Bispectrality and dual addition formulas

Tom Koornwinder

Korteweg-de Vries Institute, University of Amsterdam
T.H.Koornwinder@uva.nl
https://staff.fnwi.uva.nl/t.h.koornwinder/

R. P. Agarwal Memorial Lecture

2 November 2017, ICSFA-2017, Bikaner, Rajasthan, India

last modified: March 27, 2018

R. P. Agarwal (1925–2008)

Prof. Agarwal opens the workshop on *Special Functions and Differential Equations* held at the Institute of Mathematical Sciences in Chennai, January 1997

Plan of the lecture

1. Bispectrality and duality
2. Brief remarks about what cannot be covered
3. Dual addition formulas

Part 1. Bispectrality and duality

Duistermaat & Grünbaum, CMP (1986),
Differential Equations in the Spectral Parameter

For which $L_x = \sum_{j=0}^{\ell} L_j(x) \left(\frac{\partial}{\partial x}\right)^j$ are there nonzero $\phi_\lambda(x)$ with

$$L_x\big(\phi_\lambda(x)\big) = \lambda\, \phi_\lambda(x)$$

such that there is $M_\lambda = \sum_{r=0}^{m} M_r(\lambda) \left(\frac{\partial}{\partial \lambda}\right)^r$ with

$$M_\lambda\big(\phi_\lambda(x)\big) = \tau(x)\, \phi_\lambda(x) \ ?$$

They solved this for $L_x = -\left(\frac{\partial}{\partial x}\right)^2 + V(x)$:

- $V(x) = \alpha x + \beta, \quad \alpha \neq 0$ (Airy)
- $V(x) = \frac{c}{(x-a)^2} + b$ (Bessel)
- by finitely many rational Darboux transformations starting at $V(x) = 0$
- by finitely many rational Darboux transformations starting at $V(x) = -\frac{1}{4x^2}$

More generally, for linear operators L_x and M_λ:

$$L_x\big(\phi_\lambda(x)\big) = \sigma(\lambda)\,\phi_\lambda(x),$$
$$M_\lambda\big(\phi_\lambda(x)\big) = \tau(x)\,\phi_\lambda(x).$$

Bispectrality: spectra $\{\sigma(\lambda)\}$ of L_x and $\{\tau(x)\}$ of M_λ.

Take discrete $\lambda = n = 0, 1, 2, \ldots$, polynomials $\phi_\lambda(x) = p_n(x)$ of degree n, and a second order difference operator M_n such that

$$M_n\big(p_n(x)\big) := A_n\,p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x) = x\,p_n(x),$$

$p_0(x) = 1$, $p_{-1}(x) = 0$. By Favard's theorem, if $A_{n-1}C_n > 0$ then the $p_n(x)$ are orthogonal polynomials (OP's).

If, moreover, there is a linear second order differential operator L_x such that

$$L_x\big(p_n(x)\big) = \lambda_n\,p_n(x)$$

then classical OP's, a bispectral case of OP's.

Bochner's theorem: The classical OP's are essentially the following polynomials $p_n(x)$ with weight function $w(x)$ on (a,b):

name	$p_n(x)$	$w(x)$	(a,b)	constraints
Jacobi	$P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha(1+x)^\beta$	$(-1,1)$	$\alpha,\beta>-1$
Laguerre	$L_n^{(\alpha)}(x)$	$x^\alpha e^{-x}$	$90,\infty)$	$\alpha>-1$
Hermite	$H_n(x)$	e^{-x^2}	$(-\infty,\infty)$	

$$\text{orthogonality:}\quad \int_a^b p_m(x)\,p_n(x)\,w(x)\,dx = h_n\,\delta_{m,n}\,.$$

L_x can be factorized (with $\frac{w_1(x)}{w(x)} = 1-x^2$ or x or 1, respectively):

$$L_x(p_n(x)) = w(x)^{-1}\frac{d}{dx}\big(w_1(x)\,p_n'(x)\big) = \lambda_n\,p_n(x).$$

There result shift operator relations:

$$p_n'(x) = q_{n-1}(x),\qquad w(x)^{-1}\frac{d}{dx}\big(w_1(x)\,q_{n-1}(x)\big) = \lambda_n\,p_n(x).$$

The $q_n(x)$ are OP's on (a,b) with weight function $w_1(x)$.

Duality principle

Any **conceptual** formula or result for a family of classical OP's $p_n(x)$ should admit a corresponding formula or result where x and n have changed role.

Example 1. The factorization of

$$M_n\,(p_n(x)) := A_n\,p_{n+1}(x) + B_n\,p_n(x) + C_n\,p_{n-1}(x) = x\,p_n(x)$$

for any OP, with y fixed and $p_n(y) = 1$ for all n:

$$h_n\left(\frac{A_n}{h_n}\,\frac{p_{n+1}(x) - p_n(x)}{x - y} - \frac{A_{n-1}}{h_{n-1}}\,\frac{p_n(x) - p_{n-1}(x)}{x - y}\right) = p_n(x). \tag{1}$$

Then $q_n(x) := \frac{p_{n+1}(x)-p_n(x)}{x-y}$ are the so-called kernel polynomials.

Eqn. (1) is the dual of the previous formula for classical OP's:

$$L_x\,p_n(x) = w(x)^{-1}\,\frac{d}{dx}\big(w_1(x)\,p_n'(x)\big) = \lambda_n\,p_n(x), \tag{2}$$

where $q_n(x) = p_{n+1}'(x)$.

Intermezzo

Pochhammer symbol or shifted factorial:

$$(a)_n := a(a+1)\dots(a+n-1).$$

Hypergeometric series:

$${}_rF_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};z\right) := \sum_{k=0}^{\infty}\frac{(a_1)_k\dots(a_r)_k}{(b_1)_k\dots(b_s)_k\,k!}\,z^k.$$

If $a_i=-n$ for some i then the sum ends at $k=n$.

Duality principle (cntd.)

Example 2. Legendre polynomials $P_n(x)$, $\quad P_n(1)=1$,
$\int_{-1}^{1} P_m(x)\,P_n(x)\,dx = 0 \quad (m\neq n)$.

Linearization formula:

$$P_m(x)P_n(x) = \sum_{j=0}^{\min(m,n)} \frac{(\frac12)_j(\frac12)_{m-j}(\frac12)_{n-j}(m+n-j)!}{j!\,(m-j)!\,(n-j)!\,(\frac32)_{m+n-j}} \times \big(2(m+n-2j)+1\big)\,P_{m+n-2j}(x). \quad (3)$$

Product formula:

$$P_n(\cos\theta_1)P_n(\cos\theta_2) = \frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi,$$

or rewritten:

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{(1-x^2)(1-y^2)}}^{\sqrt{(1-x^2)(1-y^2)}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz. \quad (4)$$

Eqns (3) and (4) are duals. Both have positive kernels.
Both have group theoretic interpretations.

Duality principle — a warning

Warning
The duality principle should be used as a guiding principle, but not taken absolutely. There are counterexamples, in particular for Laguerre polynomials. See Askey & Gasper, J. Anal. Math. (1977).

Self-duality: $\phi_\lambda(x) = \phi_x(\lambda)$

Example 1. Bessel: $\phi_\lambda(x) := J_\alpha(\lambda x)$

Example 2. Krawtchouk:

$$p_n(x) = K_n(x; p, N) := {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}; p^{-1}\right) \quad (n = 0, 1, 2, \ldots, N)$$

orthogonality: $$\sum_{x=0}^{N} (K_m\, K_n\, w)(x; p, N) = \frac{(1-p)^N}{w(n; p, N)}\, \delta_{m,n}\,,$$

weights: $$w(x; p, N) := \binom{N}{x} p^x (1-p)^{N-x} \quad (0 < p < 1).$$

Duality between two families of OP's

Leonard (1982): Find all systems of OP's $p_n(x)$ on a finite or countably infinite set $\{x(m)\}$ for which there is a system of OP's $q_m(y)$ on a finite or countably infinite set $\{y(n)\}$ such that $p_n(x(m)) = q_m(y(n))$.

These include the **Racah polynomials**. Their dual is within the same family but with changed parameters:

$$\begin{aligned}
&R_n(m(m+\delta-N);\alpha,\beta,-N-1,\delta)\\
&:= {}_4F_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -m, m+\delta-N\\ \alpha+1, \beta+\delta+1, -N\end{matrix};1\right)\\
&= R_m(n(n+\alpha+\beta+1);-N-1,\delta,\alpha,\beta) \quad (m,n=0,1,\ldots,N).
\end{aligned}$$

Wilson polynomials depend on a continuous variable:

$$\begin{aligned}
W_n(x^2;a,b,c,d) &= \text{const.}\\
&\quad\times {}_4F_3\left(\begin{matrix}-n, n+a+b+c+d-1, a+ix, a-ix\\ a+b, a+c, a+d\end{matrix};1\right).
\end{aligned}$$

For going to dual within the same family put $x := i(m+a)$.

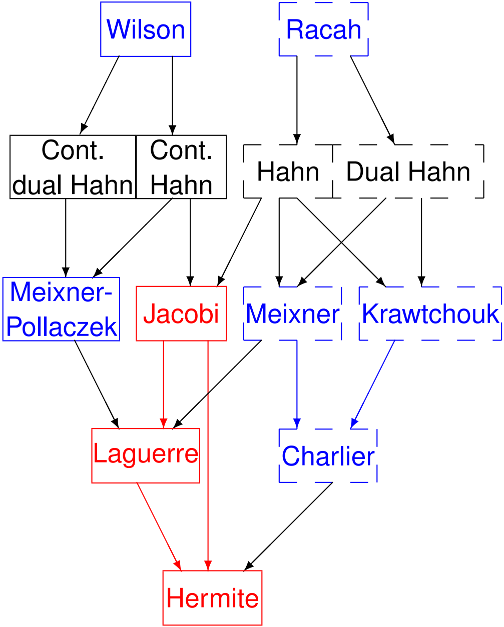
The Askey scheme classifies all OP's which are eigenfunctions of second order differential operators or second order difference operators of various types.

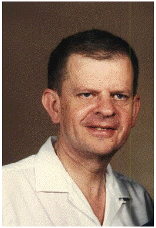
They can be expressed as terminating hypergeometric series.

Its families depend on parameters.

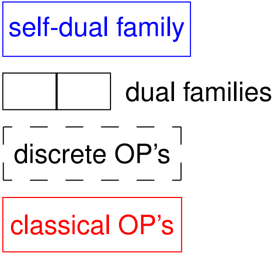
Arrows give limit transitions between the families.

Askey scheme





Dick Askey



Recall:

Duality principle

Any **conceptual** formula or result for a family of classical OP's $p_n(x)$ should admit a corresponding formula or result where x and n have changed role.

This principle is obvious for self-dual families of discrete OP's, and quite plausible for self-dual families of continuous OP's.

Limit principle

:

1. Any **conceptual** formula or result for a family of classical OP's $p_n(x)$ should fit into a **full Askey scheme** for this result.
2. The full Askey scheme for a result in item 1 should have an **extension to dual pairs** of results, where the duality on the top level of the scheme is the obvious duality.

Limit principle — a warning

Warning

Just as with the duality principle, the limit principle should be used as a guiding principle, but not taken absolutely. Certain significant results for a family somewhere in the Askey scheme may only hold under certain restrictions on the parameters, while a limit transition may leave the constrained parameter range, by which the significant result is not valid in the limit case.

Intermezzo

Assume $0 < q < 1$.

***q*-Pochhammer symbol or shifted factorial**:

$$\begin{aligned}(a;q)_n &:= (1-a)(1-aq)\ldots(1-aq^{n-1}),\\ (a_1,\ldots,a_r;q)_n &:= (a_1;1)_n\ldots(a_r;q)_n\,.\end{aligned}$$

***q*-Hypergeometric series**:

$${}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix};q,z\right) := \sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s,q;q)_k}\left((-1)^k q^{\frac{1}{2}k(k-1)}\right)^{r-s+1} z^k.$$

If $a_i = q^{-n}$ for some i then the sum ends at $k = n$.

$$\lim_{q\to 1}\frac{(q^a;q)_n}{(1-q)^n} = (a)_n.$$

Accordingly limits from q-hypergeometric series to hypergeometric series as $q \to 1$.

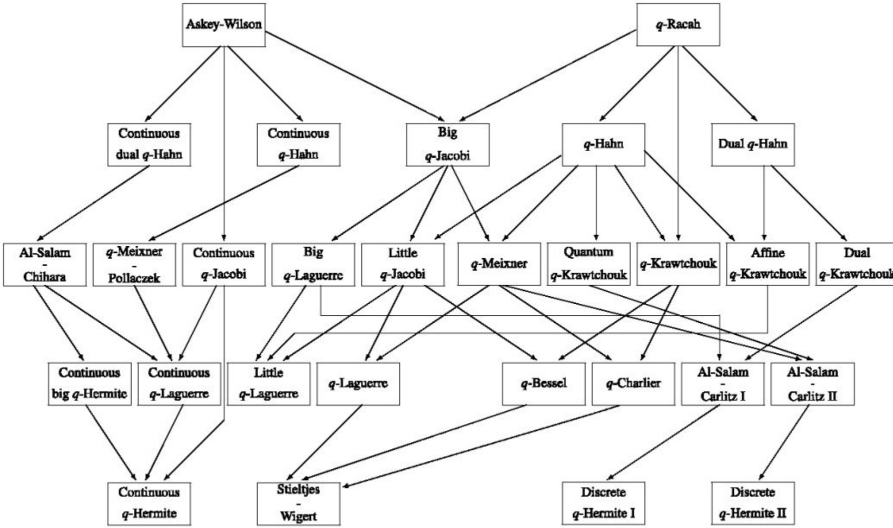
Parallel to the Askey scheme there is a q-Askey scheme with families of OP's which are eigenfunctions of second order q-difference operators of various types.

They have an expression in terms of q-hypergeometric series.

Arrows denote limits. There are also limits for $q \to 1$ from families in q-Askey scheme to families in Askey scheme.

On top of the q-Askey scheme are the **Askey-Wilson polynomials** and ***q*-Racah polynomials**.

q-Askey scheme



http://homepage.tudelft.nl/11r49/book.html

Askey-Wilson polynomials

$$R_n[z] = R_n\big[z; a,b,c,d \,|\, q\big] = \frac{p_n\big(\frac12(z+z^{-1}); a,b,c,d \,|\, q\big)}{a^{-n}(ab.ac.ad;q)_n}$$

$$:= {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, az, az^{-1} \\ ab, ac, ad \end{matrix}; q, q\right)$$

$$= \sum_{k=0}^{n} \frac{q^k}{(ab,ac,ad,q;q)_k} \, (q^{-n}, abcdq^{n-1};q)_k \, (az, az^{-1};q)_k.$$

$p_n(x; a,b,c,d \,|\, q)$ is symmetric in a,b,c,d and is orthogonal on $x \in [-1,1]$ (for suitable a,b,c,d).

Duality Define *dual parameters* $\tilde a, \tilde b, \tilde c, \tilde d$ by

$$\tilde a = (q^{-1}abcd)^{\frac12}, \quad \tilde b = ab/\tilde a, \quad \tilde c = ac/\tilde a, \quad \tilde d = ad/\tilde a.$$

Then, for $m, n = 0, 1, 2, \ldots$,

$$R_n\big[a^{-1}q^{-m}; a,b,c,d \,|\, q\big] = R_m\big[\tilde a^{-1}q^{-n}; \tilde a, \tilde b, \tilde c, \tilde d \,|\, q\big].$$

Extend the **limit principle** to the q-case.

Dual eigenvalue equations for Askey-Wilson

Askey-Wilson polynomials $R_n[z]$ are eigenfunctions of a second order q-difference operator L_z and, by the 3-term recurrence relation, of a difference operator M_n with eigenvalue $z+z^{-1}$:

$$L_z(R_n[z]) := A[z]\big(R_n[qz]-R_n[z]\big) + A[z^{-1}]\big(R_n[q^{-1}z]-R_n[z]\big)$$
$$+ (1+q^{-1}abcd)\,R_n[z] = (q^{-n}+abcdq^{n-1})\,R_n[z],$$
$$\text{with}\quad A[z] := \frac{(1-az)(1-bz)(1-cz)(1-dz)}{(1-z^2)(1-qz^2)}\,,$$

$$M_n\big(R_n[z]\big) := A_n\,R_{n+1}[z] + B_n\,R_n[z] + C_n\,R_{n-1}[z] = (z+z^{-1})\,R_n[z].$$

Duality: $$L_z\big(f[z]\big)\Big|_{z=a^{-1}q^{-m}} = \tilde a\,\tilde M_m\big(f\big[a^{-1}q^{-m}\big]\big),$$

$$L_z\Big(R_n\big[z;a,b,c,d\,|\,q\big]\Big)\Big|_{z=a^{-1}q^{-m}} = \tilde a\,\tilde M_m\Big(R_m\big[\tilde a^{-1}q^{-n};\tilde a,\tilde b,\tilde c,\tilde d\,|\,q\big]\Big),$$

where $\tilde M_m$ is the difference operator M_m with dual parameters.

Zhedanov algebra

L_z and $z+z^{-1}$ act on the space $\mathcal{P}$ of symmetric Laurent polynomials in z. Call them K_0 and K_1. They generate an abstract algebra with relations

$$(q+q^{-1})K_1K_0K_1 - K_1^2K_0 - K_0K_1^2 = B\,K_1 + C_0\,K_0 + D_0,$$
$$(q+q^{-1})K_0K_1K_0 - K_0^2K_1 - K_1K_0^2 = B\,K_0 + C_1\,K_1 + D_1,$$

called the Zhedanov algebra. Here

$$\begin{aligned}
B &:= (1-q^{-1})^2(e_3+qe_1),\\
C_0 &:= (q-q^{-1})^2,\\
C_1 &:= q^{-1}(q-q^{-1})^2 e_4,\\
D_0 &:= -q^{-3}(1-q)^2(1+q)(e_4+qe_2+q^2),\\
D_1 &:= -q^{-3}(1-q)^2(1+q)(e_1e_4+qe_3).
\end{aligned}$$

with e_1, e_2, e_3, e_4 the elementary symmetric polynomials in a, b, c, d. A special value of an explicit Casimir operator Q makes the representation on $\mathcal{P}$ faithful.

Part 2. Brief remarks about what cannot be covered

Duality for Zhedanov algebra
There is an anti-algebra isomorphism of Zhedanov algebra which sends generators K_0, K_1 and parameters a, b, c, d to aK_1, $\tilde{a}^{-1}K_0$, $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, respectively.

Askey scheme of Zhedanov algebras
There is a Zhedanov type algebra for each family in the (q-)Askey scheme. The limit arrows in these schemes also extend to these algebras.

Double affine Hecke algebras (DAHA's)
Just as the functions $\cos(\lambda x)$ are symmetrized versions of the functions $e^{i\lambda x}$, the OP's in the (q-)Askey scheme are symmetrized versions of certain non-symmetric orthogonal (Laurent) polynomials. The corresponding algebra is called **DAHA**, started by Cherednik, in which the Zhedanov algebra is embedded. There should be a (q-)Askey scheme of DAHA's.

Macdonald polynomials

Macdonald (1987) introduced q-orthogonal polynomials in n variables associated with **root systems**. They are invariant under the **Weyl group**, for instance the symmetric group. For $n = 1$ subclasses of the Askey-Wilson polynomials.

Koornwinder polynomials

K (1992) introduced a family of orthogonal symmetric Laurent polynomials in n variables associated with root system BC_n. They depend on the four Askey-Wilson parameters a, b, c, d, and on a coupling parameter t, and on q. For $n = 1$ they become the full class of Askey-Wilson polynomials.

Okounkov (1998) generalizes the Askey-Wilson expression

$$\sum_{k=0}^{n} \frac{q^k}{(ab, ac, ad, q; q)_k} \, (q^{-n}, abcdq^{n-1}; q)_k \, (az, az^{-1}; q)_k.$$

to the Koornwinder case. This involves an n-variable generalization of $(az, az^{-1}; q)_k$ as an **interpolation polynomial**. His formula implies duality.

Part 3. Dual addition formulas

Product formulas for Legendre polynomials $P_n(x)$

$\int_{-1}^1 P_m(x)\,P_n(x)\,dx = 0 \quad (m \neq n), \qquad P_n(1) = 1.$

Linearization formula:

$$P_m(x)P_n(x) = \sum_{j=0}^{\min(m,n)} \frac{(\frac{1}{2})_j(\frac{1}{2})_{m-j}(\frac{1}{2})_{n-j}(m+n-j)!}{j!\,(m-j)!\,(n-j)!\,(\frac{3}{2})_{m+n-j}} \times \big(2(m+n-2j)+1\big)\,P_{m+n-2j}(x).$$

Product formula:

$$P_n(\cos\theta_1)P_n(\cos\theta_2) = \frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\phi)\,d\phi, \tag{5}$$

or rewritten:

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{(1-x^2)(1-y^2)}}^{\sqrt{(1-x^2)(1-y^2)}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

Eqn. (5) is the constant term of a Fourier-cosine expansion of the integrand in terms of $\cos(k\phi)$. This expansion is called the *addition formula*.

Addition formula for Legendre polynomials

Addition formula:

$$
\begin{aligned}
&P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\\
&\quad = P_n(\cos\theta_1)P_n(\cos\theta_2)+2\sum_{k=1}^{n}\frac{(n-k)!\,(n+k)!}{2^{2k}(n!)^2}\\
&\qquad\times(\sin\theta_1)^k P_{n-k}^{(k,k)}(\cos\theta_1)\,(\sin\theta_2)^k P_{n-k}^{(k,k)}(\cos\theta_2)\cos(k\phi).
\end{aligned}
$$

Product formula:

$$P_n(\cos\theta_1)P_n(\cos\theta_2)=\frac{1}{\pi}\int_0^\pi P_n(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\cos\phi)\,d\phi.$$

Askey's question

Find an addition type formula corresponding to the linearization formula for Legendre polynomials just as the addition formula corresponds to the product formula.

A possible key for an answer

Chebyshev polynomials $T_n(\cos\phi) := \cos(n\phi)$. The $T_n(x)$ are OP's on [-1,1] with respect to the weight function $(1-x^2)^{-1/2}$.
The rewritten product formula

$$P_n(x)P_n(y) = \frac{1}{\pi}\int_{-\sqrt{1-x^2}\,\sqrt{1-y^2}}^{\sqrt{1-x^2}\,\sqrt{1-y^2}} \frac{P_n(z+xy)}{\sqrt{(1-x^2)(1-y^2)-z^2}}\,dz.$$

is the constant term of the Chebyshev expansion of $P_n(z+xy)$ in terms of $T_k\big(z(1-x^2)^{-1/2}(1-y^2)^{-1/2}\big)$, OP's with respect to the weight function $\big((1-x^2)(1-y^2)-z^2\big)^{-1/2}$ on the integration interval. This expansion is a rewriting of the addition formula. Can we recognize weights of discrete OP's in the coefficients of the linearization formula given below?

$$P_l(x)P_m(x) = \sum_{j=0}^{\min(l,m)} \frac{(\frac12)_j(\frac12)_{l-j}(\frac12)_{m-j}(l+m-j)!}{j!\,(l-j)!\,(m-j)!\,(\frac32)_{l+m-j}} \times \big(2(l+m-2j)+1\big)\,P_{l+m-2j}(x).$$

The answer

Yes !

The coefficients are weights of special **Racah polynomials.**
This works even with Rogers' (1896) linearization formula for **ultraspherical (Gegenbauer) polynomials**

$$R_n^{(\alpha,\alpha)}(x) := \frac{P_n^{(\alpha,\alpha)}(x)}{P_n^{(\alpha,\alpha)}(1)},$$

$$\int_{-1}^{1} R_m^{(\alpha,\alpha)}(x)\, R_n^{(\alpha,\alpha)}(x)\, (1-x^2)^\alpha\, dx = 0 \quad (m \neq n),$$

$$R_n^{(\alpha,\alpha)}(1) = 1.$$

Linearization formula:

$$R_l^{(\alpha,\alpha)}(x) R_m^{(\alpha,\alpha)}(x) = \frac{l!\, m!}{(2\alpha+1)_l (2\alpha+1)_m} \sum_{j=0}^{\min(l,m)} \frac{l+m+\alpha+\frac{1}{2}-2j}{\alpha+\frac{1}{2}}$$

$$\times \frac{(\alpha+\frac{1}{2})_j (\alpha+\frac{1}{2})_{l-j} (\alpha+\frac{1}{2})_{m-j} (2\alpha+1)_{l+m-j}}{j!\,(l-j)!\,(m-j)!\,(\alpha+\frac{3}{2})_{l+m-j}}\, R_{l+m-2j}^{(\alpha,\alpha)}(x).$$

Racah polynomials

$$
\begin{aligned}
&R_n\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\\
&\quad:= {}_4F_3\left(\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix};1\right),\quad \gamma=-N-1,\\
&\sum_{x=0}^{N}(R_mR_n)\big(x(x+\gamma+\delta+1);\alpha,\beta,\gamma,\delta\big)\,w_{\alpha,\beta,\gamma,\delta}(x)=h_{n;\alpha,\beta,\gamma,\delta}\delta_{m,n}\,,
\end{aligned}
$$

$$
\begin{aligned}
w_{\alpha,\beta,\gamma,\delta}(x)&=\frac{\gamma+\delta+1+2x}{\gamma+\delta+1}\\
&\qquad\times\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x(\delta+1)_x\,x!}\,,
\end{aligned}
$$

$$
\frac{h_{n;\,\alpha,\beta,\gamma,\delta}}{h_{0;\,\alpha,\beta,\gamma,\delta}}=\frac{\alpha+\beta+1}{\alpha+\beta+2n+1}\,\frac{(\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n\,n!}{(\alpha+1)_n(\alpha+\beta+1)_n(\beta+\delta+1)_n(\gamma+1)_n},
$$

$$
h_{0;\,\alpha,\beta,\gamma,\delta}=\sum_{x=0}^{N}w_{\alpha,\beta,\gamma,\delta}(x)=\frac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N}\,.
$$

Rewritten linearization formula:

$$R_l^{(\alpha,\alpha)}(x)R_m^{(\alpha,\alpha)}(x)=\sum_{j=0}^{m}\frac{w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)}{h_{0;\,\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}}\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$

$(l\ge m,\ \alpha>-\frac12)$. More generally evaluate

$$S_n^\alpha(l,m):=\sum_{j=0}^{m}w_{\alpha-\frac12,\alpha-\frac12,-m-1,-l-\alpha-\frac12}(j)\,R_{l+m-2j}^{(\alpha,\alpha)}(x)$$
$$\times R_n\big(j(j-l-m-\alpha-\tfrac12);\,\alpha-\tfrac12,\alpha-\tfrac12,-m-1,-l-\alpha-\tfrac12\big).$$

By Racah Rodrigues formula, summation by parts, and a difference formula for ultraspherical polynomials we get

$$S_n^\alpha(l,m)=\frac{(2\alpha+1)_{l+n}(2\alpha+1)_{m+n}(\alpha+\frac12)_{l+m}}{2^{2n}(\alpha+\frac12)_l(\alpha+\frac12)_m(2\alpha+1)_{l+m}(\alpha+1)_n^2}\,(x^2-1)^n$$
$$\times R_{l-n}^{(\alpha+n,\alpha+n)}(x)\,R_{m-n}^{(\alpha+n,\alpha+n)}(x).$$

Then Fourier-Racah inversion gives:

Dual addition formula for ultraspherical polynomials

(K, 2016)

$$R^{(\alpha,\alpha)}_{l+m-2j}(x) = \sum_{n=0}^{m} \frac{\alpha+n}{\alpha+\frac{1}{2}n} \frac{(-l)_n(-m)_n(2\alpha+1)_n}{(\alpha+1)_n^2\, n!}$$
$$\times \left(\tfrac{1}{4}(x^2-1)\right)^n R^{(\alpha+n,\alpha+n)}_{l-n}(x)\, R^{(\alpha+n,\alpha+n)}_{m-n}(x)$$
$$\times R_n\big(j(j-l-m-\alpha-\tfrac{1}{2});\, \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, -m-1, -l-\alpha-\tfrac{1}{2}\big)$$
$$(l \geq m,\ j = 0, 1, \ldots, m).$$

Compare with addition formula for ultraspherical polynomials:

$$R^{(\alpha,\alpha)}_n(xy+z) = \sum_{k=0}^{n} \binom{n}{k} \frac{\alpha+k}{\alpha+\frac{1}{2}k} \frac{(n+2\alpha+1)_k(2\alpha+1)_k}{2^{2k}(\alpha+1)_k^2} (1-x^2)^{\frac{1}{2}k}$$
$$\times R^{(\alpha+k,\alpha+k)}_{n-k}(x)(1-y^2)^{\frac{1}{2}k} R^{(\alpha+k,\alpha+k)}_{n-k}(y) R^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}_k\left(\frac{z}{\sqrt{1-x^2}\,\sqrt{1-y^2}}\right)$$

(first obtained by M. Allé in 1865, before Gegenbauer).

Renormalized q-**ultraspherical polynomials** $(0<\beta<q^{-1})$:

$$R_n^{\beta,q}(\tfrac12(z+z^{-1}))= {}_4\phi_3\left(\begin{matrix} q^{-n},\beta^2q^{n+1},q^{\frac14}\beta^{\frac12}z,q^{\frac14}\beta^{\frac12}z^{-1}\\ \beta q,-\beta q^{\frac12},-\beta q\end{matrix};q,q\right).$$

q-**Racah polynomials** $(\gamma=q^{-N-1}$, $n=0,1,\ldots,N)$:

$$R_n(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q):= {}_4\phi_3\left(\begin{matrix} q^{-n},q^{n+1}\alpha\beta,q^{-x},q^{x+1}\gamma\delta\\ q\alpha,q\beta\delta,q\gamma\end{matrix};q,q\right)$$

Orthogonality relation for q-Racah polynomials:

$$\sum_{x=0}^{N}(R_mR_n)(q^{-x}+\gamma\delta q^{x+1};\alpha,\beta,\gamma,\delta\,|\,q)\,w_{\alpha,\beta,\gamma,\delta;q}(x)=h_{n;\alpha,\beta,\gamma,\delta;q}\,\delta_{m,n}\,,$$

$$w_{\alpha,\beta,\gamma,\delta;q}(x)=\frac{1-\gamma\delta q^{2x+1}}{(\alpha\beta q)^x(1-\gamma\delta q)}\,\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_x}{(q,\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_x}\,.$$

Dual addition formula for q-ultraspherical polynomials

(K, 1 November 2017) ($0 < \beta < q^{-\frac{1}{2}},\ l \geq m,\ j = 0, 1, \ldots, m$):

$$R^{\beta,q}_{l+m-2j}(x) = \sum_{n=0}^{m} q^{\frac{1}{2}n(n+\ell+m+2)}\beta^n \frac{1-\beta^2 q^{2n}}{1-\beta^2 q^n} \frac{(q^{-\ell}, q^{-m}, q\beta^2; q)_n}{(q\beta, q\beta, q; q)_n}$$

$$\times \frac{\prod_{j=0}^{n-1}\left(4q^{j+\frac{1}{2}}\beta x^2 - (1+q^{j+\frac{1}{2}}\beta)^2\right)}{(-q^{\frac{1}{2}}\beta; q^{\frac{1}{2}})^2_{2n}} R^{q^n\beta,q}_{\ell-n}(x) R^{q^n\beta,q}_{m-n}(x)$$

$$\times R_n(q^{-j} + \beta^{-1}q^{j-\ell-m-\frac{1}{2}}; \beta q^{-\frac{1}{2}}, \beta q^{-\frac{1}{2}}, q^{-m-1}, \beta^{-1}q^{-\ell-\frac{1}{2}} \mid q).$$

For $\beta := q^\alpha$ and $q \to 1$ this tends to the dual addition formula for ultraspherical polynomials:

$$R^{(\alpha,\alpha)}_{l+m-2j}(x) = \sum_{n=0}^{m} \frac{\alpha+n}{\alpha+\frac{1}{2}n} \frac{(-l)_n(-m)_n(2\alpha+1)_n}{(\alpha+1)^2_n\, n!}$$

$$\times \left(\tfrac{1}{4}(x^2-1)\right)^n R^{(\alpha+n,\alpha+n)}_{l-n}(x)\, R^{(\alpha+n,\alpha+n)}_{m-n}(x)$$

$$\times R_n\left(j(j-l-m-\alpha-\tfrac{1}{2});\ \alpha-\tfrac{1}{2}, \alpha-\tfrac{1}{2}, -m-1, -l-\alpha-\tfrac{1}{2}\right).$$

धन्यवाद

Thank you