Jacobi polynomials associated with root systems via trigonometric Dunkl operators

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Root systems

V d-dimensional real vector space with inner product $\langle\,.\,,\,.\,\rangle$.

For $0 \neq \alpha \in V$: $\quad \check\alpha := \frac{2}{\langle\alpha,\alpha\rangle}\,\alpha, \quad s_\alpha(\beta) := \beta - \langle\beta,\check\alpha\rangle\alpha \quad (\beta \in V)$.

Root system R: finite subset of $V\backslash\{0\}$ such that

$$\forall\alpha,\beta \in R \quad s_\alpha(\beta) \in R \quad \text{and} \quad \langle\beta,\check\alpha\rangle \in \mathbb{Z}.$$

Weyl group W: group generated by the s_α $(\alpha \in R)$.

Weight lattice P : $\{\lambda \in V \mid \forall\alpha \in R \quad \langle\lambda,\check\alpha\rangle \in \mathbb{Z}\}$.

R_+ choice of positive roots; $\alpha_1,\ldots,\alpha_d$ simple roots; $s_i := s_{\alpha_i}$.

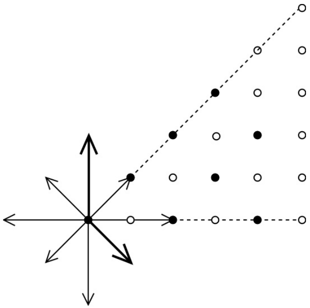
Dominant weights: $P_+ := \{\lambda \in P \mid \forall\alpha \in R_+ \quad \langle\lambda,\check\alpha\rangle \geq 0\}$.

$\lambda_1,\ldots,\lambda_d$ fundamental weights, $\langle\lambda_i,\alpha_j\rangle = \delta_{i,j}$.

Dominance partial ordering on P:

$$\lambda \geq \mu \quad \text{iff} \quad \lambda - \mu \text{ is a sum of positive roots.}$$

Example: root system C_2



simple vectors $(1, -1)$ and $(0, 2)$;

$P_+ = \{(i, j) \in \mathbb{Z}^2 \mid i \geq j \geq 0\}$;

fundamental weights $(1, 1)$ and $(1, 0)$;

the black dots are the $\mu \in P_+$ for which $\mu \leq \lambda := (4, 2)$.

Trigonometric polynomials

For $\lambda \in P$: $\quad e^\lambda(x) := e^{i\langle\lambda,x\rangle} \quad (x \in V)$.

With $(w\,f)(x) := f(w^{-1}x)$: $\quad we^\lambda = e^{w\lambda} \quad (w \in W)$.

For $\lambda \in P_+$: $\quad m_\lambda := \sum_{\mu\in W\lambda} e^\mu$, $\quad W$-invariant, $\quad m_i := m_{\lambda_i}$.

Algebra of trigonometric polynomials: $\quad \mathcal{A} := \mathrm{Span}\{e^\lambda \mid \lambda \in P\}$.

$\{m_\lambda \mid \lambda \in P_+\}$ is a basis of the space $\mathcal{A}^W$ of W-invariants in $\mathcal{A}$.

The m_i ($i = 1, \ldots, d$) generate $\mathcal{A}^W$ as an algebra.

Dual root system: $\quad R^\vee := \{\check\alpha \mid \alpha \in R\}$.

Dual root lattice $Q^\vee$ is the $\mathbb{Z}$-span of $R^\vee$.

Torus $T := V/(2\pi Q^\vee)$; $\quad x \mapsto \dot x\colon V \to T$.

If $f \in \mathcal{A}$ then $f(x) = \tilde f(\dot x)$ for suitable $\tilde f$.

$d\dot x$ Lebesgue measure on T such that $\int_T d\dot x = 1$.

Jacobi polynomials associated with R

Multiplicity function: $k \colon \alpha \mapsto k_\alpha \colon R \to [0,\infty)$

such that $k_{w\alpha} = k_\alpha$ for $w \in W$.

Weighted half sum of positive roots: $\rho_k := \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \alpha$.

Weight function:

$$\delta_k(x) := \prod_{\alpha\in R_+} \left|2\sin\left(\tfrac{1}{2}\langle\alpha,x\rangle\right)\right|^{2k_\alpha} = \prod_{\alpha\in R}\left(1-e^\alpha(x)\right)^{k_\alpha},$$

is W-invariant function on $T = V/(2\pi Q^\vee)$, independent of R_+ .

Inner product on $\mathcal{A}$: $\langle f,g\rangle_k := \int_T f(x)\,\overline{g(x)}\,\delta_k(x)\,d\dot{x}$.

k-Laplacian Δ_k on T:

$$(\Delta_k f)(\dot{x}) := \Big(\Delta + \textstyle\sum_{\alpha\in R_+} k_\alpha \cot\left(\tfrac{1}{2}\langle\alpha,x\rangle\right)\partial_\alpha\Big) f(x),$$

where $(\partial_\alpha f)(x) := \frac{d}{dt} f(x+t\alpha)\big|_{t=0}$.

$\langle \Delta_k f, g\rangle_k = \langle f, \Delta_k g\rangle_k \ \ (f,g\in\mathcal{A})$; $\ \Delta_k \circ w = w\circ \Delta_k \ \ (w\in W)$.

$\big(\Delta_k + \langle\lambda,\lambda+2\rho_k\rangle\big) m_\lambda \in \operatorname{Span}\{m_\mu \mid \mu\in P_+,\ \mu<\lambda\}$.

Jacobi polynomials associated with R (cntd)

Definition (*Jacobi polynomial associated with R*)

This is an element $P_\lambda^{(k)}$ $(\lambda \in \mathcal{P}_+)$ of $\mathcal{A}^W$ of the form

$$P_\lambda^{(k)} = \sum_{\mu \in P_+,\ \mu \le \lambda} c_{\lambda,\mu} m_\mu$$

such that $c_{\lambda,\lambda} = 1$ and

(i) $\quad \langle P_\lambda^{(k)}, m_\mu \rangle_k = 0 \quad$ if $\mu \in P_+$ and $\mu < \lambda$.

Instead of (i) we can equivalently require that

(i)′ $\quad \Delta_k P_\lambda^{(k)} = -\langle \lambda, \lambda + 2\rho_k \rangle\, P_\lambda^{(k)}$.

Clearly: $\quad \langle P_\lambda^{(k)}, P_\mu^{(k)} \rangle_k = 0 \quad$ if $\lambda < \mu$ or $\mu < \lambda \quad (\lambda, \mu \in P_+)$.

Much deeper:

Theorem (Heckman, 1987)

$\langle P_\lambda^{(k)}, P_\mu^{(k)} \rangle_k = 0 \quad$ *if* $\lambda \neq \mu \quad (\lambda, \mu \in P_+)$.

Jacobi polynomials associated with R (cntd)

$\mathcal{R}$: function algebra on V generated by $(1-e^\alpha)^{-1}$ $(\alpha \in R_+)$.

$\mathbb{D}_k$: the algebra of W-invariant differential operators D on V with coefficients in $\mathcal{R}$ such that D commutes with Δ_k.

Then: $\mathbb{D}_k$ acts on $\mathcal{A}^W$;

$\mathbb{D}_k$ is closed under adjointness with respect to $\langle\,.\,,\,.\,\rangle_k$;

$$DP_\lambda^{(k)} = -\gamma_k(D)(\lambda+\rho_k)\,P_\lambda^{(k)} \quad (D \in \mathbb{D}_k,\ \lambda \in P_+)\,.$$

Then $\gamma_k \colon \mathbb{D}_k \to \mathbb{C}^W[V]$ is an injective algebra homomorphism.

Hence $\mathbb{D}_k$ is a commutative algebra.

In particular, $\gamma_k(\Delta_k)(\mu) = \langle \mu, \mu\rangle - \langle \rho_k, \rho_k\rangle$.

Theorem (Opdam, 1988)

The map $\gamma_k \colon \mathbb{D}_k \to \mathbb{C}[V]^W$ is surjective. Hence, the algebra $\mathbb{D}_k$ has d algebraically independent generators including Δ_k.

Then the $P_\lambda^{(k)}$ are separated by eigenvalues of $D \in \mathbb{D}_k$. Full orthogonality will be a consequence.

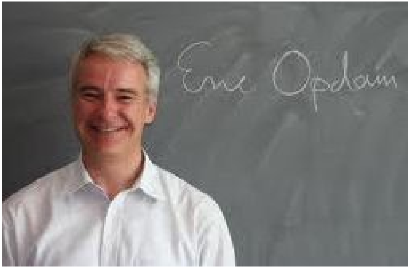
Jacobi polynomials associated with R (cntd)

Proofs of general orthogonality:

- For special parameter values: spherical functions on compact symmetric spaces. Then orthogonality by Schur.
- Generators of $\mathbb{D}_k$ explicit for BC_2 and A_2 (K, 1974).
- Generators of $\mathbb{D}_k$ explicit for A_n (Macdonald, 1987; anticipated by Sekiguchi, 1976 and Debiard, 1983).
- Generators of $\mathbb{D}_k$ explicit for BC_n (Debiard, 1987).
- Heckman (1987) for general R. He used work by Deligne (1970), Kashiwara & Oshima (1977), van der Lek (1983) on regular singularities.
- Macdonald (1987): the $q \uparrow 1$ limit of a a similar result in the q-case.
- Heckman (1991) for general R, much more elementary than in 1987, by trigonometric Dunkl operators.
- Cherednik (1991) and Opdam (1995) by using graded Hecke algebras.

Pictures

Heckman



Opdam

Dunkl

Cherednik

Dunkl type operators

For $\xi \in V$ and

- $\phi_{\rm D}(t) := t^{-1}$ (Dunkl, 1989), $\phi_{\rm D}(-t) = -\phi_{\rm D}(t)$;
- $\phi_{\rm H}(t) := \frac{1+e^{-t}}{2(1-e^{-t})}$ (Heckman, 1991), $\phi_{\rm H}(-t) = -\phi_{\rm H}(t)$;
- $\phi_{\rm C}(t) := \frac{1}{1-e^{-t}}$ (Cherednik, 1991), $\phi_{\rm C}(-t) = 1 - \phi_{\rm C}(t)$,

the Dunkl type operators

$$\mathcal{D}^{(k)}_{\xi,[\rm DHC]} := -i\,\partial_\xi + \sum_{\alpha\in R_+} k_\alpha \langle \xi,\alpha\rangle\, \phi_{[\rm DHC]}\big(i\,\langle\,.\,,\alpha\rangle\big)\,({\rm id}-s_\alpha)$$

act on $C^\infty(V)$. In particular, $\mathcal{D}^{(k)}_{\rm D}$ acts on $\mathbb{C}[V]$, $\mathcal{D}^{(k)}_{[\rm HC]}$ act on $\mathcal{A}$.

$$\mathcal{D}^{(k)}_{\xi,[\rm DH]} := -i\,\partial_\xi + \frac{1}{2}\sum_{\alpha\in R} k_\alpha \langle \xi,\alpha\rangle\, \phi_{[\rm DH]}\big(i\,\langle\,.\,,\alpha\rangle\big)\,({\rm id}-s_\alpha),$$

$$\mathcal{D}^{(k)}_{\xi,{\rm C}} := -i\,\partial_\xi + \frac{1}{2}\sum_{\alpha\in R} k_\alpha \langle \xi,\alpha\rangle\, \phi_{\rm C}\big(i\,\langle\,.\,,\alpha\rangle\big)\,({\rm id}-s_\alpha) + \frac{1}{2}\sum_{\alpha\in R_+} k_\alpha \langle \xi,\alpha\rangle\,({\rm id}-s_\alpha).$$

$$w\circ\mathcal{D}^{(k)}_{\xi,[\rm DH]}\circ w^{-1} = \mathcal{D}^{(k)}_{w\xi,[\rm DH]},\quad w\circ\mathcal{D}^{(k)}_{\xi,{\rm C}}\circ w^{-1} \neq \mathcal{D}^{(k)}_{w\xi,{\rm C}}\quad (w\in W).$$

Dunkl type operators: commutation

$$\mathcal{D}_\xi^{(k)} = -i\,\partial_\xi + \sum_{\alpha\in R_+} k_\alpha\,\langle\xi,\alpha\rangle\,\phi\big(i\,\langle\,.\,,\alpha\rangle\big)\,(\mathrm{id}-s_\alpha).$$

$$[\mathcal{D}_\xi^{(k)},\mathcal{D}_\eta^{(k)}] = \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta\,\langle\xi,\alpha\rangle\,\langle\eta,\beta\rangle \times \big[\phi\big(i\,\langle\,.\,,\alpha\rangle\big)\,(\mathrm{id}-s_\alpha),\phi\big(i\,\langle\,.\,,\beta\rangle\big)\,(\mathrm{id}-s_\beta)\big]$$

$$= \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta\,\big(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\big) \times \phi\big(i\,\langle\,.\,,\alpha\rangle\big)\,(\mathrm{id}-s_\alpha)\circ\phi\big(i\,\langle\,.\,,\beta\rangle\big)\,(\mathrm{id}-s_\beta)$$

$$= \sum_{\alpha,\beta\in R_+} k_\alpha k_\beta\,\big(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\big) \times \phi\big(i\,\langle\,.\,,\alpha\rangle\big)\,\phi\big(i\,\langle\,.\,,s_\alpha(\beta)\rangle\big)\,s_\alpha\,s_\beta$$

$$\overset{\text{[DH]}}{=} \frac14\sum_{w\in W}\ \sum_{\alpha,\beta\in R;\,s_\alpha s_\beta=w} k_\alpha k_\beta\,\big(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\big) \times\phi\big(i\,\langle\,.\,,\alpha\rangle\big)\,\phi\big(i\,\langle\,.\,,s_\alpha(\beta)\rangle\big)\,w = \frac14 \sum_{\text{simple rotations } w\in W} f_{w;\xi,\eta}(\,.\,)\,w.$$

Dunkl type operators: commutation (cntd.)

2-d reduction: R' root system in $V' = \mathbb{R}^2$ with Weyl group W'; w rotation in W'; $\phi_{[\mathrm{DH}]}(t) = t^{-1}$ or $\frac{1+e^{-t}}{2(1-e^{-t})}$; $x \in V'$.

$$f^{(k)}_{w;\xi,\eta}(x) = \sum_{\alpha,\beta\in R';\, s_\alpha s_\beta = w} k_\alpha k_\beta \left(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\right) \times \phi\big(i\,\langle x,\alpha\rangle\big)\,\phi\big(i\,\langle x,\beta\rangle\big).$$

Then $f^{(k)}_{w;\xi,\eta}$ W'-invariant. Hence $f^{(k)}_{w;\xi,\eta} = 0$ in the Dunkl case and

$$f^{(k)}_{w;\xi,\eta}(x) = \frac{1}{4} \sum_{\alpha,\beta\in R';\, s_\alpha s_\beta = w} k_\alpha k_\beta \left(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\right)$$

in the Heckman case.

Theorem

$[\mathcal{D}^{(k)}_{\xi,[\mathrm{DC}]}, \mathcal{D}^{(k)}_{\eta,[\mathrm{DC}]}] = 0,$

$[\mathcal{D}^{(k)}_{\xi,\mathrm{H}}, \mathcal{D}^{(k)}_{\eta,\mathrm{H}}] = -\frac{1}{4}\sum_{\alpha,\beta\in R_+} k_\alpha k_\beta \left(\langle\xi,\alpha\rangle\langle\eta,\beta\rangle - \langle\eta,\alpha\rangle\langle\xi,\beta\rangle\right) s_\alpha s_\beta.$

Self-adjointness

Proposition

The operators $\mathcal{D}^{(k)}_{\xi,[\mathrm{HC}]}$ *acting on* $\mathcal{A}$ *are self-adjoint with respect to* $\langle\,.\,,\,.\,\rangle_k$.

Heckman's approach

$$\mathcal{D}^{(k)}_{\xi,[\mathrm{H}]} = -i\,\partial_\xi + \sum_{\alpha\in R_+} k_\alpha \langle \xi,\alpha\rangle \, \frac{1+e^{-\alpha}}{2(1-e^{-\alpha})}\,(\mathrm{id} - s_\alpha).$$

For $\lambda \in P$ let $C(\lambda)$ be the convex hull of $W\lambda$ in P.
If $\lambda \in P_+$ then $\mu \le \lambda$ for all $\mu \in C(\lambda)$.

Proposition

Let $\lambda \in P_+$. Then $\mathcal{D}^{(k)}_{\xi,H}\, e^\lambda = \sum_{\mu\in C(\lambda)} a_{\lambda,\mu} e^\mu.$

For $\lambda \neq \mu \in W\lambda$ the coefficient $a_{\lambda,\mu}$ has degree 1 in λ if $\mu = \lambda$ and degree 0 in λ otherwise.

In particular, $a_{\lambda,\lambda} = \langle \lambda + \rho_k, \xi\rangle$ *if* $\lambda \in P_+$.

Indeed, $$\frac{1+e^{-\alpha}}{1-e^{-\alpha}}\,(1-s_\alpha)\,e^\lambda = (1+e^{-\alpha})\,\frac{1-e^{-\langle\lambda,\check\alpha\rangle\alpha}}{1-e^{-\alpha}}\,e^\lambda$$
$$= \mathrm{sgn}(\langle\lambda,\check\alpha\rangle)\Big(e^\lambda + e^{s_\alpha(\lambda)} + 2\sum_{j=1}^{|\langle\lambda,\check\alpha\rangle|-1} e^{\lambda - j\alpha}\Big).$$

Heckman's approach (cntd.)

Let $\mathcal{D}^{(k)}_{\xi,r} := \sum_{\eta\in W\xi} \big(\mathcal{D}^{(k)}_{\eta,[\mathrm{H}]}\big)^r \quad (r = 2, 3, \ldots)$. This operator is W-invariant because $w \circ \mathcal{D}^{(k)}_{\xi,[\mathrm{H}]} \circ w^{-1} = \mathcal{D}^{(k)}_{w\xi,[\mathrm{H}]}$.

If $\lambda \in P_+$ then $\quad \mathcal{D}^{(k)}_{\xi,r} e^\lambda = \sum_{\mu\in C(\lambda)} a_{\lambda,\mu} e^\mu$

For $\lambda \neq \mu \in W\lambda$ the coefficient $a_{\lambda,\mu}$ has degree $< r$ in λ if $\mu \neq \lambda$, while

$$a_{\lambda,\lambda} = \sum_{\eta\in W\xi} (\langle \lambda + \rho_k, \eta\rangle)^r + \text{terms of degee} < r \text{ in } \lambda. \qquad (*)$$

By W-invariance of $\mathcal{D}^{(k)}_{\xi,r}$, we have for $\lambda \in P_+$ and for all $w \in W$:

$\mathcal{D}^{(k)}_{\xi,r} e^{w\lambda} = \sum_{\mu\in C(\lambda)} a_{\lambda,\mu} e^{w\mu}$ with $a_{\lambda,\mu}$ independent of w.

Hence $\mathcal{D}^{(k)}_{\xi,r} m_\lambda = \sum_{\mu\leq\lambda,\, \mu\in P_+} a_{\lambda,\mu} m_\mu \;\; (\lambda \in P_+)$ with $a_{\lambda,\lambda}$ as in $(*)$.

Heckman's approach (cntd.)

$$\mathcal{D}_{\xi,r}^{(k)} m_\lambda = \sum_{\mu \le \lambda,\, \mu \in P_+} a_{\lambda,\mu} m_\mu \quad (\lambda \in P_+).$$

$$a_{\lambda,\lambda} = \sum_{\eta \in W\xi} (\langle \lambda + \rho_k, \eta \rangle)^r + \text{terms of degee} < r \text{ in } \lambda. \qquad (*)$$

Furthermore, the operator $\mathcal{D}_{\xi,r}^{(k)}$ is self-adjoint on $\mathcal{A}^W$ with respect to $\langle\, .\,,\,.\,\rangle$.

Hence $\mathcal{D}_{\xi,r}^{(k)} P_\lambda^{(k)} = a_{\lambda,\lambda} P_\lambda^{(k)}$ with $a_{\lambda,\lambda}$ as in $(*)$.

Hence the operators $\mathcal{D}_{\xi,r}^{(k)}$ $(r = 2, 3, \ldots; \xi \in V)$ acting on $\mathcal{A}^W$ mutually commute.

Consider the commutative algebra $\widetilde{\mathbb{D}}_k$ generated by the operators $\mathcal{D}_{\xi,r}^{(k)}\big|_{\mathcal{A}^W}$ $(r = 2, 3, \ldots; \xi \in V)$.

Theorem

There is a unique algebra isomorphism $\gamma_k \colon \widetilde{\mathbb{D}}_k \to \mathbb{C}[V]^W$ *such that* $DP_\lambda^{(k)} = -\gamma_k(D)(\lambda + \rho_k)\, P_\lambda^{(k)} \quad (D \in \widetilde{\mathbb{D}}_k,\ \lambda \in P_+)$.

Heckman's approach (cntd.)

The theorem implies that the $P_\lambda^{(k)}$ form an orthogonal system.

For $f \in \mathcal{A}^W$: $\quad \mathrm{ad}(f)(\mathcal{D}_{\xi,\mathrm{H}}^{(k)}) = -i\partial_\xi f$.

Hence $\qquad \mathrm{ad}(f_1)\ldots\mathrm{ad}(f_{r+1})(\mathcal{D}_{\xi,r}^{(k)}) = 0$,

in particular for $\mathcal{D}_{\xi,r}^{(k)}$ restricted to $\mathcal{A}^W$.

We can identify $\mathcal{A}^W$ with $\mathbb{C}[m_1,\ldots,m_d]$.

Hence $\mathcal{D}_{\xi,r}^{(k)}$ acting on $\mathbb{C}[m_1,\ldots,m_d]$ is in the Weyl algebra $\mathbb{C}[m_1,\ldots,m_d,\frac{\partial}{\partial m_1},\ldots,\frac{\partial}{\partial m_d}]$.

So $\mathcal{D}_{\xi,r}^{(k)}\big|_{\mathcal{A}^W}$, and more generally operators $D \in \widetilde{\mathbb{D}}_k$ coincide with partial differential operators when acting on $\mathcal{A}^W$.

Example: Laplacian

$$\left(\sum_{j=1}^{d}\big(\mathcal{D}_{e_j,H}^{(k)}\big)^2 f\right)(x) = -(\Delta f)(x) - \sum_{\alpha\in R_+} k_\alpha \cot(\tfrac12\langle\alpha,x\rangle)(\partial_\alpha f)(x)$$
$$+\frac14\sum_{\alpha\in R_+}\frac{k_\alpha\langle\alpha,\alpha\rangle}{\sin^2(\frac12\langle\alpha,x\rangle)}\big(f(x)-f(s_\alpha x)\big)$$
$$-\frac14\sum_{\alpha,\beta\in R_+} k_\alpha k_\beta\langle\alpha,\beta\rangle\cot(\tfrac12\langle\alpha,x\rangle)\cot(\tfrac12\langle\beta,x\rangle)$$
$$\times\big(f(x)-f(s_\alpha x)-f(s_\beta x)+f(s_\alpha s_\beta x)\big)$$

If $f\in\mathcal{A}^W$ then

$$\left(\sum_{j=1}^{d}\big(\mathcal{D}_{e_j,H}^{(k)}\big)^2 f\right)(x) = -(\Delta f)(x) - \sum_{\alpha\in R_+} k_\alpha \cot(\tfrac12\langle\alpha,x\rangle)(\partial_\alpha f)(x)$$

Orderings on W and P

$w \in W$ has length $\ell = \ell(w)$ if $w = s_{i_1} \dots s_{i_\ell}$ with ℓ minimal.

If $w \in W$ then $\ell(w) = |wR_+ \cap (-R_+)|$.

Let $\ell(w) = \ell$ and $w = s_{i_1} \dots s_{i_\ell}$. Then:
$-\alpha \in wR_+ \cap (-R_+) \iff s_\alpha w = s_{i_1} \dots \widehat{s_{i_k}} \dots s_{i_\ell}$ for some k.

Bruhat order on W: $u \le w$ if u has reduced expression which is subword of reduced expression of w.

Partial ordering on P: For $\lambda \in P$ let $\lambda^* \in (W\lambda) \cap P_+$ and $\lambda = w^\lambda \lambda^*$ with $\ell(w^\lambda)$ minimal. $W^{\lambda^*} := \{w^\lambda \mid \lambda \in W\lambda^*\}$.
Put $\lambda \le_W \mu$ if either $\lambda^* < \mu^*$ or $\lambda^* = \mu^*$ and $w^\lambda \ge w^\mu$.

For $\lambda \in P_+$ let $W_\lambda := \{w \in W \mid w\lambda = \lambda\}$ and let w_λ be the longest element of W_λ.

Cherednik-Opdam approach

From now on:

$$\mathcal{D}_\xi^{(k)} := -i\,\partial_\xi + \sum_{\alpha\in R_+} k_\alpha \langle \xi,\alpha\rangle \frac{1}{1-e^{-\alpha}}\,(\mathrm{id}-s_\alpha) - \langle \rho_k,\xi\rangle.$$

Proposition

$\mathcal{D}_\xi^{(k)}\, e^\lambda = a_{\lambda,\lambda} e^\lambda + \sum_{\mu<_W\lambda} a_{\lambda,\mu} e^\mu$ *with*

$a_{\lambda,\lambda} = \widetilde{\lambda} := \lambda + \sum_{\alpha\in R_+;\,\langle\lambda,\alpha\rangle>0} k_\alpha \alpha - \rho_k.$

Indeed, $\dfrac{1}{1-e^{-\alpha}}\,(1-s_\alpha)\,e^\lambda = \dfrac{1-e^{-\langle\lambda,\check\alpha\rangle\alpha}}{1-e^{-\alpha}}\,e^\lambda =$

$$\sum_{j=0}^{\langle\lambda,\check\alpha\rangle-1} e^{\lambda-j\alpha} \text{ if } \langle\lambda,\alpha\rangle\ge 0 \text{ and } \sum_{j=1}^{-\langle\lambda,\check\alpha\rangle} e^{\lambda+j\alpha} \text{ if } \langle\lambda,\alpha\rangle<0.$$

If $\langle\lambda,\alpha\rangle<0$ then $\langle\lambda^*,(w^\lambda)^{-1}\alpha\rangle<0$. Hence $(w^\lambda)^{-1}\alpha\in -R_+$.
Hence $-\alpha\in w^\lambda R_+$. Hence $w^{s_\alpha\lambda}=s_\alpha w^\lambda < w^\lambda$.

This does not work in the Heckman case.

Cherednik-Opdam approach (cntd.)

$$\widetilde{\lambda} := \lambda + \sum_{\alpha \in R_+;\, \langle \lambda, \alpha \rangle > 0} k_\alpha \alpha - \rho_k$$

Proposition

$\widetilde{\lambda} = w^\lambda \widetilde{\lambda^*}$. Furthermore, if $\lambda \in P_+$ then $\widetilde{\lambda} = w_\lambda(\lambda + \rho_k)$. Hence $\widetilde{\lambda} \neq \widetilde{\mu}$ if $\lambda \neq \mu$.

Cherednik-Opdam approach (cntd.)

Definition (non-symmetric Jacobi polynomial associated with R)

This is an element $E_\lambda^{(k)}$ ($\lambda \in \mathcal{P}$) of $\mathcal{A}$ of the form

$$E_\lambda^{(k)} = \sum_{\mu \in P;\ \mu \leq_W \lambda} c_{\lambda,\mu} e^\mu \quad \text{such that } c_{\lambda,\lambda} = 1 \quad \text{and}$$

(i) $\langle E_\lambda^{(k)}, e^\mu \rangle_k = 0$ if $\mu \in P$ and $\mu <_W \lambda$.

Instead of (i) we can equivalently require that

(i)′ $\mathcal{D}_\xi^{(k)} E_\lambda^{(k)} = \langle \widetilde{\lambda}, \xi \rangle E_\lambda^{(k)}$, where

$$\widetilde{\lambda} = \lambda + \sum_{\alpha \in R_+;\ \langle \lambda, \alpha \rangle > 0} k_\alpha \alpha - \rho_k .$$

Theorem

The $E_\lambda^{(k)}$ ($\lambda \in P$) form an orthogonal basis of $\mathcal{A}$ with respect to the inner product $\langle\, .\,,\, .\, \rangle_k$.

Cherednik-Opdam approach (cntd.)

Corollary

For $\lambda \in P_+$: $\quad P_\lambda^{(k)} = \sum_{w \in W^\lambda} w E_\lambda^{(k)} = |W_\lambda|^{-1} \sum_{w \in W} w E_\lambda^{(k)}$.

The $P_\lambda^{(k)}$ $(\lambda \in P_+)$ *form an orthogonal basis of* $\mathcal{A}^W$ *with respect to the inner product* $\langle\, .\, ,\, .\, \rangle_k$.

$\xi \mapsto \mathcal{D}_\xi^{(k)} \colon V \to \mathrm{End}(\mathcal{A})$, linear map with commuting images. This extends to algebra homomorphism

$$p \mapsto p(\mathcal{D}^{(k)}) \colon \mathbb{C}[V] \to \mathrm{End}(\mathcal{A}).$$

Proposition

If $p \in \mathbb{C}[V]^W$ *then* $p(\mathcal{D}^{(k)})$ *is* W*-invariant.*

(because in the image of the center of a graded Hecke algebra)

Cherednik-Opdam approach (cntd.)

Let $\lambda \in P_+$ and $p \in \mathbb{C}[V]^W$.

$$\begin{aligned} p(\mathcal{D}^{(k)})P_\lambda^{(k)} &= |W_\lambda|^{-1} \sum_{w\in W} p(\mathcal{D}^{(k)})wE_\lambda^{(k)} \\ &= |W_\lambda|^{-1} \sum_{w\in W} wp(\mathcal{D}^{(k)})E_\lambda^{(k)} \\ &= p(\widetilde{\lambda})\,|W_\lambda|^{-1} \sum_{w\in W^\lambda} wE_\lambda^{(k)} \\ &= p(\lambda+\rho_k)P_\lambda^{(k)}\,. \end{aligned}$$

by W-invariance of $p(\mathcal{D}^{(k)})$ and since $\widetilde{\lambda} = w_\lambda(\lambda+\rho_k)$.

Hence $\gamma_k\Big(p(\mathcal{D}^{(k)})\big|_{\mathcal{A}^W}\Big) = -p$.

Cherednik-Opdam approach (cntd.)

For $f \in \mathcal{A}^W$: $\quad \mathrm{ad}(f)(\mathcal{D}_\xi^{(k)}) = -i\partial_\xi f$.

Hence, for $p \in \mathbb{C}[V]$ of degree n and $f_1, \ldots, f_{n+1} \in \mathcal{A}^W$:

$$\mathrm{ad}(f_1)\ldots\mathrm{ad}(f_{n+1})(p(\mathcal{D}^{(k)})) = 0.$$

For $p \in \mathbb{C}[V]^W$ the operator $p(\mathcal{D}^{(k)})$ acts on $\mathcal{A}^W$, hence on $\mathbb{C}[m_1, \ldots, m_d]$.

Hence $p(\mathcal{D}^{(k)})$ acting on $\mathbb{C}[m_1, \ldots, m_d]$ is in the Weyl algebra $\mathbb{C}[m_1, \ldots, m_d, \frac{\partial}{\partial m_1}, \ldots, \frac{\partial}{\partial m_d}]$.

Graded Hecke algebra

Definition (Lusztig)

For R a reduced root system on V with set R_+ of positive roots, Weyl group W and multiplicity function k the graded Hecke algebra $\mathbf{H}$ is the algebra generated by the polynomial algebra $\mathbb{C}[V]$ and the group algebra $\mathbb{C}[W]$ with relations

$$s_i.\xi = \xi.s_i - k_i\langle\xi,\alpha_i\rangle \quad (\xi \in V,\, i = 1,\ldots,d).$$

If R is nonreduced root system with multiplicities k, let R^0 be the root system of inmultiplicable roots. If α_i is a short simple root in R then let $2\alpha_i$ with multiplicity $k_i + 2k_{2\alpha_i}$ be the corresponding simple root in R^0.

Theorem

Let $\langle\{\mathcal{D}_\xi^{(k)}\}, W\rangle$ be the algebra generated by the operators $\mathcal{D}_\xi^{(k)}$ ($\xi \in V$) and w ($w \in W$) acting on $\mathbb{C}[V]$. Then the map $\xi.w \mapsto \mathcal{D}_\xi^{(k)}\, w$ extends to an algebra isomorphism of $\mathbf{H}$ *onto $\langle\{\mathcal{D}_\xi^{(k)}\}, W\rangle$.*

Graded Hecke algebra (cntd.)

$s_i.\xi = \xi.s_i - k_i\langle\xi,\alpha_i\rangle$ translates as
$s_i\mathcal{D}_\xi s_i - \mathcal{D}_{s_i\xi} = -k_i\langle\xi,\alpha_i\rangle s_i$. Indeed, this holds:

$$\mathcal{D}_\xi^{(k)} := -i\,\partial_\xi + \sum_{\alpha\in R_+} k_\alpha\langle\xi,\alpha\rangle\,\frac{1}{1-e^{-\alpha}}\,(\mathrm{id}-s_\alpha) - \langle\rho_k,\xi\rangle.$$

Then

$$\begin{aligned} s_i\mathcal{D}_\xi s_i - \mathcal{D}_{s_i\xi} &= \sum_{\alpha\in R_+} k_\alpha\Big(\frac{\langle\xi,s_i\alpha\rangle}{1-e^{-s_i\alpha}} - \frac{\langle\xi,\alpha\rangle}{1-e^{-\alpha}}\Big) \\ &\quad - \sum_{\alpha\in R_+} k_\alpha\Big(\frac{\langle\xi,s_i\alpha\rangle}{1-e^{-s_i\alpha}}\,s_{s_i\alpha} - \frac{\langle\xi,\alpha\rangle}{1-e^{-\alpha}}\,s_\alpha\Big) \\ &\quad - \langle\rho_k,s_i\xi\rangle + \langle\rho_k,\xi\rangle. \end{aligned}$$

Graded Hecke algebra (cntd.)

Proposition

The center of $\mathbf{H}$ *equals* $\mathbb{C}[V]^W$.

Hence the center of $\langle\{\mathcal{D}_\xi^{(k)}\}, W\rangle$ *equals* $\langle\{\mathcal{D}_\xi^{(k)}\}\rangle^W$.

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