Extended abstract of the lecture

Orthogonal polynomials in several variables potentially useful in pde

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A system of orthogonal polynomials (OP's) $\{p_n\}_{n=0}^\infty$ on $\mathbb{R}$ with respect to a positive measure μ on $\mathbb{R}$ is called *classical* if there is a second order differential operator L such that $Lp_n = \lambda_n p_n$ $(n = 0, 1, 2, \ldots)$ for certain eigenvalues λ_n. By a theorem of Bochner [1] there are three families of classical OP's (up to an affine transformation of the argument of the OP):

1. Hermite: $p_n = H_n$, $d\mu(x) = e^{-x^2}\,dx$ on $\mathbb{R}$,
   $(Lf)(x) = \frac{1}{2}f''(x) - xf'(x)$, $\lambda_n = -n$.
2. Laguerre: $p_n = L_n^\alpha$, $d\mu(x) = x^\alpha e^{-x}\,dx$ on $[0, \infty)$, $\alpha > -1$,
   $(Lf)(x) = xf''(x) + (\alpha + 1 - x)f'(x)$, $\lambda_n = -n$.
3. Jacobi: $p_n = P_n^{(\alpha,\beta)}$, $d\mu(x) = (1 - x)^\alpha(1 + x)^\beta\,dx$ on $[-1, 1]$, $\alpha, \beta > -1$,
   $(Lf)(x) = (1-x^2)f''(x)+(\beta-\alpha-(\alpha+\beta+2)x)f'(x)$, $\lambda_n = -n(n+\alpha+\beta+1)$.

Let μ be a positive measure on $\mathbb{R}^d$ such that $\int_{\mathbb{R}} |x^\alpha|\,d\mu(x) < \infty$ $(\alpha \in (\mathbb{Z}_{\ge 0})^d)$ and the support of μ has nonempty interior. Let $\mathcal{P}_n$ consist of all polynomials p of degree $\le n$ such that $\int_{\mathbb{R}^d} pq\,d\mu = 0$ for all polynomials q of degree $< n$. Then $\mathcal{P}_n$ has the same dimension $\binom{n+d-1}{n}$ as the space of homogeneous polynomials of degree n in d variables. Furthermore, the spaces $\mathcal{P}_n$ $(n = 0, 1, 2, \ldots)$ are mutually orthogonal in $L^2(\mu)$. We call $\{\mathcal{P}_n\}_{n=0}^\infty$ a *system of orthogonal polynomials* with respect to the measure μ.

As a refinement of this notion we may choose an orthogonal basis $\{p_\alpha\}_{\alpha_1+\cdots+\alpha_d=n}$ for each space $\mathcal{P}_n$, and call the polynomials p_α orthogonal polynomials. Of course, there are many ways to choose such orthogonal bases.

A system $\{\mathcal{P}_n\}$ of orthogonal polynomials in d variables is called *classical* if there is a second order pdo L acting on the space of polynomials such that $\mathcal{P}_n$ is an eigenspace of L for a certain eigenvalue λ_n $(n = 0, 1, 2, \ldots)$. As a refinement there may be, apart from $L = L_1$, $d - 1$ further pdo's $L_2, \ldots, L_d$ such that $L_1, L_2, \ldots, L_d$ commute, are self-adjoint with respect to μ, and have one-dimensional joint eigenspaces. Then we have OP's p_α with $L_j p_\alpha = \lambda_\alpha^{(j)} p_\alpha$.

It was shown by Krall & Sheffer [8] and Kwon, Lee & Littlejohn [9] that there are five families of classical orthogonal polynomials in 2 variables, as follows:

1. $d\mu(x, y) = e^{-x^2-y^2}\,dx\,dy$ on $\mathbb{R}^2$, $L = \frac{1}{2}(\partial_{xx} + \partial_{yy}) - x\partial_x - y\partial_y$, $\lambda_n = -n$.
2. $d\mu(x, y) = x^\alpha y^\beta e^{-x-y}\,dx\,dy$ on $[0, \infty) \times [0, \infty)$, $\alpha, \beta > -1$,
   $L = x\partial_{xx} + y\partial_{yy} + (1 + \alpha - x)\partial_x + (1 + \alpha - y)\partial_y$, $\lambda_n = -n$.
3. $d\mu(x, y) = y^\beta e^{-x^2-y}\,dx\,dy$ on $\mathbb{R} \times [0, \infty)$, $\beta > -1$,
   $L = \frac{1}{2}\partial_{xx} + y\partial_{yy} - x\partial_x + (1 + \beta - y)\partial_y$, $\lambda_n = -n$.
4. $d\mu(x, y) = x^\alpha y^\beta (1 - x - y)^\gamma\,dx\,dy$ on $\{(x, y) \in \mathbb{R}^2 \mid x, y \ge 0,\ x + y \le 1\}$, $\alpha, \beta, \gamma > -1$, $L = x(1 - x)\partial_{xx} + y(1 - y)\partial_{yy} - 2xy\partial_{xy} + \big(\alpha + 1 - (\alpha + \beta + \gamma + 3)x\big)\partial_x + \big(\beta + 1 - (\alpha + \beta + \gamma + 3)y\big)\partial_y$, $\lambda_n = -n(n + \alpha + \beta + \gamma + 2)$.

5. $d\mu(x,y) = (1-x^2-y^2)^\alpha\,dx\,dy$ on $\{(x,y)\in\mathbb{R}^2 \mid (x^2+y^2\le 1\}$, $\alpha>-1$, $L=(1-x^2)\partial_{xx}+(1-y^2)\partial_{yy}-2xy\partial_{xy}-(2\alpha+3)(x\partial_x+y\partial_y)$, $\lambda_n=-n(n+2\alpha+2)$.

Orthogonal bases $\{p_{n,k}\}_{k=0,1,\ldots,n}$ for $\mathcal{P}_n$ $(n=0,1,2,\ldots)$ in these five cases can be obtained by Gram-Schmidt orthogonalization of the monomials $1,x,y,x^2,xy,y^2$, $\ldots,x^n,x^{n-1}y,\ldots,x^{n-k}y^k,\ldots$. The resulting polynomials are as follows.

1. $p_{n,k}(x,y)=H_{n-k}(x)H_k(y)$.
2. $p_{n,k}(x,y)=L^\alpha_{n-k}(x)L^\beta_k(y)$.
3. $p_{n,k}(x,y)=H_{n-k}(x)L^\beta_k(y)$.
4. $p_{n,k}(x,y)=P^{(\alpha,\beta+\gamma+2k+1)}_{n-k}(1-2x)\,(1-x)^kP^{(\beta,\gamma)}_k\big(1-2y/(1-x)\big)$.
5. $p_{n,k}(x,y)=P^{(\alpha+k+\frac12,\alpha+k+\frac12)}_{n-k}(x)\,(1-x^2)^{k/2}P^{(\alpha,\alpha)}_k\big(y/\sqrt{1-x^2}\,\big)$.

The expansions in monomials of these polynomials $p_{n,k}$ do not involve all monomials $x^{m-j}y^j$ with (m,j) equal or less than (n,k) in the lexicographic ordering. For classes 1, 2 and 3 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m-j\le n-k$ and $j\le k$. For classes 4 and 5 $p_{n,k}(x,y)$ only contains monomials $x^{m-j}y^j$ with $m\le n$ and $j\le k$. Furthermore, in these five cases there is a second order differental operator L_2 commuting with L which has the $p_{n,k}$ as eigenfunctions with eigenvalue only depending on k.

The OP's $p_{n,k}$ for case 4 (on the triangular region), as explicitly given above, were introduced by Proriol [10] in 1967. They were mentioned in the survey paper by Koornwinder [7] in 1975. Their special case $\alpha=\beta=\gamma=0$ (constant weight function) was rediscovered by Dubiner [2] in 1991, who was motivated by applications to finite elements. Dubiner's paper was much quoted in this context. For a while, the special functions and finite elements communities were not aware that they had a joint interest. But in 2000 Hesthaven & Teng [4] referred to Proriol's paper, while later Karniadakis & Sherwin in their book [6] had ample references to papers on special functions. Conversely, in 2001 Dunkl & Xu referred in their book [3] to Dubiner's paper.

Another important orthogonal system for case 5 on the disk is as follows.

$$R^\alpha_{m,n}(z):=\text{const.}\begin{cases}P^{(\alpha,m-n)}_n(2|z|^2-1)z^{m-n}, & m\ge n,\\ P^{(\alpha,n-m)}_m(2|z|^2-1)\overline{z}^{n-m}, & n\ge m\end{cases}$$
$$\big((m,n)\in(\mathbb{Z}_{\ge0})^2,\ z\in\mathbb{C},\ \alpha>-1\big).$$

Then $\quad R^\alpha_{m,n}(z)=\text{const.}\,z^m\overline{z}^n+\text{polynomial in } z,\overline{z}$ of lower degree.

and $\displaystyle\int_{x^2+y^2<1}R^\alpha_{m,n}(x+iy)\,\overline{R^\alpha_{k,l}(x+iy)}\,(1-x^2-y^2)^\alpha\,dx\,dy=0\quad((m,n)\neq(k,l)).$

For $\alpha=0$ these polynomials are called *Zernike polynomials*. They were introduced by Zernike [11] in 1934 for applications in optics and are still much used there. The polynomials $R^\alpha_{m,n}$ for general α first occurred in Zernike & Brinkman [12].

For numerical applications it is important that Jacobi polynomials can be approximated by polynomials which are orthogonal on finitely many equidistant points. These are the *Hahn polynomials* $Q_n(x;\alpha,\beta,N)$ $(n=0,1,\dots,N)$ satisfying

$$\sum_{x=0}^{N}(Q_nQ_m)(x;\alpha,\beta,N)\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}=0\quad(n\neq m).$$

The approximation is: $\lim_{N\to\infty}Q_n(Nx;\alpha,\beta,N)=\text{const.}\,P_n^{(\alpha,\beta)}(1-2x)$.

From the Hahn polynomials we can build polynomials (Karlin & McGregor [5])

$$Q_{n,k}(x,y;\alpha,\beta,\gamma,N):=Q_{n-k}(x;\alpha,\beta+\gamma+2k+1,N-k)\binom{N-x}{k}Q_k(y;\beta,\gamma,N-x)$$

which are orthogonal on the set $\{(x,y)\in\mathbb{Z}^2\mid x,y\geq 0,\ x+y\leq N\}$ with respect to the weights

$$w(x,y;\alpha,\beta,\gamma,N):=\binom{\alpha+x}{x}\binom{\beta+y}{y}\binom{\gamma+N-x-y}{N-x-y}.$$

They approximate the polynomials of class 4 on the triangle:

$$\lim_{N\to\infty}Q_{n,k}(Nx,Ny;\alpha,\beta,\gamma,N)=\text{const.}\,p_{n,k}^{\alpha,\beta,\gamma}(x,y),$$

which looks promising for applications.

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