Dualities in the q-Askey scheme and degenerate DAHA

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl
https://staff.fnwi.uva.nl/t.h.koornwinder/

Lecture on 10 July 2018 at XXXII International Colloquium on Group Theoretical Methods in Physics (Group32), Prague, Czech Republic

last modified: 19 July 2018

Work joint with Marta Mazzocco.

Preprint on `arXiv:1803.02775v2`.

Version v2 is much improved compared to v1..

Askey-Wilson polynomials (AW polynomials):

$$R_n[z] = R_n[z; a, b, c, d \mid q] := {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right).$$

$R_n[z] = R_n[z^{-1}]$ is symmetric Laurent polynomial of degree n.
Hence ordinary polynomial of degree n in $x = \frac{1}{2}(z + z^{-1})$.
Under constraints on parameters **orthogonal polynomials**:

$$\int_{|z|=1} R_m[z]\, R_n[z] \left| \frac{(z^2; q)_\infty}{(az, bz, cz, dz; q)_\infty} \right|^2 \frac{dz}{iz} = 0 \quad (m \neq n).$$

Eigenfunction of second order q-difference operator:

$$L_z\big(R_n[z]\big) = \lambda_n\, R_n[z], \qquad \lambda_n := q^{-n} + abcdq^{n-1},$$

where L acts on symmetric Laurent polynomials $f[z]$ by

$$(Lf)[z] = L_z\big(f[z]\big) := \big(1 + q^{-1}abcd\big) f[z] + \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)} \big(f[qz] - f[z]\big) + \frac{(a-z)(b-z)(c-z)(d-z)}{(1-z^2)(q-z^2)} \big(f[q^{-1}z] - f[z]\big).$$

Zhedanov algebra (AW algebra):

$$(K_0 f)[z] := L_z\big(f[z]\big), \qquad (K_1 f)[z] := (z + z^{-1})f[z]. \qquad (1)$$

Then

$$\begin{aligned}(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 &= B\,K_1 + C_0\,K_0 + D_0,\\ (q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 &= B\,K_0 + C_1\,K_1 + D_1\end{aligned} \qquad (2)$$

with structure constants B, C_0, C_1, D_0, D_1 explicitly given in terms of a, b, c, d, q.

The abstract algebra generated by K_0, K_1 with relations (2) is the *AW algebra*

$$\mathrm{AW} = \mathrm{AW}_{a,b,c,d;q}(K_0, K_1)$$

and (1) defines its *basic representation*.

Duality for AW polynomials and for the AW algebra

Dual parameters:

$$\tilde a=(q^{-1}abcd)^{\frac12},\quad \tilde b=ab/\tilde a,\quad \tilde c=ac/\tilde a,\quad \tilde d=ad/\tilde a.$$

Recall

$$R_n[z;a,b,c,d\,|\,q]={}_4\phi_3\left(\begin{matrix}q^{-n},q^{n-1}abcd,az,az^{-1}\\ ab,ac,ad\end{matrix};q,q\right).$$

Hence the duality relation

$$R_n\big[a^{-1}q^{-m};a,b,c,d\,|\,q\big]=R_m\big[\tilde a^{-1}q^{-n};\tilde a,\tilde b,\tilde c,\tilde d\,|\,q\big]\quad (m,n\in\mathbb{Z}_{\ge0}).$$

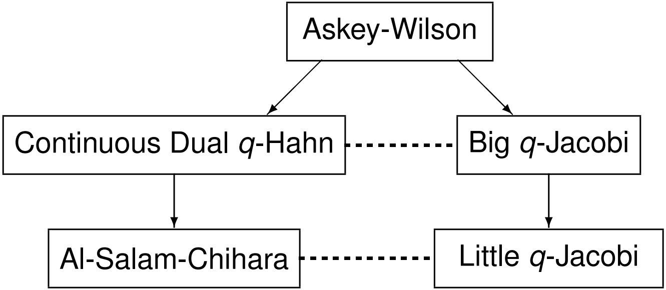
Recall the relations for the AW algebra:

$$(q+q^{-1})K_1K_0K_1-K_1^2K_0-K_0K_1^2=B\,K_1+C_0\,K_0+D_0,$$
$$(q+q^{-1})K_0K_1K_0-K_0^2K_1-K_1K_0^2=B\,K_0+C_1\,K_1+D_1$$

Hence we have for the AW algebra the duality (both isomorphism and anti-isomorphism):

$$\mathrm{AW}_{a,b,c,d;q}(K_0,K_1)\simeq \mathrm{AW}_{\tilde a,\tilde b,\tilde c,\tilde d;q}(aK_1,\tilde a^{-1}K_0),$$

Duality in a small part of the *q*-Askey scheme



Marta Mazzocco's motivation

The Painlevé differential equations are eight non-linear ODE's whose solutions are encoded by points in the so-called monodromy manifolds (a different manifold for each Painlevé equation). Each of these monodromy manifolds carries a natural Poisson structure which quantizes to a special degeneration of the AW algebra that regulates a specific family in the q-Askey scheme.

Continous dual q-Hahn and Big q-Jacobi:

Askey-Wilson:

$$R_n[z;a,b,c,d\,|\,q] = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n-1}abcd, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right),$$

$$R_n\big[a^{-1}q^{-m};a,b,c,d\,|\,q\big] = R_m\big[\tilde{a}^{-1}q^{-n};\tilde{a},\tilde{b},\tilde{c},\tilde{d}\,|\,q\big] \quad \text{(duality)}.$$

Continuous dual q-Hahn:

$$\begin{aligned} R_n[z;a,b,c\,|\,q] &= \lim_{d\to 0} R_n[z;a,b,c,d\,|\,q] \\ &= {}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ ab, ac \end{matrix}; q, q\right). \end{aligned}$$

Big q-Jacobi:

$$\begin{aligned} P_n(x;a,b,c;q) &= \lim_{\lambda\to 0} R_n[\lambda^{-1}x;\lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda\,|\,q] \\ &= {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ aq, cq \end{matrix}; q, q\right). \end{aligned}$$

Duality between Continuous dual q-Hahn and Big q-Jacobi:

$$R_n\big[a^{-1}q^{-m};a,b,c\,|\,q\big] = P_m(q^{-n};q^{-1}ab, ab^{-1}, q^{-1}ac;q).$$

Al-Salam-Chihara and Little q-Jacobi

Al-Salam-Chihara is Continuous dual q-Hahn for $c = 0$:

$$R_n[z; a, b, 0 \,|\, q] = {}_3\phi_2\left(\begin{matrix} q^{-n}, az, az^{-1} \\ ab, 0 \end{matrix}; q, q\right).$$

Little q-Jacobi is Big q-Jacobi for $c = 0$:

$$P_n(x; a, b, 0; q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}ab, x \\ aq, 0 \end{matrix}; q, q\right).$$

Duality between Al-Salam-Chihara and Little q-Jacobi:

$$R_n\left[a^{-1}q^{-m}; a, b, 0 \,|\, q\right] = P_m(q^{-n}; q^{-1}ab, ab^{-1}, 0; q).$$

Degenerate AW-algebras

Continuous dual q-Hahn:

$$\mathrm{AW}^{\mathrm{CDqH}}_{a,b,c;q}(K_0,K_1)=\lim_{d\to 0}\mathrm{AW}_{a,b,c,d;q}(K_0,K_1)$$

Big q-Jacobi:

$$\mathrm{AW}^{\mathrm{BqJ}}_{a,b,c;q}(K_0,K_1)=\lim_{\lambda\to 0}\mathrm{AW}_{\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda;q}(K_0,\lambda^{-1}K_1).$$

Duality between both algebras:

$$\mathrm{AW}^{\mathrm{CDqH}}_{a,b,c;q}(K_0,K_1)\simeq\mathrm{AW}^{\mathrm{BqJ}}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}(aK_1,K_0).$$

For the duality beteween Al-Salam-Chihara and Little q-Jacobi put $c=0$.

Remark

Our dualities between orthogonal polynomials in the q-Askey scheme are not completely bispectral. Instead one can identify, for instance, Al-Salam-Chihara polynomials for base q^{-1} with dual little q-Jacobi polynomials, see for instance Atakishyiev & Klimyk.
On the level of AW algebras this would need

$$\mathrm{AW}_{a,b,c,d;q}(K_0,K_1)\simeq \mathrm{AW}_{a^{-1},b^{-1},c^{-1},d^{-1};q^{-1}}\left(\frac{q}{abcd}K_0,K_1\right).$$

and its limit cases.

Askey-Wilson DAHA (Sahi; Noumi & Stokman)

Algebra $\mathcal{H}$ with generators $T_1, T_0, \check{T}_1, \check{T}_0$ and with relations

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0, \\
(T_0 + q^{-1}cd)(T_0 + 1) &= 0, \\
(a\check{T}_1 + 1)(b\check{T}_1 + 1) &= 0, \\
(c\check{T}_0 + q)(d\check{T}_0 + q) &= 0, \\
\check{T}_1 T_1 T_0 \check{T}_0 &= 1.
\end{aligned}
$$

Put $\check{T}_0 = T_0^{-1} Z, \quad \check{T}_1 = Z^{-1} T_1^{-1}, \quad T_0 = T_1^{-1} Y$.
Then we get the presentation $\mathcal{H}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle$:

$$
\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0, \\
(T_1^{-1} Y + q^{-1}cd)(T_1^{-1} Y + 1) &= 0, \\
(aZ^{-1} T_1^{-1} + 1)(bZ^{-1} T_1^{-1} + 1) &= 0, \\
(c + qZ^{-1} T_1^{-1} Y)(d + qZ^{-1} T_1^{-1} Y) &= 0, \\
ZZ^{-1} = 1 &= Z^{-1} Z.
\end{aligned}
$$

Basic representation of $\mathcal{H}$ on Laurent polynomials

$$
\begin{aligned}
(Zf)[z] &= z\,f[z],\\
(T_1 f)[z] &= \frac{(a+b)z-(1+ab)}{1-z^2}\,f[z]+\frac{(1-az)(1-bz)}{1-z^2}\,f[z^{-1}],\\
(Yf)[z] &= \frac{z\big(1+ab-(a+b)z\big)\big((c+d)q-(cd+q)z\big)}{q(1-z^2)(q-z^2)}\,f[z]\\
&\quad+\frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,f[qz]\\
&\quad+\frac{(1-az)(1-bz)\big((c+d)qz-(cd+q)\big)}{q(1-z^2)(1-qz^2)}\,f[z^{-1}]\\
&\quad+\frac{(c-z)(d-z)\big(1+ab-(a+b)z\big)}{(1-z^2)(q-z^2)}\,f[qz^{-1}].
\end{aligned}
$$

Non-symmetric AW polynomials:

$$E_n[z;a,b,c,d\,|\,q] := R_n[z;a,b,c,d\,|\,q] - \frac{a(1-q^n)(1-q^{n-1}cd)(1-az)(1-bz)}{q^{n-1}(1-qab)(1-ab)(1-ac)(1-ad)z} R_{n-1}[z;qa,qb,c,d\,|\,q] \qquad (n \ge 0,\ (1-q^n)E_{n-1} := 0 \text{ for } n=0),$$

$$E_{-n}[z;a,b,c,d\,|\,q] := R_n[z;a,b,c,d\,|\,q] - \frac{(1-q^n ab)(1-q^{n-1}abcd)(1-az)(1-bz)}{q^{n-1}b(1-qab)(1-ab)(1-ac)(1-ad)z} R_{n-1}[z;qa,qb,c,d\,|\,q] \qquad (n \ge 1),$$

Symmetric AW polynomial R_n is linear combination of E_n and E_{-n}.

$$YE_n = q^{n-1}abcd\,E_n \qquad (n = 0,1,2,\ldots),$$

$$YE_{-n} = q^{-n}\,E_{-n} \qquad (n = 1,2,\ldots),$$

$$(Y + q^{-1}abcdY^{-1})R_n = LR_n = \lambda_n R_n \qquad (n = 0,1,2,\ldots).$$

Duality for non-symmetric AW polynomials:

$$E_n\big[z_{a,q}(m)^{-1}; a, b, c, d \,|\, q\big] = E_m\big[z_{\tilde a,q}(n)^{-1}; \tilde a, \tilde b, \tilde c, \tilde d \,|\, q\big] \quad (m, n \in \mathbb{Z}),$$

where

$$\begin{aligned} z_{a,q}(n) &:= aq^n \qquad &(n \ge 0),\\ z_{a,q}(-n) &:= a^{-1}q^{-n} \qquad &(n > 0).\end{aligned}$$

Duality for the Askey-Wilson DAHA:

$$\mathcal{H}_{a,b,c,d;q}\langle T_1, Y, Z^{-1}\rangle \simeq \mathcal{H}_{\tilde a,\tilde b,\tilde c,\tilde d;q}\langle T_1, aZ^{-1}, \tilde a^{-1}Y\rangle,$$

an anti-isomorphism.

DAHA degeneration from AW to continuous dual q-Hahn (Mazzocco)

Write the AW DAHA as $\mathcal{H}_{a,b,c,d;q}\langle T_1, Y, Y^{-1}, Z, Z^{-1}\rangle$,
rescale $Y^{-1} = qc^{-1}d^{-1}Y'$, and let $d \to 0$.
Then we get the degenerate DAHA for continuous dual q-Hahn

$$\mathcal{H}^{\mathrm{CDqH}}_{a,b,c;q}\langle T_1, Y, Y', Z, Z^{-1}\rangle = \lim_{d\to 0} \mathcal{H}_{a,b,c,d;q}\langle T_1, Y, qc^{-1}d^{-1}Y', Z, Z^{-1}\rangle$$

with generators T_1, Y, Y', Z, Z^{-1} and relations

$$\begin{aligned}
(T_1 + ab)(T_1 + 1) &= 0,\\
T_1^{-1}Y + Y'T_1 + 1 &= 0,\\
(aZ^{-1}T_1^{-1} + 1)(bZ^{-1}T_1^{-1} + 1) &= 0,\\
qZ^{-1}T_1^{-1}Y + Y'T_1Z + c &= 0,\\
YY' = 0 = Y'Y, \quad ZZ^{-1} &= 1 = Z^{-1}Z.
\end{aligned}$$

Further degeneration to Al-Salam-Chihara by $c \to 0$.

DAHA degeneration from AW to big q-Jacobi (Mazzocco)

In $\mathcal{H}_{a,b,c,d;q}\langle T_1, Y, Y^{-1}, Z, Z^{-1}\rangle$ rescale $(a,b,c,d) \to (\lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda)$, $Z \to \lambda^{-1}X$, $Z^{-1} \to \lambda^{-1}X'$, and let $\lambda \to 0$.
Then we get the degenerate DAHA for big q-Jacobi:

$$\mathcal{H}^{\mathrm{BqJ}}_{a,b,c;q}\langle T_1, Y, Y^{-1}, X, X'\rangle = \lim_{\lambda\to 0} \mathcal{H}_{\lambda, qa\lambda^{-1}, qc\lambda^{-1}, bc^{-1}\lambda; q}\langle T_1, Y, Y^{-1}, \lambda^{-1}X, \lambda^{-1}X'\rangle$$

with generators T_1, Y, Y^{-1}, X, X' and with relations

$$\begin{aligned} (T_1 + qa)(T_1 + 1) &= 0, \\ (T_1^{-1}Y + b)(T_1^{-1}Y + 1) &= 0, \\ T_1X + qaX'T_1^{-1} + qa &= 0, \\ bY^{-1}T_1X + qX'T_1^{-1}Y + qc &= 0, \\ XX' = 0 &= X'X. \end{aligned}$$

Further degeneration to little q-Jacobi by $c \to 0$.

Duality for degenerate DAHA's (Mazzocco)

$$\mathcal{H}^{\mathrm{CDqH}}_{a,b,c;q}\langle T_1, X', a^{-1}b^{-1}X, aY^{-1}, a^{-1}Y\rangle \simeq \mathcal{H}^{\mathrm{BqJ}}_{q^{-1}ab,ab^{-1},q^{-1}ac;q}\langle T_1, Y, Y^{-1}, X, X'\rangle.$$

Also for $c \to 0$.

Non-symmetric dual q-Hahn polynomials:

$$E_n[z;a,b,c\,|\,q]=\lim_{d\to 0}E_n[z;a,b,c,d\,|\,q].$$

In basic representation of $\mathcal{H}^{\rm CDqH}_{a,b,c;q}\langle T_1,Y,Y',Z,Z^{-1}\rangle$:

$$\begin{aligned} YE_n&=0 && (n=0,1,2,\ldots),\\ YE_{-n}&=q^{-n}\,E_{-n} && (n=1,2,\ldots),\end{aligned}$$

$$\begin{aligned} Y'E_n&=q^{-n}a^{-1}b^{-1}\,E_n && (n=0,1,2,\ldots),\\ Y'E_{-n}&=0 && (n=1,2,\ldots).\end{aligned}$$

Also for $c\to 0$.

Non-symmetric big q-Jacobi polynomials

Recall that

$$P_n(x;a,b,c;q)=\lim_{\lambda\to 0} R_n[\lambda^{-1}x;\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda\,|\,q].$$

$E_{\pm n}[z;a,b,c,d\,|\,q]$ is linear combination of $R_n[z;a,b,c,d\,|\,q]$ and $z^{-1}(1-az)(1-bz)R_{n-1}[z;qa,qb,c,d\,|\,q]$.
So the non-symmetric big q-Jacobi polynomial would be the limit for $\lambda\to 0$ of a linear combination of
$R_n[\lambda^{-1}x;\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda\,|\,q]$ and
$x^{-1}(1-x)(1-qax)R_{n-1}[\lambda^{-1}x;q\lambda,q^2a\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda\,|\,q]$.
This would introduce undesired dependencies.

Non-symmetric AW polynomials as vector-valued polynomials (K & Bouzeffour)

A way around. First write the non-symmetric AW polynomials as 2-vector-valued symmetric Laurent polynomials, and then take limit to non-symmetric big q-Jacobi polynomials as 2-vector-valued ordinary polynomials.

$$f[z] = f_1[z] + az^{-1}(1-az)(1-bz)f_2[z] \;\leftrightarrow\; \vec{f}[z] = \begin{pmatrix} f_1[z] \\ f_2[z] \end{pmatrix},$$

where $f_1[z]$ and $f_2[z]$ are symmetric Laurent polynomials.

Then

$$\overrightarrow{E_n}[z] = \begin{pmatrix} R_n[z; a, b, c, d \mid q] \\ -\dfrac{\sigma(n) R_{n-1}[z; qa, qb, c, d \mid q]}{(1-qab)(1-ab)(1-ac)(1-ad)} \end{pmatrix} \quad (n \ge 0),$$

$$\overrightarrow{E_{-n}}[z] = \begin{pmatrix} R_n[z; a, b, c, d \mid q] \\ -\dfrac{\sigma(-n) R_{n-1}[z; qa, qb, c, d \mid q]}{(1-qab)(1-ab)(1-ac)(1-ad)} \end{pmatrix} \quad (n > 0),$$

where

$$\sigma(n) := q^{1-n}(1-q^n)(1-q^{n-1}cd) \qquad (n \ge 0),$$
$$\sigma(-n) := (ab)^{-1} q^{1-n}(1-q^n ab)(1-q^{n-1}abcd) \quad (n > 0),$$

and where $\sigma(n) R_{n-1} = \mathrm{const.}\,(1-q^n) R_{n-1} := 0$ for $n = 0$.

For f a Laurent polynomial and A an operator acting on Laurent polynomials:

$$Af \leftrightarrow \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \mathbf{A}\,\vec{f}\,,$$

where the A_{ij} are operators acting on symmetric Laurent polynomials. Then

$$\mathbf{T_1} = \begin{pmatrix} -ab & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},$$

where the Y_{ij} are q-difference operators acting on symmetric Laurent polynomials.

$$\mathbf{Z} = \frac{1}{ab-1} \begin{pmatrix} a+b-z-z^{-1} & -a(1-az)(1-az^{-1})(1-bz)(1-bz^{-1}) \\ a^{-1} & ab(z+z^{-1})-(a+b) \end{pmatrix}.$$

Vector-valued non-symmetric big q-Jacobi:

$$\overrightarrow{E_n}(x;a,b,c;q) := \lim_{\lambda\to 0} \overrightarrow{E_n}[\lambda^{-1}x;\lambda,qa\lambda^{-1},qc\lambda^{-1},bc^{-1}\lambda\,|\,q] \quad (n\in\mathbb{Z}).$$

Then

$$\overrightarrow{E_n}(x) = \begin{pmatrix} P_n(x;a,b,c\,|\,q) \\ -\dfrac{q^{1-n}(1-q^n)(1-q^nb)}{(1-qa)(1-q^2a)(1-qc)}\,P_{n-1}(qx,q^2a,b,qc;q) \end{pmatrix} \quad (n\ge 0),$$

$$\overrightarrow{E_{-n}}(x) = \begin{pmatrix} P_n(x;a,b,c\,|\,q) \\ -\dfrac{q^{-n}(1-q^{n+1}a)(1-q^{n+1}ab)}{a(1-qa)(1-q^2a)(1-qc)}\,P_{n-1}(qx,q^2a,b,qc;q) \end{pmatrix} \quad (n>0),$$

where $(1-q^n)P_{n-1} := 0$ for $n=0$. Also for $c\to 0$.

Can also take limit of the AW **Y** matrix operator, has the $\overrightarrow{E_{\pm n}}$ as eigenfunctions.

Duality between non-symmetric continuous dual q-Hahn and non-symmetric big q-Jacobi:

$$E_n(z_{a,q}(m)^{-1}; a, b, c \,|\, q) = \begin{pmatrix} 1 & \mu_{ab,q}(n) \end{pmatrix} \overrightarrow{E_m}(q^{-n}; q^{-1}ab, ab^{-1}, q^{-1}ac; q) \qquad (m, n \in \mathbb{Z}), \tag{3}$$

where

$$z_{a,q}(n) := aq^n \qquad (n \geq 0),$$
$$z_{a,q}(-n) := a^{-1}q^{-n} \qquad (n > 0),$$

and

$$\mu_{ab,q}(n) := abq^{-n}(1 - q^n) \qquad (n \geq 0),$$
$$\mu_{ab,q}(-n) := q^{-n}(1 - q^n ab) \qquad (n > 0).$$

In (3) on the left nonsymmetric continuous dual q-Hahn polynomials and on the right row vector times column vector of non-symmetric big q-Jacobi polynomials.

Remark

$$\overrightarrow{E_n}(x;c,ab/c,a) = \begin{pmatrix} P_n(x;a,b,c\mid q) \\ -\dfrac{q^{1-n}(1-q^n)(c-q^nab)}{c(1-qa)(1-qc)(1-q^2c)}\,P_{n-1}(qx,qa,qb,q^2c;q) \end{pmatrix} \quad (n\geq 0),$$

$$\overrightarrow{E_{-n}}(x;c,ab/c,a) = \begin{pmatrix} P_n(x;a,b,c\mid q) \\ -\dfrac{q^{-n}(1-q^{n+1}c)(1-q^{n+1}ab)}{c(1-qa)(1-qc)(1-q^2c)}\,P_{n-1}(qx,qa,qb,q^2c;q) \end{pmatrix} \quad (n>0)$$

q-shifts in parameters here are different from the ones in $\overrightarrow{E_n}(x;a,b,c;q)$.

No clear limit of corresponding degenerate DAHA for $c\to 0$.