Zhedanov's Askey-Wilson algebra, Cherednik's double affine Hecke algebras, and bispectrality. Lecture 3: Double affine Hecke algebras in the rank one case

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Hankel transform

Normalized Bessel function:

$$\mathcal{J}_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}x^2)^k}{(\alpha+1)_k\,k!} = {}_0F_1\left(\begin{matrix}-\\ \alpha+1\end{matrix}; -\tfrac{1}{4}x^2\right).$$

$$\mathcal{J}_\alpha(x) = \mathcal{J}_\alpha(-x), \quad \mathcal{J}_\alpha(0) = 1, \quad \mathcal{J}_{-\frac{1}{2}}(x) = \cos x, \quad \mathcal{J}_{\frac{1}{2}}(x) = \frac{\sin x}{x}.$$

Eigenfunctions:

$$\left(\frac{d^2}{dx^2} + \frac{2\alpha+1}{x}\frac{d}{dx}\right)\mathcal{J}_\alpha(\lambda x) = -\lambda^2\,\mathcal{J}_\alpha(\lambda x).$$

Hankel transform pair:

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_0^\infty f(x)\mathcal{J}_\alpha(\lambda x)x^{2\alpha+1}\,dx, \\ f(x) = \displaystyle\frac{1}{2^{2\alpha+1}\Gamma(\alpha+1)^2}\int_0^\infty \widehat{f}(\lambda)\mathcal{J}_\alpha(\lambda x)\lambda^{2\alpha+1}\,d\lambda. \end{cases}$$

Non-symmetric Hankel transform

Non-symmetric Bessel function:

$$\mathcal{E}_\alpha(x) := \mathcal{J}_\alpha(x) + \frac{i\,x}{2(\alpha+1)}\,\mathcal{J}_{\alpha+1}(x), \quad \text{so} \quad \mathcal{E}_{-\frac{1}{2}}(x) = e^{ix}.$$

Non-symmetric Hankel transform pair:

$$\begin{cases} \widehat{f}(\lambda) = \displaystyle\int_{-\infty}^{\infty} f(x)\,\mathcal{E}_\alpha(-\lambda x)\,|x|^{2\alpha+1}\,dx, \\ f(x) = \displaystyle\frac{1}{2^{2(\alpha+1)}\Gamma(\alpha+1)^2}\int_{-\infty}^{\infty} \widehat{f}(\lambda)\,\mathcal{E}_\alpha(\lambda x)\,|\lambda|^{2\alpha+1}\,d\lambda. \end{cases}$$

Differential-reflection operator:

$$(Yf)(x) := f'(x) + (\alpha+\tfrac{1}{2})\,\frac{f(x)-f(-x)}{x}$$

(Dunkl operator for root system A_1).

Eigenfunctions:

$$Y(\mathcal{E}_\alpha(\lambda\,.\,)) = i\,\lambda\,\mathcal{E}_\alpha(\lambda\,.\,)\,.$$

Askey-Wilson polynomials

Recall the Askey-Wilson polynomials $P_n[z]$ as monic symmetric Laurent polynomials:

$$P_n[z] = P_n[z; a, b, c, d \mid q] = P_n\big(\tfrac{1}{2}(z + z^{-1})\big)$$
$$:= \frac{(ab, ac, ad; q)_n}{a^n(abcdq^{n-1}; q)_n} \, {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix} ; q, q \right).$$

They satisfy

$$(LP_n)[z] = \lambda_n \, P_n[z] \quad \text{with} \quad \lambda_n = q^{-n} + abcdq^{n-1} \quad \text{and}$$
$$(Lf)[z] := A[z] \, f[qz] + A[z^{-1}] \, f[q^{-1}z] - (A[z] + A[z^{-1}]) \, f[z]$$
$$+ (1 + q^{-1}abcd) f[z]$$
$$\text{with} \quad A[z] := \frac{(1 - az)(1 - bz)(1 - cz)(1 - dz)}{(1 - z^2)(1 - qz^2)}.$$

Non-symmetric Askey-Wilson polynomials

Assumptions $q \neq 0, \quad q^m \neq 1 \ (m = 1, 2, \ldots)$,
$a, b, c, d \neq 0, \quad abcd \neq q^{-m} \ (m = 0, 1, 2, \ldots)$,
$\{a, b\} \cap \{a^{-1}, b^{-1}\} = \emptyset$.

In terms of

$$\begin{aligned} P_n[z] &= P_n[z; a, b, c, d \mid q], \\ Q_n[z] &:= a^{-1} b^{-1} z^{-1} (1 - az)(1 - bz)\, P_{n-1}[z; qa, qb, c, d \mid q] \end{aligned}$$

the nonsymmetric Askey-Wilson polynomials are defined by:

$$E_{-n} := \frac{ab}{ab-1} (P_n - Q_n) \qquad (n = 1, 2, \ldots), \qquad E_0[z] := 1,$$

$$E_n := \frac{(1 - q^n ab)(1 - q^{n-1} abcd)}{(1 - ab)(1 - q^{2n-1} abcd)}\, P_n - \frac{ab(1 - q^n)(1 - q^{n-1} cd)}{(1 - ab)(1 - q^{2n-1} abcd)}\, Q_n \qquad (n = 1, 2, \ldots).$$

Eigenfunctions of q-difference-reflection operator

Let

$$
\begin{aligned}
(Yf)[z] := {} & \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\, f[z] \\
& + \frac{(1-az)(1-b)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)} f[qz] \\
& + \frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\, f[z^{-1}] \\
& + \frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\, f[qz^{-1}],
\end{aligned}
$$

then

$$
\begin{aligned}
YE_{-n} &= q^{-n}\, E_{-n} \qquad && (n = 1, 2, \ldots), \\
YE_n &= q^{n-1} abcd\, E_n \qquad && (n = 0, 1, 2, \ldots).
\end{aligned}
$$

These come from I. Cherednik's theory of double affine Hecke algebras associated with root systems, extended by S. Sahi to the type $(C_l^\vee, C_l)$. Here his case $l = 1$ is considered.

Double affine Hecke algebra of type $(C_1^\vee, C_1)$

This is the algebra $\tilde{\mathfrak{H}}$ generated by Z, Z^{-1}, T_1, T_0 with relations $ZZ^{-1} = 1 = Z^{-1}Z$ and

$$(T_1 + ab)(T_1 + 1) = 0, \qquad (T_0 + q^{-1}cd)(T_0 + 1) = 0,$$
$$(T_1 Z + a)(T_1 Z + b) = 0, \quad (qT_0 Z^{-1} + c)(qT_0 Z^{-1} + d) = 0.$$

This algebra acts faithfully on the linear space $\mathcal{A}$ of Laurent polynomials:

$$(Zf)[z] := z\, f[z],$$
$$(T_1 f)[z] := \frac{(a+b)z - (1+ab)}{1-z^2}\, f[z] + \frac{(1-az)(1-bz)}{1-z^2}\, f[z^{-1}],$$
$$(T_0 f)[z] := \frac{q^{-1}z((cd+q)z - (c+d)q)}{q-z^2}\, f[z] - \frac{(c-z)(d-z)}{q-z^2}\, f[qz^{-1}].$$

Then $Y = T_1 T_0$.

Eigenspaces of T_1

- T_1 acting on $\mathcal{A}$ has eigenvalues $-ab$ and -1.
- $T_1 f = -ab\, f \iff f \in \mathcal{A}_{\rm sym}$ (symmetric Laurent polynomial).
- $T_1 f = -f \iff f[z] = z^{-1}(1-az)(1-bz)g[z]$ for some $g \in \mathcal{A}_{\rm sym}$.

Let A be an operator on $\mathcal{A}$. Write
$f[z] = f_1[z] + z^{-1}(1-az)(1-bz)f_2[z] \quad (f_1, f_2 \in \mathcal{A}_{\rm sym})$.
Then we can write

$$(Af)[z] = (A_{11} f_1 + A_{12} f_2)[z] + z^{-1}(1-az)(1-bz)(A_{21} f_1 + A_{22} f_2)[z],$$

where the A_{ij} are operators on $\mathcal{A}_{\rm sym}$. So

$$f \leftrightarrow (f_1, f_2), \quad A \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Rewriting the eigenvalue equation for E_n in matrix form

$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{-n}\right)\begin{pmatrix} P_n[z;a,b,c,d \mid q] \\ -a^{-1}b^{-1}P_{n-1}[z;qa,qb,c,d \mid q] \end{pmatrix} = 0,$$

$$\left(\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} - q^{n-1}abcd\right) \times \begin{pmatrix} (1-q^n ab)(1-q^{n-1}abcd)\,P_n[z;a,b,c,d \mid q] \\ -(1-q^n)(1-q^{n-1}cd)\,P_{n-1}[z;qa,qb,c,d \mid q] \end{pmatrix} = 0.$$

Here

$$Y_{11} = \frac{ab(1+q^{-1}cd) - abL_{a,b,c,d;q}}{1-ab},$$

$$Y_{22} = \frac{-ab(1+q^{-1}cd) + q^{-1}L_{aq,bq,c,d;q}}{1-ab},$$

where $L_{a,b,c,d;q}$ is the second order q-difference operator L having the $p_n[z;a,b,c,d \mid q]$ as eigenfunctions.

Also: $Y + q^{-1}abcdY^{-1} = \begin{pmatrix} L_{a,b,c,d;q} & 0 \\ 0 & q\,L_{aq,bq,c,d;q} \end{pmatrix}$.

The off-diagonal operators Y_{21} and Y_{12}

$$(Y_{21}g)[z] = \frac{z(c-z)(d-z)\left(g[q^{-1}z]-g[z]\right)}{(1-ab)(1-z^2)(1-qz^2)} + \frac{z(1-cz)(1-dz)\left(g[qz]-g[z]\right)}{(1-ab)(1-z^2)(1-qz^2)},$$

$$\begin{aligned}(Y_{12}h)[z] = {} & \frac{ab(a-z)(b-z)(1-az)(1-bz)}{(1-ab)z(q-z^2)(1-qz^2)} \\ & \times \left((cd+q)(1+z^2)-(1+q)(c+d)z\right) h[z] \\ & - \frac{ab(a-z)(b-z)(c-z)(d-z)(aq-z)(bq-z)}{q(1-ab)z(1-z^2)(q-z^2)}\, h[q^{-1}z] \\ & - \frac{ab(1-az)(1-bz)(1-cz)(1-dz)(1-aqz)(1-bqz)}{q(1-ab)z(1-z^2)(1-qz^2)}\, h[qz].\end{aligned}$$

An equivalent form for the eigenvalue equations

The eigenvalue equations for E_n and for E_{-n} are equivalent to the four equations

$$
\begin{aligned}
&L_{a,b,c,d;q}P_n[\,.\,;a,b,c,d\mid q]\\
&\qquad\qquad = (q^{-n}+abcdq^{n-1})P_n[\,.\,;a,b,c,d\mid q],\\
&L_{qa,qb,c,d;q}P_{n-1}[\,.\,;qa,qb,c,d\mid q]\\
&\qquad\qquad = (q^{-n+1}+abcdq^{n})P_{n-1}[\,.\,;qa,qb,c,d\mid q],\\
&Y_{21}P_n[\,.\,;a,b,c,d\mid q]\\
&\qquad\qquad = -\,\frac{(q^{-n}-1)(1-cdq^{n-1})}{1-ab}\,P_{n-1}[\,.\,;qa,qb,c,d\mid q],\\
&Y_{12}P_{n-1}[\,.\,;qa,qb,c,d\mid q]\\
&\qquad\qquad = -\,\frac{ab(q^{-n}-ab)(1-abcdq^{n-1})}{1-ab}\,P_n[\,.\,;a,b,c,d\mid q].
\end{aligned}
$$

Note that Y_{21} and Y_{12} act as shift operators.

Askey-Wilson polynomials: orthogonality

Askey-Wilson polynomials $P_n[z]$ satisfy the orthogonality relation

$$\frac{1}{4\pi i}\oint_C P_n[z]\,P_m[z]\,w[z]\,\frac{dz}{z} = h_n\,\delta_{n,m}, \quad \text{where}$$

$$w(z) := \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty},$$

$$h_0 = \frac{(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},$$

$$\widetilde{h}_n := \frac{h_n}{h_0} = \frac{(q, ab, ac, ad, bc, bd, cd; q)_n}{(abcd; q)_{2n}(q^{n-1}abcd; q)_n}.$$

Here C is the unit circle traversed in positive direction with deformations to separate the sequences of poles converging to zero from the sequences of poles diverging to ∞.

For suitable a, b, c, d this can be rewritten as an orthogonality relation for the $P_n(x)$ with respect to a positive measure μ supported on $[-1, 1]$ (or on its union with a finite discrete set).

Askey-Wilson polynomials: orthogonality (continued)

Let $\langle\,.\,,\,.\,\rangle_{a,b,c,d;q}$ be the Hermitian inner product on $\mathcal{A}_{\rm sym}$ such that the $P_n[\,.\,;a,b,c,d\mid q]$ are orthogonal in the familiar way:

$$\langle P_n[\,.\,;a,b,c,d\mid q],P_m[\,.\,;a,b,c,d\mid q]\rangle_{a,b,c,d;q}=\widetilde{h}_n^{a,b,c,d;q}\,\delta_{n,m}\,,$$

where

$$\widetilde{h}_n^{a,b,c,d;q}=\frac{(q,ab,ac,ad,bc,bd,cd;q)_n}{(abcd;q)_{2n}(q^{n-1}abcd;q)_n}\,.$$

(Assume that a,b,c,d are such that $\langle\,.\,,\,.\,\rangle_{a,b,c,d;q}$ and $\langle\,.\,,\,.\,\rangle_{qa,qb,c,d;q}$ are positive definite.) Then

$$\frac{\widetilde{h}_n^{a,b,c,d;q}}{\widetilde{h}_{n-1}^{qa,qb,c,d;q}}=\frac{(1-q^n)(1-q^{n-1}cd)}{(1-q^nab)(1-q^{n-1}abcd)}\times\frac{(1-ab)(1-qab)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)}\,.$$

Orthogonality relations: vector-valued case

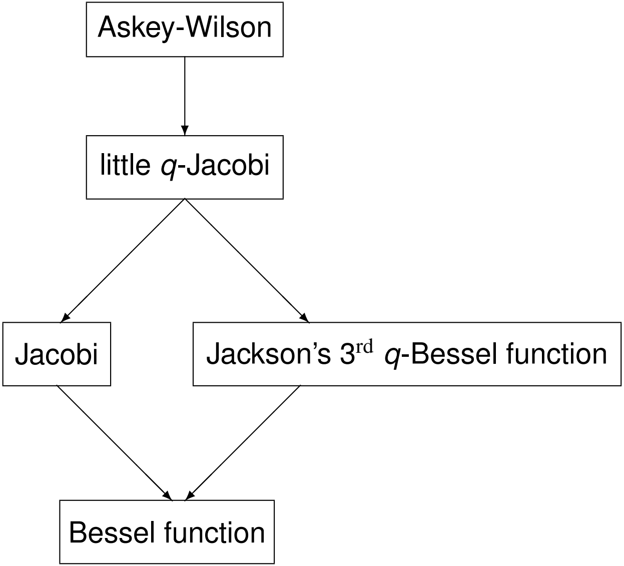
For g_1, h_1, g_2, h_2 symmetric Laurent polynomials define an hermitian inner product

$$\langle (g_1,h_1),(g_2,h_2)\rangle := \langle g_1,g_2\rangle_{a,b,c,d;q} - \langle h_1,h_2\rangle_{qa,qb,c,d;q}$$
$$ab \times \frac{(1-ab)(1-qab)(1-ac)(1-ad)(1-bc)(1-bd)}{(1-abcd)(1-qabcd)}\,.$$

Then the E_n ($n \in \mathbb{Z}$) in vector-valued form are orthogonal with respect to this inner product. If $ab < 0$ then the inner product is positive definite.

In earlier papers (Sahi, Noumi & Stokman, Macdonald's 2003 book) a biorthogonality was given in the form of a contour integral, and there were no results on positive definiteness of the inner product.

Going down in the (q-)Askey scheme



Comment on previous sheet

I gave only a very limited selection of possible paths downwards in the (q-)Askey scheme. Moreover I moved out of this scheme by taking limits to the non-polynomial orthogonal system of Jackson's q-Bessel functions (see Koornwinder & Swarttouw, 1992 for this limit), or to a generalized orthogonal system of classical Bessel functions.

The machinery of an algebra given by generators and relations acting on the space of 2-vector-valued polynomials and giving rise to an explicit orthogonal system of eigenfunctions has good limits when going from Askey-Wilson polynomials to little q-Jacobi polynomials, from little q-Jacobi polynomials to Jacobi polynomials, and from Jacobi polynomials to Bessel functions (arriving at the Dunkl operator as a 2×2 matrix operator acting on 2-vectors of symmetric functions).

However, in the limit from little q-Jacobi polynomials to Jackson's third q-Bessel functions undesired degenerations occur. Work on this is still in progress.

Zhedanov's algebra $AW(3)$

- generators K_0, K_1, K_2,
- structure constants B, C_0, C_1, D_0, D_1,
- relations

$$
\begin{aligned}
[K_0, K_1]_q &= K_2,\\
[K_1, K_2]_q &= BK_1 + C_0 K_0 + D_0,\\
[K_2, K_0]_q &= BK_0 + C_1 K_1 + D_1.
\end{aligned}
$$

Put for the structure constants:

$$
\begin{aligned}
B &:= (1 - q^{-1})^2 (e_3 + q e_1),\\
C_0 &:= (q - q^{-1})^2,\\
C_1 &:= q^{-1}(q - q^{-1})^2 e_4,\\
D_0 &:= -q^{-3}(1-q)^2(1+q)(e_4 + q e_2 + q^2),\\
D_1 &:= -q^{-3}(1-q)^2(1+q)(e_1 e_4 + q e_3).
\end{aligned}
$$

Central extension of $AW(3)$

Let the algebra $\widetilde{AW}(3)$ be generated by K_0, K_1, T_1 such that T_1 commutes with K_0, K_1 and with further relations

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0,\\
(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 &= B\,K_1 + C_0\,K_0 + D_0\\
&\quad + E\,K_1(T_1+ab) + F_0(T_1+ab),\\
(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 &= B\,K_0 + C_1\,K_1 + D_1\\
&\quad + E\,K_0(T_1+ab) + F_1(T_1+ab),
\end{aligned}
$$

where

$$
\begin{aligned}
E &:= -q^{-2}(1-q)^3(c+d),\\
F_0 &:= q^{-3}(1-q)^3(1+q)(cd+q),\\
F_1 &:= q^{-3}(1-q)^3(1+q)(a+b)cd.
\end{aligned}
$$

Basic representation of $\widetilde{AW}(3)$

The following element $\widetilde{Q}$ commutes with all elements of $\widetilde{AW}(3)$:

$$\begin{aligned}\widetilde{Q} :=&(K_1K_0)^2-(q^2+1+q^{-2})K_0(K_1K_0)K_1\\ &+(q+q^{-1})K_0^2K_1^2+(q+q^{-1})(C_0K_0^2+C_1K_1^2)\\ &+\big(B+E(T_1+ab)\big)\big((q+1+q^{-1})K_0K_1+K_1K_0\big)\\ &+(q+1+q^{-1})\big(D_0+F_0(T_1+ab)\big)K_0\\ &+(q+1+q^{-1})\big(D_1+F_1(T_1+ab)\big)K_1+G(T_1+ab),\end{aligned}$$

where G can be explicitly specified.

$\widetilde{AW}(3)$ acts on $\mathcal{A}$ such that K_0, K_1, T_1 act as $Y+q^{-1}abcdY^{-1}$, $Z+Z^{-1}$, T_1, respectively, in the basic representation of $\tilde{\mathfrak{H}}$ on $\mathcal{A}$. This action is called the *basic representation* of $\widetilde{AW}(3)$ on $\mathcal{A}$.

Then $\widetilde{Q}$ acts as the constant

$$\begin{aligned}Q_0 = q^{-4}(1-q)^2\big(&q^4(e_4-e_2)+q^3(e_1^2-e_1e_3-2e_2)\\ &-q^2(e_2e_4+2e_4+e_2)+q(e_3^2-2e_2e_4-e_1e_3)+e_4(e_1-e_2)\big).\end{aligned}$$

A faithful representation on $\mathcal{A}$

Definition

$\widetilde{AW}(3, Q_0)$ is the algebra $\widetilde{AW}(3)$ with additional relation $\widetilde{Q} = Q_0$.

Theorem (THK, 2007)

$\widetilde{AW}(3, Q_0)$ has the elements

$$K_0^n (K_1 K_0)^i K_1^m T_1^j \quad (m, n = 0, 1, 2, \ldots, \quad i, j = 0, 1)$$

as a linear basis.

The polynomial representation of $\widetilde{AW}(3, Q_0)$ on $\mathcal{A}$ is faithful.

$\widetilde{AW}(3, Q_0)$ has an injective embedding in $\tilde{\mathfrak{H}}$ such that $K_0 \to Y + q^{-1} abcd Y^{-1}$, $K_1 \to Z + Z^{-1}$, $T_1 \to T_1$.

The spherical subalgebra

Definition

The *spherical subalgebra* of $\tilde{\mathfrak{H}}$ consists of all linear operators A_{11} on $\mathcal{A}_{\rm sym}$ obtained from all linear operators $\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ in the 2×2 matrix realization of the basic representation of $\tilde{\mathfrak{H}}$.

Theorem (THK, 2008)

The spherical subalgebra is indeed an associative algebra with identity. It coincides with the algebra $AW(3, Q_0)$ (defined in lecture 2) in its basic representation.

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