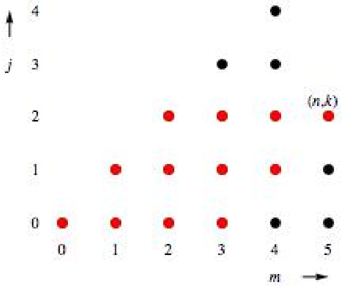
$$1, x, y, x^2, xy, y^2, \ldots, x^n, x^{n-1}y, \ldots, x^{n-k}y^k, \ldots,$$

i.e., of the monomials $x^{m-j}y^j$ with the (m, j) lexicographically ordered. The classical OP's $p_{n,k}(x, y)$ can also be obtained in this way. However, not all (m, j) below (n, k) are then needed for obtaining $p_{n,k}$.

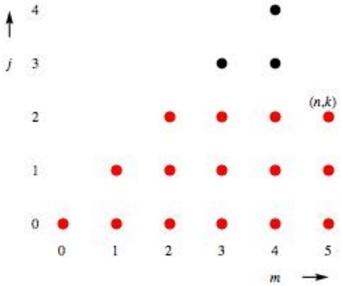


Orthogonalization, cntd.

In cases 1, 2, 3
$p_{n,k}(x,y) = p_{n-k}(x)q_k(y)$
only contains
monomials $x^{m-j}y^j$ with
$m-j \le n-k$ and $j \le k$
(red dots in the picture).



In case 4 $p_{n,k}(x,y)$ of form
$p^k_{n-k}(x)(1-x)^k q_k(\frac{y}{1-x})$
only contains
monomials $x^{m-j}y^j$
with $m \le n$ and $j \le k$
(red dots in the picture).
Similarly for case 5.



Orthogonalization, cntd.

Thus $p_{n,k}$ of case 4 can already be characterized as a polynomial

$$p_{n,k}(x,y) = \sum_{j=0}^{k} \sum_{m=j}^{n} c_{m,j} x^{m-j} y^j$$

such that $c_{n,k} \neq 0$ and

$$\int_{\mathbb{R}^2} p_{n,k}(x,y) x^{m-j} y^j \, d\mu(x,y) = 0$$

if $0 \le j \le k$, $j \le m \le n$, $(m,j) \neq (n,k)$.
For an arbitrary measure μ on $\mathbb{R}^2$ this would not guarantee that always $\int_{\mathbb{R}^2} p_{n,k}(x,y) p_{m,j}(x,y) \, d\mu(x,y) = 0$ for $(n,k) \neq (m,j)$, but for the special measure of case 4 it yields full orthogonality.

More general triangle polynomials

Let the polynomials $q_n(x)$ be orthogonal with respect to a weight function $w_2(x)$ on $(0,1)$ and let for $k = 0,1,2,\ldots$ the polynomials $p_n^k(x)$ be orthogonal with respect to a weight function $w_1(x)(1-x)^{2k+1}$ on $(0,1)$. Then the polynomials

$$p_{n,k}(x,y) := p_{n-k}^k(x)(1-x)^k q_k(y/(1-x))$$

are orthogonal on the triangular region $x, y > 0$, $x + y < 1$ with respect to the weight function

$$w(x,y) := w_1(x)w_2(y/(1-x)).$$

These OP's generalize the classical OP's of case 4. The expansion of $p_{n,k}(x,y)$ in monomials has the same type for all generalized triangle prolynomials.
The recurrence relations are also of simple nature:

$$x\,p_{n,k} \in \mathrm{Span}\{p_{m,k}\}_{|m-n|\le 1} \qquad \text{(3 terms)},$$
$$y\,p_{n,k} \in \mathrm{Span}\{p_{m,j}\}_{|m-n|\le 1,\ |j-k|\le 1} \qquad \text{(9 terms)}.$$

Other orthogonal bases: Disk polynomials

$$R_{m,n}^{\alpha}(z) := \text{const.} \begin{cases} P_n^{(\alpha,m-n)}(2|z|^2-1)z^{m-n}, & m \geq n, \\ P_m^{(\alpha,n-m)}(2|z|^2-1)\overline{z}^{n-m}, & n \geq m \end{cases}$$

$$((m,n) \in (\mathbb{Z}_{\geq 0})^2,\ z \in \mathbb{C},\ \alpha > -1).$$

$R_{m,n}^{\alpha}(z) = \text{const.}\, z^m \overline{z}^n +$ polynomial in $z, \overline{z}$ of lower degree.

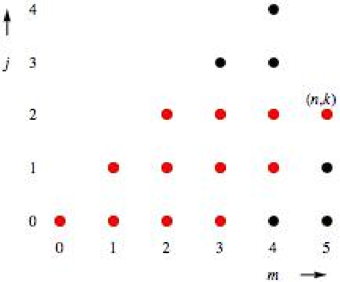
$$\int_{x^2+y^2<1} R_{m,n}^{\alpha}(x+iy)\,\overline{R_{k,l}^{\alpha}(x+iy)}\,(1-x^2-y^2)^{\alpha}\,dx\,dy = 0$$

$$((m,n) \neq (k,l)).$$

For $\alpha = 0$ called *Zernike polynomials*, introduced by Zernike (1934) in view of applications in optics. Also in Dubiner (1991). Still much used, for instance in design of chip machines at ASML. Zernike got Nobel prize in physics in 1953. Disk polynomials for general α first in Zernike & Brinkman (1935).

Monomial bases and biorthogonal systems

For OP's in 2 variables of the form $p_{n,k}(x) = p_{n-k}(x)q_k(y)$ we have seen that only the monomials corresponding to red dots occur. In particular, $p_{n,k} \in \mathcal{P}_n$ such that $p_{n,k}(x,y) = const.x^{n-k}y^k +$ polynomial of lower degree.



For general OP's in 2 variables we can also define a basis of such $p_{n,k}$ for $\mathcal{P}_n$ (a *monomial basis*), but then, for fixed n, the $p_{n,k}$ are usually not mutually orthogonal. We may then consider a basis of $q_{n,k}$ ($k = 0, 1, \ldots, n$) for $\mathcal{P}_n$ which is *biorthogonal* to the $p_{n,k}$, i.e., $\int_{\mathbb{R}^2} p_{n,k}(x,y)p_{n,j}(x,y)\,d\mu(x,y) = 0 \quad (k \neq j)$.

Appell gave explicit biorthogonal systems on the disk and for $\gamma = 0$ on the triangle (1882). Fackerell & Littler (1974) gave them for general α, β, γ.

Positive convolution structures

Let $\{p_\alpha\}$ be a complete orthogonal system in $L^2(X,\mu)$:

$$\int_X p_\alpha(x)\,p_\beta(x)\,d\mu(x) = \omega_\alpha^{-1}\,\delta_{\alpha,\beta}.$$

Suppose that there are positive measures $\nu_{x,y}$ $(x,y\in X)$ such that for all α and all $x,y\in X$ there holds the *product formula*

$$p_\alpha(x)\,p_\alpha(y) = \int_X p_\alpha(z)\,d\nu_{x,y}(z).$$

Then the following *generalized convolution* is positive:

$$\Big(\sum_\alpha a_\alpha\,\omega_\alpha p_\alpha\Big) * \big(\sum_\alpha b_\alpha\,\omega_\alpha\,p_\alpha\big) := \sum_\alpha a_\alpha\,b_\alpha\,\omega_\alpha\,p_\alpha.$$

This is the case for Jacobi polynomials normalized by $P_n^{(\alpha,\beta)}(1)=1$ while $\alpha\ge\beta\ge-\frac{1}{2}$ (Gasper, 1972).
It is also the case for triangle polynomials normalized by $p_{n,k}^{\alpha,\beta,\gamma}(0,1)=1$ while $\alpha\ge\beta+\gamma+1$ and $\gamma\ge\beta\ge-\frac{1}{2}$ (K & Schwartz, 1997).

Classical orthogonal polynomials on the simplex

Define the simplex T^d in $\mathbb{R}^d$ by

$$0 \le x_d \le x_{d-1} \ldots \le x_1 \le 1.$$

The OP's on T^d with respect to the weight function

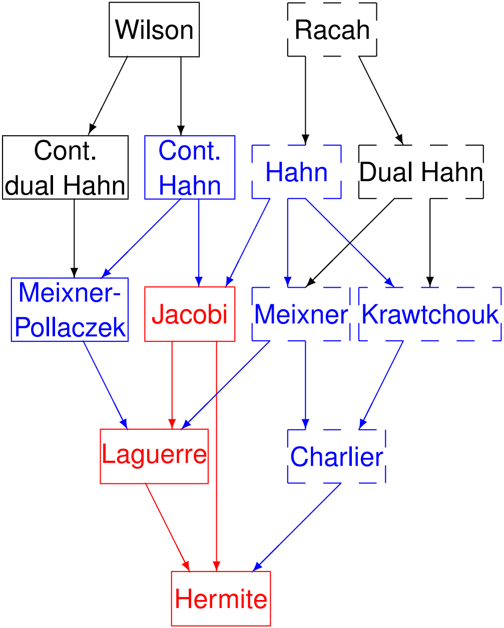
$$w(x_1, \ldots, x_d) := (1 - x_1)^{\alpha_1} (x_1 - x_2)^{\alpha_2} \ldots (x_{d-1} - x_d)^{\alpha_d} \, x_d^{\alpha_{d+1}}$$

are classical. An orthogonal basis $\left\{p_{n_1,\ldots,n_d}^{\alpha_1,\ldots,\alpha_{d+1}}\right\}_{n=n_1 \ge n_2 \ge \ldots \ge n_d}$ of $\mathcal{P}_n$ can be defined recursively by

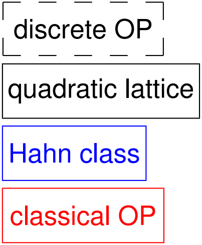
$$p_{n_1,\ldots,n_d}^{\alpha_1,\ldots,\alpha_{d+1}}(x_1, \ldots, x_d) = P_{n_1-n_2}^{(\alpha_1, \alpha_2+\cdots+\alpha_{d+1}+2n_2+d-1)}(2x_1 - 1) \\ \times \, x_1^{n_2} \, p_{n_2,\ldots,n_d}^{\alpha_2,\ldots,\alpha_{d+1}} \left(\frac{x_2}{x_1}, \ldots, \frac{x_d}{x_1}\right)$$

(Kalnins, Miller & Tratnik, 1991). Here $p_n^{\alpha,\beta}(x) := P_n^{(\alpha,\beta)}(2x - 1)$. Then the recursion for $d = 2$ essentially gives the triangle polynomials.

Askey scheme



Dick Askey



Some limits in the Askey scheme

Jacobi $\to$ *Laguerre* $\quad (\beta \to \infty)$

$$P_n^{(\alpha,\beta)}(1 - 2\beta^{-1}x) \to L_n^{\alpha}(x), \quad x^{\alpha}(1 - \beta^{-1}x)^{\beta} \to x^{\alpha}e^{-x}.$$

triangle $\to$ *product Laguerre* $\quad (\gamma \to \infty)$

$$\begin{aligned} &p_{n,k}^{\alpha,\beta,\gamma}(\gamma^{-1}x, \gamma^{-1}y) \\ &\quad = P_{n-k}^{(\alpha,\beta+\gamma+2k+1)}(1 - 2\gamma^{-1}x)(1 - \gamma^{-1}x)^k P_k^{(\beta,\gamma)}\big(1 - 2y/(\gamma - x)\big) \\ &\qquad\qquad \to L_{n-k}^{\alpha}(x)L_k^{\beta}(y), \end{aligned}$$

$$x^{\alpha}y^{\beta}\big(1 - \gamma^{-1}(x + y)\big)^{\gamma} \to x^{\alpha}y^{\beta}e^{-x-y}.$$

Some limits in the Askey scheme, cntd.

Pochhammer symbol:

$$(a)_n := a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}\,.$$

Hahn polynomials $\quad Q_n(x;\alpha,\beta,N) \quad (n=0,1,\dots,N)$:

$$\sum_{x=0}^{N}(Q_nQ_m)(x;\alpha,\beta,N)\,\frac{(\alpha+1)_x}{x!}\,\frac{(\beta+1)_{N-x}}{(N-x)!}=0 \quad (n\neq m).$$

Hahn $\to$ *Jacobi* $\quad (N\to\infty)$

$$Q_n(Nx;\alpha,\beta,N)\to P_n^{(\alpha,\beta)}(1-2x)$$

$$\frac{1}{N}\sum_{x\in\{0,\frac{1}{N},\frac{2}{N},\dots,1\}} f(x)\,\frac{\Gamma(Nx+\alpha+1)}{N^\alpha\,\Gamma(Nx+1)}\,\frac{\Gamma(N(1-x)+\beta+1)}{N^\beta\,\Gamma(N(1-x)+1)}$$

$$\to\int_0^1 f(x)\,x^\alpha\,(1-x)^\beta\,dx.$$

Discrete triangle polynomials

OP's on the set $\{(x,y)\in\mathbb{Z}^2 \mid x,y\ge 0,\ x+y\le N\}$ with respect to the weights

$$w(x,y;\alpha,\beta,\gamma,N) := \frac{(\alpha+1)_x}{x!}\,\frac{(\beta+1)_y}{y!}\,\frac{(\gamma+1)_{N-x-y}}{(N-x-y)!}$$

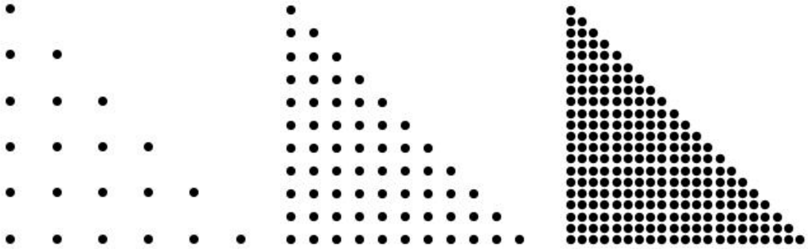
(for $\alpha=\beta=\gamma=0$ constant weights). There is a second order difference operator in x and y for which the spaces $\mathcal{P}_n$ are eigenspaces. An orthogonal basis of $\mathcal{P}_n$ is given by

$$Q_{n,k}(x,y;\alpha,\beta,\gamma,N) := Q_{n-k}(x;\alpha,\beta+\gamma+2k+1,N-k) \\ \times(-N+x)_k\,Q_k(y;\beta,\gamma,N-x)$$

(Karlin & McGregor, 1964, 1975; Rahman, 1981).

$$\lim_{N\to\infty} Q_{n,k}(Nx,Ny;\alpha,\beta,\gamma,N) = p_{n,k}^{\alpha,\beta,\gamma}(x,y).$$

From discrete to continuum triangle



Some literature

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